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Weak Semiprojectivity for Purely Infinite C^*-Algebras

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Abstract. We prove that a separable, nuclear, purely infinite, simple C^*-algebra satisfying the universal coefficient theorem is weakly semiprojective if and only if its K-groups are direct sums of cyclic groups.

Introduction

The first definition of semiprojectivity for C^*-algebras was given by Effros and Kaminker in the context of noncommutative shape theory [3]. A more restrictive definition was given by Blackadar [1]. Loring introduced a third definition, which he termed *weak semiprojectivity*, in his investigations of stability problems for C^*-algebras defined by generators and relations [8]. Recently, Neubüser introduced a slew of variants, the most important being what he called *asymptotic semiprojectivity* [9]. Using the authors' initials to represent the above notions, the implications among them are: B $\Rightarrow$ N $\Rightarrow$ EK, L.

All versions of semiprojectivity are of the following form: $*$-homorphisms into inductive limit C^*-algebras can be lifted (in some sense) to a finite stage of the limit (the precise definitions may be found in Section 1 and in the references). As a consequence, among the first (and easiest) examples for which semiprojectivity was established are the Cuntz–Krieger algebras. This drew attention to the class of separable, nuclear, purely infinite simple C^*-algebras, now commonly referred to as *Kirchberg* algebras [11]. Kirchberg, and independently Phillips, have shown that in the presence of the universal coefficient theorem, K-theory is a complete invariant for Kirchberg algebras [10]. Blackadar proved [2] that for such algebras, finitely generated K-theory is necessary for semiprojectivity in the sense of [3]. He conjectured that for these algebras, finitely generated K-theory is sufficient for semiprojectivity in the sense of [1], and he proved this for the case of free K_0 and trivial K_1. Szymański extended this to the case that $\operatorname{rank} K_1 \leq \operatorname{rank} K_0$ [17], and in [14] semiprojectivity was proved whenever K_1 is free. The conjecture remains open in the case that K_1 has torsion. The methods used in all previous work on the conjecture rely upon explicit models for these algebras, constructed from directed graphs. In another direction, Neubüser used abstract methods to show that (for the algebras under consideration) finitely generated K-theory is equivalent to asymptotic semiprojectivity.

In this paper we study weak semiprojectivity for UCT-Kirchberg algebras. We prove that such an algebra is weakly semiprojective if and only if its K-groups are direct sums of cyclic groups. The key difficulty lies in dealing with torsion in K_1, where

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we are forced to use tensor products of known semiprojectives. Semiprojectivity is badly behaved with respect to tensor products, and we rely on Neubüser's result to get started. Our contribution is thus in extending to the case where the K-theory is not finitely generated. Another crucial technical aid is an alternative characterization of weak semiprojectivity, due to Eilers and Loring [4].

Our method of proof uses explicit models for the C^*-algebras constructed from a hybrid object which is partly a directed graph and partly a 2-graph (in the sense of [6].) The construction of this object, and the proof that it defines a UCT-Kirchberg algebra having the desired K-theory and given by suitable generators and relations, appears in [15].

The outline of the paper is as follows. In Section 1 we prove the necessity in the main theorem. This involves a kind of finite approximation property for abelian groups. In Section 2 we prove the main theorem. During the final stages of writing an earlier draft of this paper we learned of Huaxin Lin's preprint of [7], where the same theorem is proved by different means.

1 Direct Sums of Cyclic Groups

The definition of weak semiprojectivity that follows is not Loring's original one, but was proved to be equivalent to it [4, Theorem 3.1].

Definition 1.1 The C^*-algebra A is called *weakly semiprojective* if given a C^*-algebra B with ideals $I_1 \subseteq I_2 \subseteq \cdots \subseteq I = \overline{\bigcup_k I_k}$, a $*$-homomorphism $\pi\colon A \to B/I$, a finite set $M \subseteq A$, and $\epsilon > 0$, there exist n and a $*$-homomorphism $\phi\colon A \to B/I_n$ such that

$$\|\pi(x) - \nu_n \circ \phi(x)\| < \epsilon \quad \text{for } x \in M,$$

where $\nu_n\colon B/I_n \to B/I$ is the quotient map.

It is sometimes convenient to replace the increasing sequence of ideals by a directed family.

We remark that if M and ϵ are omitted, and it is required that $\pi = \nu_n \circ \phi$, then we recover Blackadar's definition of semiprojectivity. Neubüser's definition of asymptotic semiprojectivity can be obtained by omitting M and ϵ and replacing ϕ by a point-norm continuous path ϕ_t such that for every $x \in A$, $\lim_t \|\pi(x) - \nu_n \circ \phi_t(x)\| = 0$.

Definition 1.2 An abelian group G has *Property C* (for *cyclic*, see Proposition 1.5) if for every finite set $F \subseteq G$, there exist a finitely generated abelian group K, and homomorphisms $\alpha\colon G \to K$, $\beta\colon K \to G$ such that $\beta \circ \alpha(x) = x$ for all $x \in F$.

Lemma 1.3 *Let A be a UCT-Kirchberg algebra. If A is weakly semiprojective, then $K_*(A)$ has Property C.*

Proof By Kirchberg,[1] $A = \overline{\bigcup A_n}$, $A_n \subseteq A_{n+1}$, where each A_n is a UCT-Kirchberg

[1] *The classification of purely infinite C^*-algebras using Kasparov's theory.* Ms., 1994, to appear in Fields Institute Communication series.

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algebra with finitely generated K-theory. We modify the mapping telescope construction slightly (see [8]). Let

$$B = \left\{ f \in C([0,1], A) \;\middle|\; f(t) \in A_n \text{ for } t \geq \frac{1}{n} \right\},$$
$$J_n = \left\{ f \in B \;\middle|\; f|_{[0,1/n]} = 0 \right\},$$
$$J = \overline{\bigcup J_n} = \left\{ f \in B \;\middle|\; f(0) = 0 \right\}.$$

Then

$$B/J_n \cong \left\{ f|_{[0,1/n]} \;\middle|\; f \in B \right\}, \quad B/J \cong A.$$

Now let $[x_1], \dots, [x_k] \in K_*(A)$. Then by weak semiprojectivity there are n and $\phi\colon A \to B/J_n$ such that

$$\|\phi(x_i)(0) - x_i\| < 1, \quad 1 \leq i \leq k.$$

Since $\phi(x_i)(1/n)$ is homotopic to $\phi(x_i)(0)$, we have that

$$[x_i] = [\phi(x_i)(1/n)], \quad 1 \leq i \leq k.$$

Let $\alpha\colon K_*(A) \to K_*(A_n)$ be given by $\alpha([x]) = [\phi(x)(1/n)]$, and let $\beta\colon K_*(A_n) \to K_*(A)$ be induced from the inclusion. Then $\beta \circ \alpha([x_i]) = [x_i]$ for $1 \leq i \leq k$. ■

Lemma 1.4 *Let G be a countable abelian group with Property C. Then G/G_{tor} is free.*

Proof By [5, Exercise 52], it suffices to show that every finite rank subgroup of G/G_{tor} is free. So let $H \subseteq G/G_{\text{tor}}$ be a subgroup of finite rank. Put $\overline{H} = \pi^{-1}(H)$, where π is the quotient map of G onto G/G_{tor}. Let $e_1, \dots, e_r$ be a basis for H. Then we may write $\mathbf{Z}^r \subseteq H \subseteq \mathbf{Q}^r$ (relative to this basis). Let $\overline{e_1}, \dots, \overline{e_r} \in \overline{H}$ with $\pi(\overline{e_i}) = e_i$, $1 \leq i \leq r$. Let $K, \alpha\colon G \to K$ and $\beta : K \to G$ be as in Property C, with $\beta \circ \alpha(\overline{e_i}) = \overline{e_i}$ for $1 \leq i \leq r$.

We claim that $\ker(\alpha|_{\overline{H}}) \subseteq G_{\text{tor}}$. To see this, let $y \in \ker(\alpha|_{\overline{H}})$. Choose $N \in \mathbf{Z}$ such that $N\pi(y) \in \mathbf{Z}^r$. We may write

$$\pi(Ny) = \sum_{i=1}^{r} c_i e_i, \quad c_i \in \mathbf{Z}.$$

Then $z = Ny - \sum_{i=1}^{r} c_i \overline{e_i} \in \ker \pi = G_{\text{tor}}$. Thus

$$0 = N\beta \circ \alpha(y) = \beta \circ \alpha(Ny) = \beta \circ \alpha(z) + \sum_{i=1}^{r} c_i \overline{e_i}.$$

But since $\beta \circ \alpha(G_{\text{tor}}) \subseteq G_{\text{tor}}$, we may apply π to the last equation to get $0 = \sum_{i=1}^{r} c_i e_i$. It follows that $c_i = 0$ for all i, so that $Ny = z \in G_{\text{tor}}$. Hence $y \in G_{\text{tor}}$.

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Next we claim that $\ker(\pi_K \circ \alpha|_{\overline{H}}) = G_{\text{tor}}$, where π_K is the quotient map of K onto K/K_{tor}. To see this, first note that the containment $\supseteq$ is obvious. For the other containment, let $y \in \ker(\pi_K \circ \alpha|_{\overline{H}})$. Then $\alpha(y) \in K_{\text{tor}}$, so $N\alpha(y) = 0$ for some $N \in \mathbf{Z} \setminus \{0\}$. Then $Ny \in \ker(\alpha|_{\overline{H}})$, so $Ny \in G_{\text{tor}}$ by the previous claim. Hence $y \in G_{\text{tor}}$.

Finally, it follows from the last claim that $\pi_K \circ \alpha|_{\overline{H}}$ induces an injection $H \to K/K_{\text{tor}}$, which implies that H is free. ∎

Proposition 1.5 *Let G be a countable abelian group. Then G has Property C if and only if G is a direct sum of cyclic groups.*

Proof It is clear that a direct sum of cyclic groups has Property C. Conversely, by Lemma 1.4, $G \cong G_{\text{tor}} \oplus G/G_{\text{tor}}$, where G/G_{tor} is free, and hence a direct sum of (infinite) cyclic groups. Since $G_{\text{tor}} = \bigoplus_p G_p$, where G_p is the p-primary component of G_{tor}, it suffices to prove that G_p is a direct sum of cyclic groups. By [5, Theorem 11], it suffices to prove that G_p contains no element of infinite height. To see this, let $x \in G_p \setminus \{0\}$. Choose K, $\alpha\colon G \to K$, and $\beta\colon K \to G$ as in Property C so that $\beta \circ \alpha(x) = x$. We have $\alpha(x) \in K_p$, the p-primary component of K. Let n be the maximal height of elements of K_p. Now if $x = p^j y$ in G, then

$$x = \beta \circ \alpha(x) = \beta \circ \alpha(p^j y) = \beta(p^j \alpha(y)) = 0, \quad \text{if } j > n.$$

Therefore $j \le n$, and so x is of finite height. ∎

Corollary 1.6 *Let A be a UCT-Kirchberg algebra. If A is weakly semiprojective, then $K_*(A)$ is a direct sum of cyclic groups.*

2 The Main Theorem

We now wish to prove the converse of Corollary 1.6, establishing weak semiprojectivity for any UCT-Kirchberg algebra whose K-theory is a direct sum of cyclic groups. To do this we will use the models for UCT-Kirchberg algebras constructed in [15]. For each $k \ge 2$ let H_k be the directed graph shown in Figure 1.

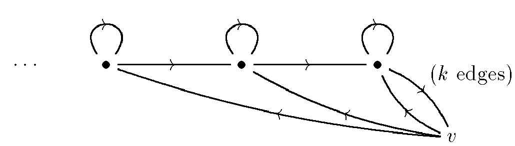

Figure 1: H_k.

One easily checks that $K_*\mathcal{O}(E_k) = (\mathbf{Z}/(k), 0)$ (see [16]). We let H_∞ denote the usual directed graph of the Cuntz algebra $\mathcal{O}_\infty$: one vertex with denumerably many loops. Finally we let $\overline{H}_\infty$ denote the graph shown in figure 2. Again one can easily check that $K_*\mathcal{O}(\overline{H}_\infty) = (0, \mathbf{Z})$. We remark that the graphs H_k, H_∞ and $\overline{H}_\infty$ have a

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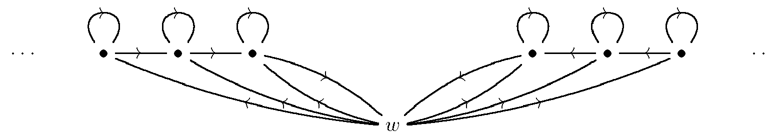

Figure 2: $\overline{H}_\infty$.

distinguished vertex emitting infinitely many edges, as required for the construction in [15].

We now construct a UCT-Kirchberg algebra having as K-theory a prescribed direct sum of cyclic groups. Let $G^i = (G^i_0, G^i_1)$ for $i = 0, 1, \dots$, where for each i, one of G^i_0, G^i_1 is a cyclic group and the other is the zero group. By the Künneth formula [12], G^i is equal either to $K_*(\mathcal{O}(H_{k_i}) \otimes \mathcal{O}(H_\infty)$ or to $K_*(\mathcal{O}(H_{k_i}) \otimes \mathcal{O}(\overline{H}_\infty))$ for some $k_i \in \{2, 3, \dots, \infty\}$. Let $E_i = H_{k_i}$ and $F_i = H_\infty$ or $\overline{H}_\infty$ so that $G^i = K_*(\mathcal{O}(E_i) \otimes \mathcal{O}(F_i))$. As in [15], we let Ω denote the hybrid object constructed from the product 2-graphs $E_i \times F_i$ and the connecting 1-graphs D_i. By [15, Theorem 4.8], $C^*(\Omega)$ is a UCT-Kirchberg algebra with K-theory equal to $\bigoplus_i G^i$. Let $A = C^*(\Omega)$.

We briefly recall the definition of the C^*-algebra $C^*(\Omega)$ from [15]. First let us recall the definitions of the C^*-algebras of a directed graph in a form convenient for this purpose. A *directed graph* E consists of two sets, E^0 (the *vertices*) and E^1 (the *edges*), together with two maps $o, t \colon E^1 \to E^0$ (*origin* and *terminus*). We let $\mathcal{O}(E)$ denote the C^*-algebra of E. It is the universal C^*-algebra defined by generators $\{P_a \mid a \in E^0\}$ and $\{S_e \mid e \in E^1\}$ with the *Cuntz–Krieger relations*:

- $\{P_a \mid a \in E^0\}$ are pairwise orthogonal projections.
- $S_e^* S_e = P_{t(e)}$, for $e \in E^1$.
- $o(e) = o(f) \Rightarrow S_e S_e^* + S_f S_f^* \le P_{o(e)}$, for $e, f \in E^1$ with $e \neq f$.
- $0 < \# E^1(a) < \infty \Rightarrow P_a = \sum\{S_e S_e^* \mid o(e) = a\}$, for $a \in E^0$,

where in the fourth relation we use the notation $E^1(a)$ to denote the set of edges with origin a. (These are a variant of the relations given in [13, Theorem 2.21].)

The relationship between the C^*-algebras of a graph and a subgraph are crucial to our methods. We refer to [13]. The results are as follows. Let E be a graph and let F be a subgraph of E. We let $S = S(F)$ be the set of vertices in F^0 that do not emit more edges in E than in F. We let $\mathcal{TO}(F, S)$ denote the relative Toeplitz Cuntz–Krieger algebra of F in E. It is the universal C^*-algebra defined by generators $\{P_a \mid a \in F^0\}$ and $\{S_e \mid e \in F^1\}$ with the relations (as above) for $\mathcal{O}(F)$, modified by requiring the fourth relation only if $a \in S$. Then $\mathcal{TO}(F, S)$ is the C^*-subalgebra of $\mathcal{O}(E)$ generated by the projections and partial isometries associated to the vertices and edges of F [13, Theorem 2.35].

The hybrid object Ω is constructed from a directed graph D (see Figure 3), and the sequence of product 2-graphs $E_i \times F_i$ [6]. We let v_i, respectively w_i, denote the distinguished vertex in E_i, respectively F_i, emitting infinitely many edges, and we form Ω by attaching $E_i \times F_i$ to D by identifying u_i with (v_i, w_i). By a *vertex* of Ω we mean an element of $\bigcup_i (E_i^0 \times F_i^0) \cup D^0$, where we identify u_i and (v_i, w_i). By an *edge* we mean an element of $\bigcup_i \big((E_i^1 \times F_i^0) \cup (E_i^0 \times F_i^1)\big) \cup D^1$. The C^*-algebra of Ω is

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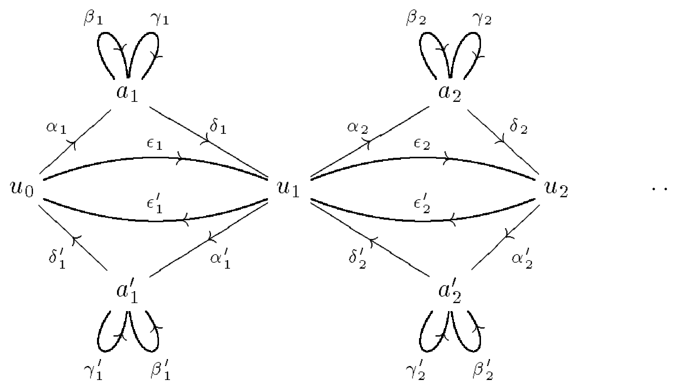

Figure 3: D

defined by generators and relations as follows. We let $\mathcal{S}$ denote the set of symbols

$$\{P_x \mid x \text{ is a vertex}\} \cup \{S_y \mid y \text{ is an edge}\}.$$

We let $\mathcal{R}$ denote the following set of relations on $\mathcal{S}$.

(i) P_x is a projection for every vertex x; S_y is a partial isometry for every edge y.

(ii) For every $a \in E_i^0$, the projections for $\{a\} \times F_i^0$ and the partial isometries for $\{a\} \times F_i^1$ satisfy the Cuntz–Krieger relations corresponding to the graph F_i (see the discussion at the end of the Section 1).

(iii) For every $b \in F_i^0$, the projections for $E_i^0 \times \{b\}$ and the partial isometries for $E_i^1 \times \{b\}$ satisfy the Cuntz–Krieger relations corresponding to the graph E_i.

(iv) The projections for D^0 and the partial isometries for D^1 satisfy the Toeplitz–Cuntz–Kriger relations which correspond to the graph D and the vertices $\{a_i, a_i' : i \geq 0\}$.

(v) If μ and ν are edges of types D and $E_i \times F_i$, respectively, then $S_\mu^* S_\nu = 0$.

(vi) For all $e \in E_i^1$ and $f \in F_i^1$ we have

$$S_{(o(e),f)}\, S_{(e,t(f))} = S_{(e,o(f))}\, S_{(t(e),f)},$$
$$S_{(t(e),f)}\, S_{(e,t(f))}^* = S_{(e,o(f))}^*\, S_{(o(e),f)}.$$

Then $A = C^*(\Omega) = C^*\langle \mathcal{S}, \mathcal{R}\rangle$ is the universal C^*-algebra given by these generators and relations.

Lemma 2.1 *The C^*-algebra $A = C^*(\Omega)$ is weakly semiprojective.*

Proof Let $\Omega_{(n)}$ be the subobject of Ω consisting of $E_0 \times F_0, \dots, E_n \times F_n$, and the portion of D having subscript less than or equal to n. (We use parentheses in order to avoid confusion with the notation of [15].) Theorem 4.8 of [15] applies to $\Omega_{(n)}$, so that $A_n = C^*(\Omega_{(n)})$ is a UCT-Kirchberg algebra with finitely generated K-theory. Note that the generators and relations defining $C^*(\Omega_{(n)})$ are the same in $C^*(\Omega_{(n)})$ as in $C^*(\Omega)$ (this is essentially because of the infinite valence of the vertex u_n). Thus A_n is a C^*-subalgebra of A, and $A = \overline{\bigcup_n A_n}$. By [9, Satz 6.12], A_n is uniformly asymptotically semiprojective. It follows from [4, Theorem 3.1] that A_n is weakly semiprojective.

Let B be a C^*-algebra with ideals $I_1 \subseteq I_2 \subseteq \cdots \subseteq I = \overline{\bigcup_k I_k}$, and let $\pi\colon A \to B/I$ be a $*$-homomorphism. Let $M \subseteq A$ be a finite set, and let $\epsilon > 0$. Choose n and a finite set $M' \subseteq A_{n-1}$, such that $d(x, M') < \epsilon/2$ for all $x \in M$. Since A_n is weakly semiprojective, there are k and a $*$-homorphism $\phi_0\colon A_n \to B/I_k$, such that

$$\|\pi(x') - \nu_k \circ \phi_0(x')\| < \epsilon/2 \quad \text{for } x' \in M',$$

where $\nu_k\colon B/I_k \to B/I$ is the quotient map. We will construct a $*$-homomorphism $\phi\colon A \to B/I_k$ extending $\phi_0|_{A_{n-1}}$. Then it will follow that

$$\|\pi(x) - \nu_k \circ \phi(x)\| < \epsilon \quad \text{for } x \in M,$$

concluding the proof.

Let $p = P_{u_{n+1}}$ and $q = P_{u_n}$, the projections in A corresponding to the vertices u_{n+1} and u_n. The hereditary subalgebra pAp of A contains a hereditary subalgebra, C, isomorphic to A. (This follows easily from the pure infiniteness of A. See the proof of [14, Theorem 3.12].) Let $\psi_1\colon A \to C$ be a $*$-isomorphism. Since the inclusion of C into A induces the identity in K-theory, it follows that ψ_{1*} is an automorphism of $K_*(A)$. It follows from [10, Theorem 4.2.1] that there is a $*$-automorphism α of A with $\alpha_* = \psi_{1*}$. Let $\psi_2 = \psi_1 \circ \alpha^{-1}$. Then $\psi_2\colon A \to C$ is a $*$-isomorphism, and ψ_{2*} is the identity in K-theory. Let $x \in A$ be a partial isometry with $x^*x = q$ and $xx^* = \psi_2(q)$. Increasing k if necessary, we may find a partial isometry $z \in B/I_k$ with $z^*z = \phi_0(q)$ and $zz^* = \phi_0 \circ \psi_2(q)$. We define $\phi\colon A \to B/I_k$ by defining it on the generators $\mathcal{S}$ of A [15, Definition 3.3]:

$$\phi(s_y) = \begin{cases} \phi_0(s_y) & y \in \Omega_{n-1}, \\ \phi_0 \circ \psi_2(s_y) & y \notin \Omega_{n-1} \text{ and } o(y),\ t(y) \neq u_n, \\ (\phi_0 \circ \psi_2(s_y))z^* & y \notin \Omega_{n-1} \text{ and } t(y) = u_n,\ o(y) \neq u_n, \\ z(\phi_0 \circ \psi_2(s_y)) & y \notin \Omega_{n-1} \text{ and } o(y) = u_n,\ t(y) \neq u_n, \\ z(\phi_0 \circ \psi_2(s_y))z^* & y \notin \Omega_{n-1} \text{ and } o(y) = t(y) = u_n. \end{cases}$$

It is easy to see that the elements $\phi(s_y)$ satisfy the relations $\mathcal{R}$ of [15, Theorem 3.3], and hence ϕ defines a $*$-homomorphism. ∎

Theorem 2.2 *Let A be a UCT-Kirchberg algebra. Suppose that $K_*(A)$ is a direct sum of cyclic groups. Then A is weakly semiprojective.*

Proof As in the proof of [14, Theorem 3.12], it suffices to prove that if A is unital and $\mathcal{K} \otimes A$ is weakly semiprojective, then A is weakly semiprojective. (We are relying on the classification theory of [10], as well as the theorem of [18] that nonunital separable, simple, purely infinite C^*-algebras are stable.) Let $u_1, u_2, \ldots \in A$ with $u_i^* u_j = \delta_{ij}$. Put $A_0 = \overline{\operatorname{span}}\{u_i A u_j^*\}$. Then A_0 is isomorphic to $\mathcal{K} \otimes A$.

Let $A = B/I$, where I is the closure of a directed family of ideals $\mathcal{L}$ of B. (Since A is simple, we may dispense with the homomorphism π of Definition 1.1.) For $J \in \mathcal{L}$, we let $\pi\colon B \to B/I$, $\pi_J\colon B \to B/J$ denote the quotient maps. Put $B_0 = \pi^{-1}(A_0)$. We will use [4, Theorem 3.1]. So let $F \subseteq A$ be a finite set, and let $\epsilon > 0$. We may assume that $\epsilon < 1$ and that $1 \in F$. Choose $\gamma < 1$ such that $3\gamma\|x\| < \epsilon$ for all $x \in F$. Since A_0 is weakly semiprojective by hypothesis, there is a $*$-homomorphism $\psi_{00}\colon A_0 \to B/J$ such that

$$\|\pi \circ \psi_{00}(x) - x\| < \gamma \quad \text{for } x \in u_1 F u_1^*.$$

In particular, we have $\|\pi \circ \psi_{00}(u_1u_1^*) - u_1u_1^*\| < \gamma$. Choose $v \in B$ with $\pi(v) = u_1$. By increasing J if necessary, we may assume that $\pi_J(v)$ is an isometry. Since

$$\|\pi(vv^*) - \pi \circ \psi_{00}(u_1u_1^*)\| < \gamma,$$

we may assume, again by increasing J if necessary, that $\|\pi_J(vv^*) - \psi_{00}(u_1u_1^*)\| < \gamma$. Then there exist $s, t \in B$ such that

$$\pi_J(s^*s) = \pi_J(vv^*), \quad \pi_J(ss^*) = \psi_{00}(u_1u_1^*),$$
$$\pi_J(t^*t) = 1 - \pi_J(vv^*), \quad \pi_J(tt^*) = 1 - \psi_{00}(u_1u_1^*),$$
$$\|\pi_J(s) - \pi_J(vv^*)\| < \gamma, \quad \|\pi_J(t) - (1 - \pi_J(vv^*))\| < \gamma.$$

Let $z = s + t$. Then $\pi_J(z)$ is unitary, and $\|\pi_J(z) - 1\| < \gamma$. Define $\psi_0\colon A_0 \to B/J$ by

$$\psi_0(x) = \pi_J(z)^*\psi_{00}(x)\pi_J(z).$$

Then ψ_0 is a $*$-homomorphism, and

$$(*) \qquad \begin{aligned} \|\pi \circ \psi_0(x) - x\| &= \|\pi(z)^*\pi \circ \psi_{00}(x)\pi(z) - x\| \\ &\le 2\|\pi(z) - 1\|\,\|x\| + \|\pi \circ \psi_{00}(x) - x\| \\ &\le 2\gamma\|x\| + \gamma < \epsilon, \quad \text{for } x \in u_1Fu_1^*, \end{aligned}$$

$$(**) \qquad \begin{aligned} \psi_0(u_1u_1^*) &= \pi_J(z)^*\psi_{00}(u_1u_1^*)\pi_J(z) \\ &= \pi_J(s)^*\psi_{00}(u_1u_1^*)\pi_J(s) \\ &= \pi_J(vv^*). \end{aligned}$$

Now define $\psi\colon A \to B/J$ by $\psi(x) = \pi_J(v)^*\psi_0(u_1xu_1^*)\pi_J(v)$. By $(**)$ we have that ψ is a $*$-homomorphism. For $x \in F$, we have

$$\begin{aligned} \|\pi \circ \psi(x) - x\| &= \|u_1^*\pi \circ \psi_0(u_1xu_1^*)u_1 - x\| \\ &= \|u_1^*(\pi \circ \psi_0(u_1xu_1^*) - u_1xu_1^*)u_1\| \\ &\le \|\pi \circ \psi_0(u_1xu_1^*) - u_1xu_1^*\| \\ &< \epsilon, \quad \text{by } (*). \end{aligned}$$

■

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