TRACE RINGS OF GENERIC MATRICES ARE UNIQUE FACTORIZATION DOMAINS

by LIEVEN LE BRUYN*

(Received 3 August, 1984)

Trace rings of generic matrices are U.F.D. A. W. Chatters and D. A. Jordan defined in **[0]** a unique factorization ring to be a prime ring in which every height one prime ideal is principal. In this note we will prove that the trace ring of m generic $n \times n$-matrices satisfies this condition.

Throughout this note, k will be a field of characteristic zero. Consider the polynomial ring $S = k[t^l_{ij}; 1 \le i, j \le n, 1 \le l \le m]$ and the $n \times n$ matrices $X_l = (t^l_{ij})$ in $M_n(S)$. The k-subalgebra of $M_n(S)$ generated by $\{X_l; 1 \le l \le m\}$ is called the ring of m generic $n \times n$ matrices $G_{m,n}$. Adjoining to it the traces of all its elements we obtain the trace ring $\mathbb{T}_{m,n}$ of m $n \times n$ generic matrices, cfr. e.g. **[1]**.

The main aim of this note is to prove:

THEOREM 1. *Height one prime ideals of $\mathbb{T}_{m,n}$ are cyclic.*

M. Artin and A. Schofield **[2]** have proved that $\mathbb{T}_{m,n}$ is always a maximal order, cfr. **[3]**. The following two results liberate their proof from Hilbert–Mumford theory.

LEMMA 2. *Let B be a unique factorization domain and G a group of automorphisms of B s.t. $H^1(G, B^*) = 1$. If A is the fixed ring of B under G, then:*

(a) *$A \subset B$ satisfies no blowing up*

(b) *A is a unique factorization domain.*

Proof. (a) Let $P = Bp$ be a height one prime of B s.t. $P \cap A \neq 0$. Then P has a finite orbit under G, say $\{Bp, Bp_1, \ldots, Bp_k\}$ (take an element $a \in P \cap R$ write $a = p^k q_1^{l_1} \ldots q_m^{l_m}$, then $\sigma(Bp) \in \{Bp, Bq_i\}$). This shows that for every $\sigma \in G$ there exists a unit $f_\sigma \in B^*$ s.t. $\sigma(p \cdot p_1 \ldots p_k) = f_\sigma \cdot pp_1 \ldots p_k \{f_\sigma; \sigma \in G\}$ is clearly a 1-cocycle, so by assumption there exists a unit $\alpha \in B^*$ s.t. $f_\sigma = \sigma(\alpha)\alpha^{-1}$ for every $\sigma \in G$. Replace p by $p' = \alpha^{-1} . p$, then $p'p_1 \ldots p_k \in A$. Therefore, any element $0 \neq \alpha \in P \cap A$ can be written as $a = (p'p_1 \ldots p_k)^l q_1^{l_1} \ldots q_l^{l_l}$, i.e., $a \in (p'p_1 \ldots p_k)A$. So, $P \cap A = (p'p_1 \ldots p_k)A$ and therefore $A \subset B$ satisfies no blowing up (b). By (a) and [4] we know that the natural map $Cl(A) \to Cl(B)$ is a homomorphism. Suppose Q is a nonprincipal height one prime of A, then $(BQ)^{**} = Bp_1^{l_1} \ldots p_m^{l_m}$ for irreducible elements $p_i \in B$. Clearly, $Q = Bp_1 \cap A$ which is a principal ideal by the proof of part (a), done.

Let us return to trace rings. Procesi **[6]** has shown that there exists an action of $PGL_n(k)$ on S and $M_n(S)$ s.t.

(1) $PGL_n(k)$ acts trivially on k

(2) The fixed ring of $M_n(S)$ under $PGL_n(k)$ equals $\mathbb{T}_{m,n}$

(3) The fixed ring of S under $PGL(k)$ equals $R_{m,n}$, the center of $\mathbb{T}_{m,n}$.

* Work supported by an NFWO/FNRS-grant.

COROLLARY 3. *The extension $R_{m,n} \subset S$ satisfies no blowing up and $R_{m,n}$ is a unique factorization domain.*

Proof. Because $PGL_n(k)$ acts trivially on k, $H^1(PGL_n(k), S^*) = H^1(PGL_n(k), k^*) = Hom(PGL_n(k), k^*) = 1$ because $PGL_n(k)$ is a simple group.

It follows immediately from this result that $\mathbb{T}_{m,n}$ is reflexive as an $R_{m,n}$-module. A reflexive order Λ over a normal domain R is said to be a reflexive Azumaya algebra, cfr. e.g. [**8**], if the natural map:

$$\phi : (\Lambda \otimes_R \Lambda^{opp})^{**} \to End_R(\Lambda)$$

is an isomorphism. It is fairly easy to show that for every divisorial Λ-ideal I (i.e. a fractional Λ-ideal which is reflexive as an R-module) $I = \Lambda(I \cap R)^{**}$. We are now in a position to prove Theorem 1.

***Proof of Theorem* 1.** The proof of the Artin–Schofield theorem shows that the localization of $\mathbb{T}_{m,n}$ at every central height one prime ideal p is an Azumaya algebra (except if $m = n = 2$). This shows that ϕ_p is an isomorphism for every $p \in X^{(1)}(R)$. $\mathbb{T}_{m,n}$ being a reflexive $R_{m,n}$-module, this yields that $\mathbb{T}_{m,n}$ is a reflexive Azumaya algebra and the theorem follows from Corollary 3 and the remark above.

If $m = n = 2$, then the only height one prime which is not centrally generated is $\mathbb{T}_{2,2}(XY - YX)$ which is cyclic, done.

We will give two applications of this result:

THEOREM 4 (Montgomery). *Every automorphism of $G_{m,n}$ which leaves the center invariant is the identity.*

Proof. By the Skolem–Noether theorem such an automorphism is given by conjugation with a normalizing element of $G_{m,n}$, h. $G_{m,n} \subset \mathbb{T}_{m,n}$ being a central extension, h is also a normalizing element of $\mathbb{T}_{m,n}$, i.e. $\mathbb{T}_{m,n} . h$ is a divisorial $\mathbb{T}_{m,n}$-ideal.

If m or $n \neq 2$, this entails that $h = \gamma . c$ for some $\gamma \in \mathbb{T}_{m,n} = k$ and c in the field of fractions of $R_{m,n}$, done.

If $m = n = 2$, the only noncentral normalizing element of $\mathbb{T}_{m,n}$ is $XY - YX$. This element does not normalize $G_{2,2}$, done.

If Λ is a maximal order over a normal domain R in some central simple algebra Σ, we denote by $h(\Lambda)$ the (pointed) set of left Λ-module isomorphism classes of left fractional Λ-ideals which are reflexive R-modules and with $t_R(\Sigma)$ we denote the conjugacy classes of maximal R-orders in Σ.

THEOREM 5. *There is a one-to-one correspondence between $h(\mathbb{T}_{m,n})$ and $t_{R_{m,n}}(Q(\mathbb{T}_{m,n}))$. (m and n not both equal to 2).*

Proof. The correspondence is given by assigning to an (isomorphism class) of a left fractional $\mathbb{T}_{m,n}$-ideal L, its right order

$$O_r(L) = \{X \in Q(\mathbb{T}_{m,n}) : Lx \subset L\}$$

https://doi.org/10.1017/S0017089500006261 Published online by Cambridge University Press

This map is well defined and epimorphic because for any maximal $R_{m,n}$-order in $Q(\mathbb{T}_{m,n})$ we can take:

$$L = (\Lambda :_l \mathbb{T}_{m,n}) = \{x \in Q(\mathbb{T}_{m,n}) : x\Lambda \subset \mathbb{T}_{m,n}\}$$

cfr., e.g., [**4**]. Now suppose L and L' are left fractional $\mathbb{T}_{m,n}$-ideals s.t. $O_r(L) = \alpha^{-1} . \Theta_r(L')\alpha$, then replacing L' by $L'\alpha$ we may assume that $\Theta_r(L) = \Theta_r(L') = \Lambda$.

Let $M = ((\Lambda :_r \mathbb{T}_{m,n})L)^{**}$ and $M' = ((\Lambda :_r \mathbb{T}_{m,n})L')^{**}$, then M and M' are twosided divisorial Λ-ideals, i.e., $M = M' . c$ for some element c in the field of fractions of $R_{m,n}$ (because Λ is also a reflexive Azumaya algebra and therefore every twosided divisorial Λ-ideal is generated by a central element).

Finally

$$L = ((\mathbb{T}_{m,n} :_r \Lambda)M)^{**} = ((\mathbb{T}_{m,n} :_r \Lambda)M'c)^{**} = L'c$$

finishing the proof.

REMARK. Even if one restricts attention to projective left $\Delta_{m,n}$-ideals, some of them are not free (cfr. [**7**] in $m = n = 2$ case and similarly in $m = 3$, $n = 2$ case [**5**]), so there are maximal orders over $R_{m,n}$ not conjugated to $\mathbb{T}_{m,n}$.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ANTWERP, UIA
UNIVERSITEITSPLEIN 1
2610 WILRIJK, BELGIUM

https://doi.org/10.1017/S0017089500006261 Published online by Cambridge University Press