Relating constructions and properties through duality

DAVID L. FARNSWORTH, STEVEN J. KILNER

1. *Introduction*

Our goal is to find new constructions and properties of parabolas. Our strategy is to display the steps in a known construction or property, and then to take the dual of the steps in order to create a new construction or property.

Two decisions are made for simplicity and sufficient generality. One is that throughout, in the original x, y-space, the equation of the parabola is the *standard form*

$$C : y = \frac{x^2}{4p}, \tag{1}$$

with $p > 0$. Any parabola can be converted to (1) by applying linear transformations (see [1, pp. 666-673]).

The second choice is the specific duality transformation from among the many duality transformations that are available (see [2, pp. 20-23]). We utilise the transformation that is based upon the form $ax + by = 1$ for lines. It is the same duality transformation that was successfully used in [3] to find a new construction for points of a parabola by applying a duality transformation to the well-known triangle-construction method for tangent lines of a parabola. Henceforth, *dual* refers only to this duality.

2. *The duality transformation*

The duality transformation that we use is a mapping between a pair of two-dimensional spaces, which maps points to lines and lines to points with the exception of the origin and lines containing the origin. For clarity and to help to distinguish between these two spaces, we use x, y-coordinates for one and u, v-coordinates for the other. The dual of line L is denoted L', and the dual of point P is denoted P'.

Each line not containing the origin can be expressed uniquely in the form $ax + by = 1$. We may therefore unambiguously define the dual of the line $L : ax + by = 1$ to be the point $L' : (a, b)$ and the dual of the point $P : (a, b)$ to be the line $P' : au + bv = 1$.

Through this duality, one can associate a differentiable curve with its *dual* curve, because points of tangency and tangent lines of a curve in one space are dual to tangent lines and points of tangency, respectively, of the dual curve in the dual space (see [4, pp. 385-388]). Applying the dual to the points of a given differentiable curve results in a family of lines whose envelope is the dual curve in the dual space. Similarly, applying the dual to the family of tangent lines to a differentiable curve results in the points of its dual curve.

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This duality is involutive or reflexive, that is, the inverse transformation follows the same rules, and the transformation of the transformation is the identity (see [4, pp. 24-25]). Because of this property, it can be unimportant which space is identified as the dual space and which the original space.

Theorem 1 establishes the identity of the parabola C' that is dual to parabola C in (1), with the exceptions of the origins. The origins can be made to correspond by continuity.

Theorem 1 (Dual of C): Consider C for $x \neq 0$. Then, except for $u = 0$,

$$C' : \{(u, v) : v = -pu^2\}. \tag{2}$$

Proof: For any $a \neq 0$, the tangent line to (1) at $\left(a, \frac{a^2}{4p}\right)$ is $\frac{2}{a}x - \frac{4p}{a^2}y = 1$. The dual to this line is the point $\left(\frac{2}{a}, \frac{-4p}{a^2}\right)$, which satisfies (2). All points of C', except the origin, are of this form.

As we apply the duality transformation, we want to be able to make sense of the concepts of angle, parallel, perpendicular, and distance in the dual space. With the inclusion of the origin, which has no dual, the set of all points in $\mathbb{R}^2$, equipped with the dot product, gives rise to Euclidean 2-space and the usual definitions of angle, parallel, perpendicular, and distance. By defining a sum, scalar multiplication, and an inner product on the set of all lines of the form $ax + by = 1$, including the special *degenerate line* $0x + 0y = 1$, we establish an inner product space that is isomorphic to Euclidean 2-space. This space possesses the usual definitions of angle, parallel, and perpendicular associated with lines, and also a definition of distance between lines and a way of adding and scaling them.

These ideas are fruitful. For example, perpendicularity of lines in one space implies perpendicularity of the dual points in the dual space. This new perpendicularity can be used to shorten derivations, because it contains a number of steps that do not need to be explicitly indicated. Also, new insights can be revealed.

Definition 1 (Sum of lines, scalar multiplication of a line, inner product of lines, and distance between lines): Let $a, b, c, d, k \in \mathbb{R}^2$, and consider the lines $M : ax + by = 1$ and $N : cx + dy = 1$.

(i) *Sum of lines*:
$M \oplus N = (ax + by = 1) \oplus (cx + dy = 1) = \big((a + c)x + (b + d)y = 1\big)$.

(ii) *Scalar multiplication of a line*:
$k \otimes M = k \otimes (ax + by = 1) = (kax + kby = 1)$.

(iii) *Inner product of lines*: $\langle M, N\rangle = \langle ax + by = 1, cx + dy = 1\rangle = ac + bd$.

(iv) *Distance between lines*: $D(M, N) = \sqrt{(a - c)^2 + (b - d)^2}$.

The inner product in (iii) induces the definition of distance between lines in (iv) and can be used to obtain one of the two supplementary angles

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between a pair of lines. Because the inner product of a pair of lines is equal to the dot product of the corresponding dual points, both distances and angles are preserved by duality. In particular, two non-zero points are perpendicular (parallel) if and only if their dual lines are perpendicular (parallel). Also, all parallel lines, excepting those containing the origin, have dual points lying on a line through the origin.

Theorem 2 (Sum of lines, scalar multiplication of a line, and distance between lines through duality): For lines M and N, the dual relationships for the sum of lines, scalar multiplication of a line, and distance between two lines are

$$(M \oplus N)' = M' + N', (k \otimes M)' = kM', \text{ and } \mathrm{D}(M, N) = \mathrm{D}(M', N'),$$

where the distance function D is understood from its arguments of points or lines.

Proof: For points $P : (a, b)$ and $Q : (c, d)$ in x, y-space, addition yields the point $R = P + Q = (a + c, b + d)$. The dual of R is $R' : (a + c)u + (b + d)v = 1$. The dual lines are $P' : au + bv = 1$ and $Q' : cu + dv = 1$, so that $R' = P' \oplus Q'$. Thus addition of points and addition of lines are dual operations. Proofs of the other two formulas follow similarly.

3. *The pedal-curve construction and its dual construction*

Construction 1 is called the pedal-curve construction. See [5, pp. 33-34], [6, p. 126]. Construction 1 can be used to supply all the tangent lines to C, except the tangent line at the origin. The parabola C is the envelope of these lines. The dual Construction 2 supplies all the points of C, except the origin.

Construction 1 (Pedal-curve construction for a tangent line L to C): See Figure 1. Take a point $Q : (a, 0)$, $a \neq 0$. The focus is $F : (0, p)$. The line $M : \frac{1}{a}x + \frac{1}{p}y = 1$ contains Q and F. Construct the line $L : \frac{1}{a}x - \frac{p}{a^2}y = 1$, which is perpendicular to M and contains Q. Line L is tangent to C at $x = 2a$. The x-axis supplies the tangent line at the vertex, where $a = 0$.

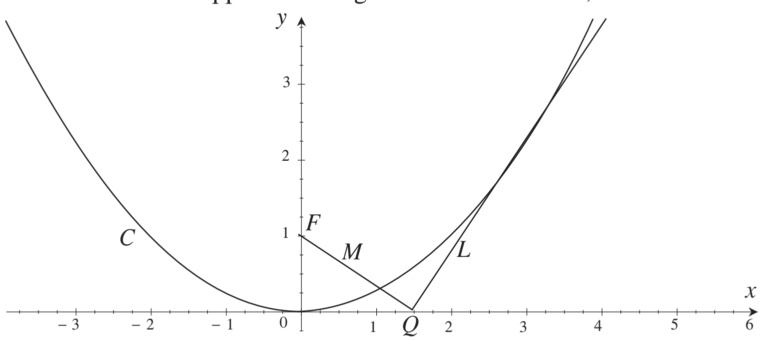

FIGURE 1: The pedal-curve construction for a tangent line to C

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Construction 2 (Dual construction for a point of C): See Figure 2. Take the lines $Q' : au = 1$ and $F' : pv = 1$, which intersect in point $M' : \left(\frac{1}{a}, \frac{1}{p}\right)$. Construct the point $L' : \left(\frac{1}{a}, -\frac{p}{a^2}\right)$, which is perpendicular to M' and on the line Q'. In Figure 2, the perpendicular dashed line segments show this perpendicularity. Point L' is contained in C.

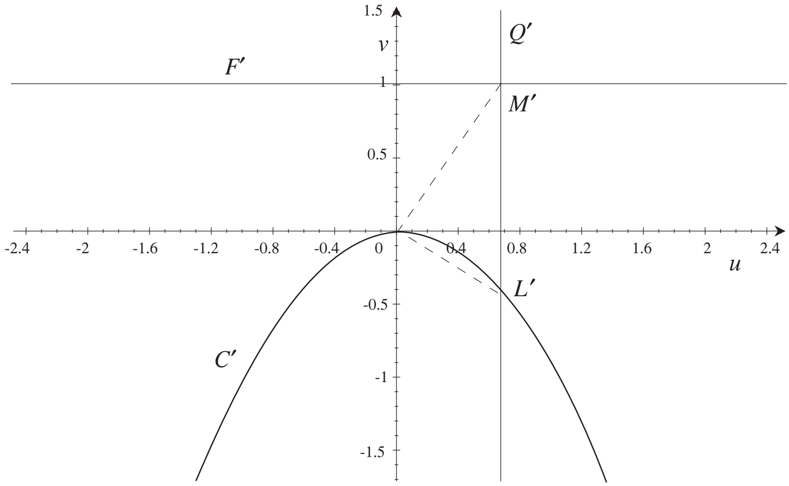

FIGURE 2: The duality transformation of the pedal-curve construction

4. *St Vincent's construction and its dual construction*

Construction 3, which is attributed to Gregory of Saint-Vincent, also known as St Vincent, in 1647, is for all points of the parabola C, excepting the origin (see [7, p. 37]). St Vincent's construction is very specific. It uses a fixed point on the y-axis, and, for each non-zero point on the x-axis, a point of parabola (1) is found. The dual Construction 4 gives all the tangent lines of C', except the line at the origin.

Construction 3 (St Vincent's construction for a point of C): Refer to Figure 3. Take a point $Q : (a, 0)$, $a \neq 0$, on the x-axis. The fixed point is $R : (0, -4p)$ with $p > 0$. The line $M : \frac{1}{a}x - \frac{1}{4p}y = 1$ contains points Q and R. Erect the line $L : \frac{1}{a}x + \frac{4p}{a^2}y = 1$, which is perpendicular to the line M at Q. The intersection of L with the y-axis is $S : \left(0, \frac{a^2}{4p}\right)$. Adding points Q and S gives $Q + S = P : \left(a, \frac{a^2}{4p}\right)$, which is a point of C. The set of all such points, along with (0, 0), is C.

Construction 4 (Dual construction for a line of C): Refer to Figure 4. For fixed $p > 0$, plot the lines $Q' : au = 1$ with $a \neq 0$ and $R' : -4pv = 1$ and the point $M' : \left(\frac{1}{a}, -\frac{1}{4p}\right)$, which is the intersection of Q' and R'. Point $L' : \left(\frac{1}{a}, \frac{4p}{a^2}\right)$ is perpendicular to point M' and is on line Q'. The perpendicular dashed line segments in Figure 4 show how to locate L' geometrically. Plot

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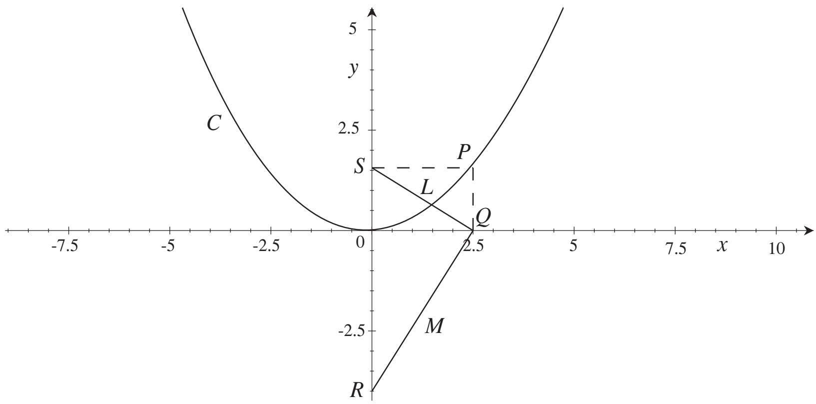

FIGURE 3: St Vincent's construction for a point of C

the line $S' : \left(\frac{a^2}{4p}\right)v = 1$. Performing the dual operation to $Q + S$ in Construction 3 gives $Q' \oplus S' = P' : au + \frac{a^2}{4p}v = 1$, which is tangent to C' at $u = \frac{2}{a}$. The family of all such lines, along with the line $v = 0$, has C' as its envelope.

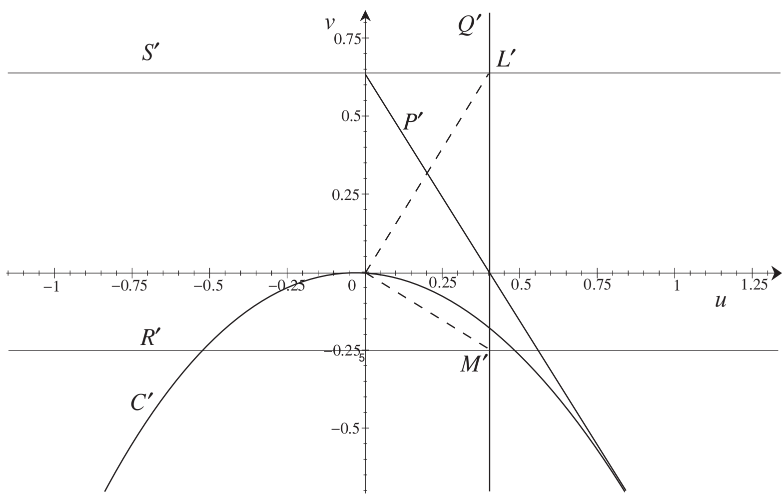

FIGURE 4: The duality transformation of St Vincent's construction

5. *The collinearity of certain points and the dual property*

Property 1 concerns all chords of the parabola C that have a fixed slope. The tangent lines to C at the termini of each such chord intersect at a point. The set of all those points lies on a vertical line that is determined by the slope of the chords. Property 2, which is dual to Property 1, says that each and every point which is dual to a line containing a chord in Property 1, on a line through the origin and external to the parabola, produces two tangent

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lines to the parabola. It concludes that the line through the contact points with the parabola contains a fixed point, which corresponds to the vertical line in Property 1.

Property 1 (Intersections of tangent lines to C at the endpoints of parallel chords are collinear): See Figure 5. Let L be one of the parallel lines of slope c meeting C in two distinct points. The termini of the chord on L are $P\ :\ \left(a,\ \frac{a^2}{4p}\right)$ and $Q\ :\ \left(b,\ \frac{b^2}{4p}\right)$, the equation of L is

$$L\ :\ \frac{a+b}{ab}x - \frac{4p}{ab}y = 1,$$

and the tangent lines at P and Q are $M : \frac{2}{a}x - \frac{4p}{a^2}y = 1$ and $N : \frac{2}{b}x - \frac{4p}{b^2}y = 1$. Lines M and N meet at $R\ :\ \left(\frac{a+b}{2},\ \frac{ab}{4p}\right)$. The slope of L is $\frac{a+b}{4p} = c$. The x-coordinate of the intersection points, such as R, of all pairs of tangent lines is $2pc$, which is a constant. Thus, all these points of intersection are on $T\ :\ x = 2pc$. The arrows in Figure 5 show how the location of the intersection point changes along the line T as one varies the chord of slope c in the indicated direction.

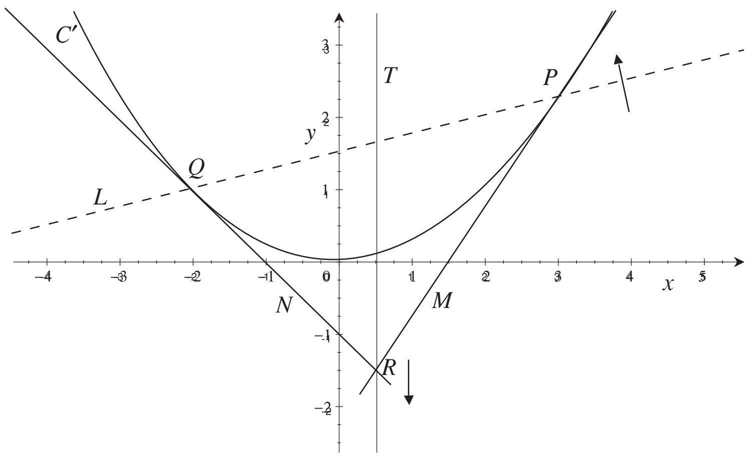

FIGURE 5: All pairs of tangent lines at the endpoints of parallel chords meet on line T

Property 2 (Dual property of Property 1): See Figure 6. By duality, the lines P' and Q' are tangent to C' at points M' and N'. Line R' contains the points $M'\ :\ \left(\frac{2}{a},\ -\frac{4p}{a^2}\right)$ and $N'\ :\ \left(\frac{2}{b},\ -\frac{4p}{b^2}\right)$ and has equation $\frac{a+b}{2}u + \frac{ab}{4p}v = 1$. Because point R is on line T in Property 1, point $T'\ :\ \left(\frac{2}{a+b},\ 0\right)$ is on line R'. This is true for all points R generated by lines that are parallel to L. At the same time, because points Q and P are on line L in Property 1, the intersection of lines P' and Q' is point L'. All lines that are parallel to L in Property 1 have dual points that are collinear with the origin. The equation of that line, which contains L', is $v = -\frac{2p}{c}u$. As point L' moves along this line, the line R' is rotated about the fixed point T'.

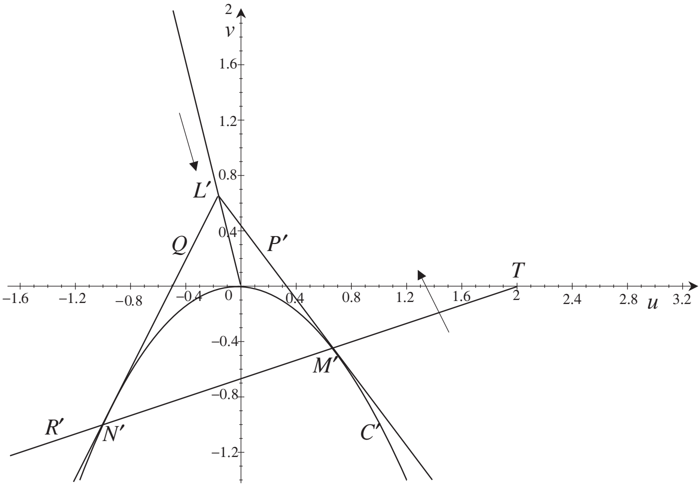

FIGURE 6: The duality transformation of Property 1

6. *The reflection property and its dual property*

The reflection or focusing property of parabolas is Property 3. It says that rays which are inside the curve and parallel to the parabola's axis are reflected at the point of contact to the focus using the Law of Reflection. See [8, p. 752]. Property 4 is the dual of Property 3.

Property 3 (Reflection or focusing property): Refer to Figure 7. For parabola C, consider the portions of the lines $R : \frac{1}{a}x = 1$, $a \neq 0$, and

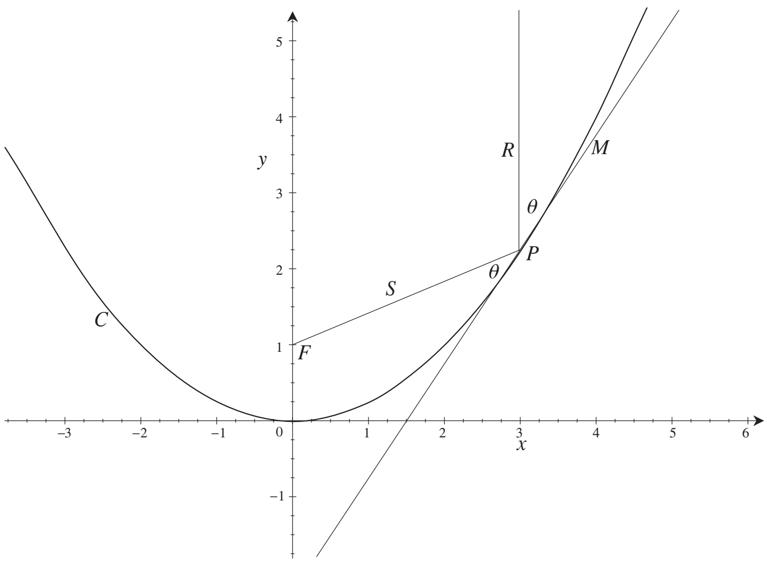

FIGURE 7: The reflection property

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$S : \frac{4p^2 - a^2}{4p^2 a}x + \frac{1}{p}y = 1$, which are interior to C and meet C at point $P : \left(a, \frac{a^2}{4p}\right)$. The tangent line to C at P is $M : \frac{2}{a}x - \frac{4p}{a^2}y = 1$. The Law of Reflection requires the equality of the two angles interior to C between R and M and between S and M (labelled θ in Figure 7). The angle θ between R and M is equal to the angle between R and M determined by the inner product given in Definition 1, while the angle θ between lines S and M is the supplement of the angle between lines S and M determined by the inner product. This fact must be taken into account when we look at the dual property. For all values of a, line S contains the focus $F : (0, p)$.

Property 4 (Dual of the reflection property): See Figure 8. Consider parabola C', points M' and R', and lines P' and F'. Point R' is on the u-axis, and points M' and R' are on line P'. The angle between R' and M' is θ while the angle between S' and M' is the supplementary angle $180° - \theta$ where S' is on P'. By considering the three lines through the origin that separately contain R', M' and S' two equal angles of measure θ can be identified. Duality with Property 3 implies that the three lines P', OS' and F' meet at the point S'.

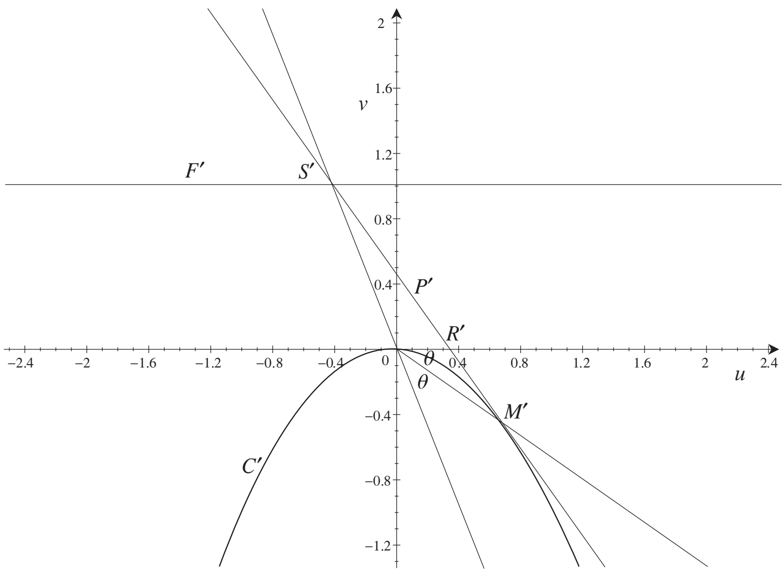

FIGURE 8: Dual to the reflection property

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STEVEN J. KILNER
Department of Mathematics,
1000 East Henrietta Road,
Monroe Community College,
Rochester, NY, 14623 USA
e-mail: *skilner@monroecc.edu*
DAVID L. FARNSWORTH
School of Mathematics and Statistics,
84 Lomb Memorial Drive,
Rochester Institute of Technology,
Rochester, NY 14623 USA
e-mail: *dlfsma@rit.edu*