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An E_8 Correspondence for Multiplicative Eta-Products

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Abstract. We describe an E_8 correspondence for the multiplicative eta-products of weight at least 4.

There are two E_8 correspondences due to McKay. The first is part of the well-known *McKay correspondence* [15], linking finite subgroups of $\mathrm{SU}(2)$, Coxeter–Dynkin diagrams, and Kleinian singularities. The second correspondence is known as *McKay's Monstrous E_8 observation* [2, p. 528]. This second correspondence is a link between certain rational conjugacy classes of the Monster finite simple group and the nodes of the affine E_8 diagram. The Monstrous classes concerned are those which consist of elements that arise as the product of two involutions of type $2A$ in the monster. Glauberman and Norton [7] extended McKay's observation to the diagrams obtained by deleting one node of the E_8 diagram. Lam, Yamada, and Yamauchi [13, 14] and Lam and Yamauchi [12] made a connection between McKay's second correspondance and certain coset subalgebras of the lattice vertex operator algebra $V_{\sqrt{2}E_8}$. The second author [6] has also recently shown that the discrete groups associated by moonshine with McKay's monstrous classes may be associated with the affine E_8 diagram.

In this note we provide a variation on this theme, showing that the 9 multiplicative η-products of weight at least 4 may be associated with the nodes of the affine E_8 diagram. These 9 classes naturally correspond to 9 conjugacy classes in the Mathieu group M_{24} and we observe that, as for the monster, the 9 classes concerned are those whose elements arise as the product of two involutions of a certain conjugacy class of elements of order 2 in M_{24}, *viz.*, the class consisting of permutations of cycle shape $2^8 1^8$. We also show how McKay's Monstrous E_8 observation may be recovered from the correspondence presented here via a "super-theta" construction.

Recall that a sequence $\lambda = (\lambda_1, \dots, \lambda_k)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$, is called a *partition* of the number $N = \lambda_1 + \lambda_2 + \cdots + \lambda_k$. The numbers $\lambda_1, \lambda_2, \dots, \lambda_k$ are called the *parts* of the partition λ. The number of parts, k, of λ is called the *length* of the partition. For example $\lambda = (8, 8, 4, 4)$ is a partition of 24 into 4 parts. This partition may conveniently be denoted $8^2 4^2$ in an obvious exponential notation. To each partition λ we can associate an η-product, $\eta_\lambda(z) = \prod_i \eta(\lambda_i z)$. So that, for example, the η-product associated with the partition $8^2 4^2$ is $\eta(8z)^2 \eta(4z)^2$.

Dummit, Kisilevsky, and McKay [5] have classified all the multiplicative η-products. There are 9 such products of weight at least 4. These are listed in Table 1, which is taken from [5].

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Table 1: Multiplicative η-products of weight at least 4

Partition	Weight	Level	χ
$6^2 3^2 2^2 1^2$	4	6	
$5^4 1^4$	4	5	
$4^4 2^4$	4	8	
3^8	4	9	
$4^4 2^2 1^4$	5	4	$\left(\frac{-1}{d}\right)$
$3^6 1^6$	6	3	
2^{12}	6	4	
$2^8 1^8$	8	2	
1^{24}	12	1	

For any partition λ, let $n(\lambda)$ be the largest part of λ, and let $h(\lambda)$ be the smallest part. Set $N(\lambda) = n(\lambda)h(\lambda)$. Then each η-product η_λ in Table 1 is a modular form on $\Gamma_0(N(\lambda))$ with some character χ. (See [5] for details.)

For each positive integer m, let $\omega(m)$ be the number of distinct prime divisors of m. Then for any λ such that $h(\lambda)$ divides $n(\lambda)$, define the *valence* $v(\lambda)$ to be $\omega(n/h) + 1$, and define the *parity* $p(\lambda)$ to be $(-1)^{n/h+h}$. It will be convenient to identify the η-product $\eta_\lambda(z)$ with the partition λ, so that we may write $n(\eta_\lambda)$ for $n(\lambda)$, and similarly for h, v, and p.

Let $\mathcal{M}$ be the set of multiplicative η-products of weight at least 4. The significance of the parameters n and h for the η-products of $\mathcal{M}$ is given by the following proposition.

Proposition 1 *Suppose that η_λ is an element of $\mathcal{M}$. Then the subgroup of* $\mathrm{SL}(2,\mathbb{R})$ *which fixes η_λ up to a* 24*-th root of unity is the group $\Gamma_0(n(\lambda)|h(\lambda))^+$ in the notation of Conway and Norton* [4].

Proof Let $n = n(\lambda)$, $h = h(\lambda)$, and $f(z) = \eta_\lambda(z)$. By [5], f is a modular form on $\Gamma_0(N)$, where $N = nh$. (See also Newman [16].) Let G be the subgroup of $\mathrm{SL}(2,\mathbb{R})$ which fixes f as a modular form. Since f, or its square, is of even weight and is not a polynomial, it follows from a result of Knopp [10] that G is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$.

Now by a result of Siegel [17], the area of a fundamental domain of G is bounded below by $\pi/21$ and hence the index of $\Gamma_0(N)$ in G is finite. It follows that G is commensurable with $\mathrm{SL}(2,\mathbb{Z})$. By a result of Helling [8,9] (see also [3]), every subgroup of $\mathrm{SL}(2,\mathbb{R})$, which is commensurable with $\mathrm{SL}(2,\mathbb{Z})$, is conjugate to a subgroup of a group of the form $\Gamma_0(K)^+$ for some square-free integer K. The notation is that of Conway and Norton where $\Gamma_0(K)^+$ is $\Gamma_0(K)$ extended by all the Atkin–Lehner elements. We will call these groups *Helling groups*. Since K is square-free, $\Gamma_0(K)^+$ is the normalizer of $\Gamma_0(K)$ in $\mathrm{SL}(2,\mathbb{R})$ [1,4]. In the notation of Conway and Norton, if h

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divides n, then the group $\Gamma_0(n|h)^+$ is the conjugate $A^{-1}\Gamma_0(n/h)^+A$, where

$$A = h^{-1/2}\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}.$$

By Helling's theorem, these groups are maximal discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$.

We consider first the cases for which $h \neq 1$. The two partitions 2^{12} and 3^8 correspond to $\eta(2z)^{12}$ and $\eta(3z)^8$, respectively. By the transformation properties of the eta function, these are fixed, up to 24-th roots of unity, by two conjugates of the modular group, *viz.*, $\Gamma_0(2|2)$ and $\Gamma_0(3|3)$, respectively. Since these are maximal discrete groups, they are the groups which fix these forms up to 24-th roots of unity.

The partition $4^4 2^4$ corresponds to $\eta(4z)^4\eta(2z)^4$, which is a form on $\Gamma_0(8)$. This group is a subgroup of $\Gamma_0(4|2)^+$, which is generated over $\Gamma_0(8)$ by the following two matrices.

$$\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}, \quad 2^{-3/2}\begin{pmatrix} 0 & -1 \\ 8 & 0 \end{pmatrix}.$$

It is straightforward to verify that $\eta(4z)^4\eta(2z)^4$ is invariant up to 24-th roots of unity under these transformations. Thus the group which fixes this form up to 24-th roots of unity is $\Gamma_0(4|2)^+$, as again this group is a maximal discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ by Helling's result.

For the cases with $h = 1$, it is straightforward to verify that each f is invariant, up to 24-th roots of unity, under the appropriate Atkin–Lehner elements, using the known transformation properties of $\eta(z)$. This shows that f is fixed up to 24-th roots of unity by $\Gamma_0(n)^+$. With one exception, all the n arising are square-free. So these are maximal discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$, and hence are the groups fixing the corresponding η_λ up to 24-th roots of unity. The exception is $\lambda = 4^4 2^2 1^4$. The corresponding f is fixed by $\Gamma_0(4)^+$, which is $\Gamma_0(4)$ extended by the element

$$\begin{pmatrix} 0 & -1/2 \\ 2 & 0 \end{pmatrix}.$$

The question then is whether or not f is fixed by a larger group in this case.

The group $\Gamma_0(4)^+$ is conjugate in $\mathrm{SL}(2,\mathbb{R})$ to $\Gamma_0(2)$. For $\Gamma_0(2)$ to be a subgroup of a Helling group $\Gamma_0(K)^+$ say, the area of a fundamental domain of $\Gamma_0(K)^+$ must be an integral multiple of π, since the area of a fundamental domain of $\Gamma_0(2)$ is π. The only possibilities for K then are $K = 1, 2, 5, 6$. For $K = 5$ or 6 the ratio is 1, but this is impossible since $\Gamma_0(2)$ has cusp number equal to 2, while $\Gamma_0(5)^+$ and $\Gamma_0(6)^+$ have cusp number equal to 1. Thus $\Gamma_0(2)$ can only be a subgroup of a conjugate of $\mathrm{SL}(2,\mathbb{Z})$ or $\Gamma_0(2)^+$, with index 3 or 2, respectively. So the only larger groups which are candidates to be the fixing groups up to 24-th roots of unity of f are conjugates of $\mathrm{SL}(2,\mathbb{Z})$ and conjugates of $\Gamma_0(2)^+$. These groups have cusp number equal to 1. This would imply for a suitable choice of local parameter that f would have the same Fourier coefficients at each cusp. However, an explicit calculation of the expansions of f at, say, $1/2$ and ∞ shows that this is not the case. We conclude that the fixing group is $\Gamma_0(4)^+$, as required. ∎

We next show that there is an affine E_8 diagram associated with the elements of $\mathcal{M}$. The main ideas of the proof follow [6].

Proposition 2 *Let $\mathcal{M}_1$ be the nodes of parity* 1 *and $\mathcal{M}_{-1}$ be the nodes of parity* -1. *Then there is a unique graph with vertex set $\mathcal{M}$, such that*

- *the valence of the node η_λ is $v(\lambda)$;*
- *for each node $x \in \mathcal{M}$, we have $2n(x) = \sum_{y \in \mathrm{Adj}(x)} n(y)$, where* $\mathrm{Adj}(x)$ *is the set of nodes adjacent to x;*
- *if $x \in \mathcal{M}_{-1}$, then the nodes adjacent to x are in $\mathcal{M}_1$.*

The graph is the affine E_8 graph shown in Figure 1.

We call the second condition here *condition L*. We call the third condition the *parity condition*.

Proof The node 1^{24} has $v(1^{24}) = 1$ and $n(1^{24}) = 1$, and so its adjacent node is either 2^{12} or $2^8 1^8$. However, $v(2^{12}) = 1$ and $n(2^{12}) = 2$, which would give a single component of the graph that would not satisfy condition L. Thus the node adjacent to 1^{24} is $2^8 1^8$. A similar argument resolves the placement of the nodes 3^8 and $3^6 1^6$. The only remaining ambiguity is the placement of the nodes $4^4 2^4$ and $4^4 2^2 1^4$, but this is resolved by the parity condition. ∎

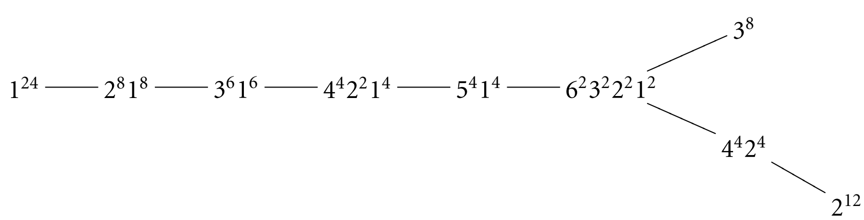

Figure 1

Associated with each η-product in $\mathcal{M}$ is a conjugacy class in M_{24} whose cycle type is λ. The associated classes share a property similar to the monstrous classes of McKay's monstrous E_8 correspondence.

Proposition 3 *The conjugacy classes of M_{24} of cycle type λ, where η_λ is a multiplicative η-product of weight at least* 4, *are those classes whose elements are the product of two involutions of cycle shape* $2^8 1^8$.

Proof This is a character calculation. ∎

Replacing each η-product with the discrete group of Proposition 1 gives rise to the E_8 diagram of Figure 2. (For convenience we omit the "Γ_0" in the name of the group.) This diagram is remarkably similar to McKay's monstrous E_8 correspondence: the node $\Gamma_0(2|2)$ is replaced by $\Gamma_0(2)$ in McKay's correspondence, and for $h \neq 1$, the groups appearing in Figure 2 are what Conway and Norton call the "eigen-groups"

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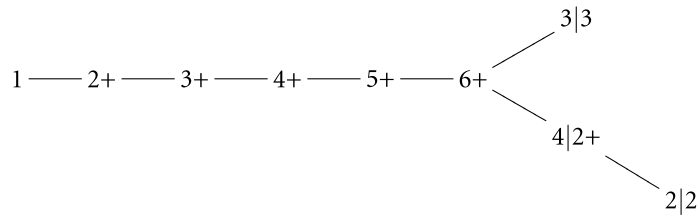

Figure 2

of the corresponding Hauptmodul, *i.e.*, the groups which fix the Hauptmodul up to a root of unity, rather than the fixing groups of these Hauptmoduls.

Note that the definition of the parity of a node used in [6] is based on whether or not the inequality $\mathrm{Index}(G \cap \Gamma_0(2) : \Gamma_0(2)) \leq 2$ is satisfied. As $\Gamma_0(2|2) \cap \Gamma_0(2) = \Gamma_0(2)$, this inequality is satisfied by both $\Gamma_0(2)$ and $\Gamma_0(2|2)$. However, although $\Gamma_0(2|2)$ contains $\Gamma_0(4)$ normally and $\Gamma_0(2)$ non-normally, the index is not a power of 2 and so the groups of Figure 2 are not a solution to the problem posed in [6] which requires the index to be a power of 2.

A second connection with McKay's monstrous E_8 can be made via a conjecture of Conway and Norton connecting elements of M_{24} with monstrous moonshine. For an element $m \in M_{24}$ considered as an automorphism of the Leech lattice, the conjecture, proved by Kondo and Tasaka [11], states that Θ_m/η_m is a Hauptmodul attached to a rational conjugacy class of the monster, where Θ_m is the theta function of the sublattice of the Leech lattice fixed by m, and η_m is the η-product attached to m.

Once again, the resulting labelling of the E_8 diagram is that of McKay's observation, except for the node corresponding to 2^{12} which is associated with the monstrous class $4A$ with fixing group $\Gamma_0(4)^+$. It is interesting to note that the only two partitions which give rise, in this construction, to the same Monstrous class are $4^4 2^2 1^4$ and 2^{12}. Both correspond to the Monstrous class 4A (see, for example, [11, Table 2]).

There is, however, a way of obtaining the "correct" monstrous class, $2B$, for the 2^{12} node by the following "super" construction. The Hauptmoduls for the monstrous classes $2B$ and $4A$ have q-coefficients which are the same in absolute value. The coefficients for $4A$ are non-negative and those of $2B$ alternate in sign.

Suppose C_λ is the subcode of the Golay code invariant under the action of an element of class λ. We can ask if C_λ is a "blow-up" of some code of smaller length. More precisely, suppose D is a code of length m and k is a positive integer. Then we can consider the length mk code $D + \cdots + D$ (k times). Call the diagonal embedding of D in this code the *k-fold blow-up* of D. Then C_λ is an h-fold blow-up for $h = h_\lambda$ for each λ (and h is in each case the largest positive integer with this property).

Let $C' = C'_\lambda$ be the code from which $C = C_\lambda$ is obtained by h-fold blow-up. Then C' is doubly-even for all λ except when $\lambda = 2^{12}$. If $h = 1$, then $C' = C$ which is certainly doubly-even, being a subcode of the Golay code. If $h = 3$, then $\lambda = 3^8$ and C' is the Hamming code. If $h = 2$ and $\lambda = 4^4 2^4$, then C' is a code with weight enumerator $1 + 7X^4 + 7X^8 + X^{12}$. Finally if $h = 2$ and $\lambda = 2^{12}$, then C' is the unique self-dual even code of length 12.

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An even code admits a natural $\mathbb{Z}/2\mathbb{Z}$-grading: the 0-graded part being the subcode consisting of doubly even codewords. The case $\lambda = 2^{12}$ is just the case that C' is non-trivially $\mathbb{Z}/2\mathbb{Z}$-graded.

Kondo and Tasaka [11] explain how to compute Θ_λ using weight enumerators for $C(\lambda)$. Repeating this construction, but with the weight enumerators replaced by super-weight enumerators, now has the effect of replacing the monstrous class $4A$ with the class $2B$ and thus recovering McKay's monstrous E_8 correspondence exactly.

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