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A LOCALIZATION OF $R[x]$

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1. Introduction. Throughout this paper, R will be a commutative integral domain with identity and x an indeterminate. If $f \in R[x]$, let $c_R(f)$ denote the ideal of R generated by the coefficients of f. Define $S_R = \{f \in R[x]: c_R(f) = R\}$ and $U_R = \{f \in R(x): c_R(f)^{-1} = R\}$. For $a, b \in R$, write $(a \underset{R}{:} b) = \{r \in R: rb \in (a)\}$. When no confusion may result, we will write $c(f)$, S, U, and $(a:b)$. It follows that both S and U are multiplicatively closed sets in $R[x]$ [**7**, Proposition 33.1], [**17**, Theorem F], and that $R[x]_S \subseteq R[x]_U$.

The ring $R[x]_S$, denoted by $R(x)$, has been the object of study of several authors (see for example [**1**], [**2**], [**3**], [**12**]). An especially interesting paper concerning $R(x)$ is that of Arnold's [**3**], where he, among other things, characterizes when $R(x)$ is a Prüfer domain. We shall make special use of his results in our work.

In § 2 we determine conditions on the ring R so that $R(x) = R[x]_U$. A complete characterization of this property is given for Noetherian domains in Proposition 2.2. In particular, we prove that if R is a Noetherian domain, then $R(x) = R[x]_U$ if and only if depth $(R) \leqq 1$. Some sufficient conditions for $R(x) = R[x]_U$ are that R be treed (Proposition 2.5), or that $\mathscr{P}(R)$ (see § 2 for definitions) be finite (Proposition 2.9).

The main results of this paper occur in § 3. We prove that if R is either a GCD-domain, an integrally closed coherent domain, or a Krull domain, then $R[x]_U$ is a Bezout domain. As is well known [**7**, Theorem 32.7], the Kronecker function ring R^K of an integrally closed domain R is a Bezout domain. Hence, it would seem likely that R^K and $R[x]_U$ would coincide for many rings R. However, this is not the case. In fact, $R^K = R[x]_U$ if and only if R is a Prüfer domain. In general there is no containment relation between R^K and $R[x]_U$ (Remark 3.3). Finally, we apply the results of § 3 to § 1 to obtain new characterizations of Prüfer domains, Bezout domains, and Dedekind domains (Corollary 3.2).

2. When does $R(x) = R[x]_U$? Since $R(x) \subseteq R[x]_U$, it is natural to consider when this inclusion is strict or not. We shall indicate some classes of domains establishing both possibilities. In Section 3, we will further pursue this topic as an application of the results of that section.

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A useful result, which we shall frequently employ, is due to Tang and is stated as follows:

THEOREM 2.0 [**17**, Theorem *E*]. *Let $\mathscr{P}(R) = \{P \in \mathrm{Spec}(R)$: P is minimal over $(a:b)$ for some $a, b \in R\}$.*

(a) *For a finitely generated ideal I of R, $I \subseteq P$ for some $P \in \mathscr{P}(R)$ if and only if $I^{-1} \neq R$; and*

(b) $R = \bigcap_{P \in \mathscr{P}(R)} R_P$.

It is immediate from this result that $U = R[x] \backslash \bigcup_{P \in \mathscr{P}(R)} P[x]$. Hence, since $S = R[x] \backslash \bigcup_{M \in \mathrm{Max}(R)} M[x]$ [**7**, Proposition 33.1], it is clear that $R(x) = R[x]_U$ if and only if $S = U$. Whence, we shall focus our attention on determining when the inclusion $S \subseteq U$ is strict or not.

Our first lemma provides us with a workable necessary condition. For a ring R, the notation depth $(R) \leqq 1$ shall mean that R does not contain any R-sequences of length greater than one.

LEMMA 2.1. *If $S = U$, then* depth $(R) \leqq 1$.

Proof. Suppose that the depth $(R) > 1$, and let $\{a, b\}$ be an R-sequence of length two in R. Since $S = U$, we have that

$$\bigcup_{P \in \mathscr{P}(R)} P[x] = \bigcup_{M \in \mathrm{Max}(R)} M[x].$$

It follows that the element $a + bx \in P[x]$ for some $P \in \mathscr{P}(R)$. Hence, $I = (a, b) \subseteq P$ for some $P \in \mathscr{P}(R)$. However, $I^{-1} = R$ [**13**, Exercise 1, p. 102], and this contradicts Theorem 2.0.

Our next result characterizes Noetherian domains for which $R(x) = R[x]_U$.

PROPOSITION 2.2. *The following are equivalent for a Noetherian domain R:*

(a) depth $(R) \leqq 1$;

(b) Max $(R) \subseteq \mathscr{P}(R)$;

(c) $S = U$;

(d) $M^{-1} \neq R$ *for each* $M \in$ Max (R);

(e) $P^{-1} \neq R$ *for each* $P \in$ Spec (R);

(f) $I^{-1} \neq R$ *for each proper ideal* I *of* R.

Proof. It is clear that (b) $\Rightarrow$ (c), (c) $\Rightarrow$ (*a*) (Lemma 2.1), (d) $\Leftrightarrow$ (e) $\Leftrightarrow$ (f) and (d) $\Leftrightarrow$ (b) (Theorem 2.0). To complete the proof it suffices to show that (a) $\Rightarrow$ (b). We assume R is not a field. Let $M \in$ Max (R), and let a be a nonzero element of M. We claim $M \in \mathrm{Ass}(R/aR)$. (Recall that for a finitely generated R-module N, $\mathrm{Ass}(N) = \{P \in \mathrm{Spec}(R)$: P is the annihilator of some $m \in N$, $m \neq 0\}$.) Suppose not. Then

$$M \nsubseteq \bigcup_{P \in (\mathrm{Ass}\, R/aR)} P,$$

for otherwise $M = P$ for some $P \in \operatorname{Ass}(R/aR)$. Let

$$b \in M \setminus \cup_{P \in \operatorname{Ass}(R/aR)} P,$$

and note that $\{a, b\}$ forms an R-sequence in M. This contradiction substantiates our claim. Hence $M = (a:c)$ for some $c \in R$, and thus $M \in \mathscr{P}(R)$.

COROLLARY 2.3. *The following are equivalent for an integrally closed Noetherian domain R:*

(i) *R is a Dedekind domain;*

(ii) $\dim(R) \leqq 1$ *(Krull dimension);*

(iii) $S = U$.

Proof. Certainly $(a) \Leftrightarrow (b)$ and $(b) \Rightarrow (c)$. To complete the proof we show that $(c) \Rightarrow (b)$. (The following proof will play a useful role in the more general setting of Theorem 3.1 (c).) Suppose $S = U$ and assume M is a maximal ideal of R of height greater than one. Select $0 \neq a \in M$ and let $P_1, \ldots, P_t$ be the height one prime ideals of R containing a. Choose $b \in M \setminus \cup_{i=1}^{t} P_i$, and observe that a and b belong to no common height one prime ideal of R. Hence $(a:b) = (a)$ [**8**, p. 205], and so $\{a, b\}$ forms an R-sequence in M. Therefore $M \notin \mathscr{P}(R)$ (Theorem 2.0). This contradicts Proposition 2.2.

Remark 2.4. Several of the implications in Proposition 2.2 do not depend on the Noetherian assumption. The following diagram shows which implications are true and which implications are false in the general integral domain case.

$$\begin{tikzcd}
(a) \arrow[r, shift left, "/" marking] \arrow[d, shift right, "/" marking] & (b) \arrow[l, shift left] \arrow[r, shift left] \arrow[d, shift right, "/" marking] & (c) \arrow[l, shift left, "/" marking] \arrow[d, shift left, "/" marking] \arrow[ll, bend right] \\
(f) \arrow[u, shift right] \arrow[r, leftrightarrow] & (e) \arrow[u, shift right] \arrow[r, leftrightarrow] & (d) \arrow[u, shift left]
\end{tikzcd}$$

Proof. Clearly $(b) \Rightarrow (c)$, $(c) \Rightarrow (a)$ (Lemma 2.1), $(d) \Rightarrow (a)$, and $(d) \Leftrightarrow (e) \Leftrightarrow (f)$.

$(d) \Rightarrow (b)$: Let $M \in \operatorname{Max}(R)$ such that $M^{-1} \neq R$. Let $a/b \in M^{-1} \setminus R$. Then $aM \subseteq (b)$, and so $M = (b:a)$. Hence $M \in \mathscr{P}(R)$.

$(a) \not\Rightarrow (b)$ and $(c) \not\Rightarrow (b)$: Let V be a valuation ring with infinitely many prime ideal such that the union of the nonmaximal primes equals the maximal ideal. Clearly V satisfies (a) and (c), but does not satisfy (b).

$(b) \not\Rightarrow (d)$: Let W be a finite dimensional valuation ring with a non-finitely generated maximal ideal M. Clearly, $M \in \mathscr{P}(W)$, but the argument above that was used to show $(d) \Rightarrow (b)$ gives us that $M^{-1} = W$.

We conclude this remark with an open question. Does $(a) \Rightarrow (c)$?

Later in this section we shall apply some of the results just considered. For now, we will continue with the theme of determining classes of domains where $S = U$.

A large class of domains where $S = U$ is the class of Prüfer domains. The next proposition provides us with a class of domains strictly containing Prüfer domains and satisfying $S = U$. Recall that a domain R is said to be *treed* in case $\mathrm{Spec}(R)$, as a partially ordered set under inclusion, is a tree.

PROPOSITION 2.5. *If R is treed, then $S = U$.*

Proof. We claim that

$$\bigcup_{P \in \mathscr{P}(R)} P[x] = \bigcup_{M \in \mathrm{Max}(R)} M[x].$$

Since the left-hand side is obviously contained in the right-hand side, let f be a member of the right-hand side. So $f \in M[x]$ for some $M \in \mathrm{Max}(R)$, and we may assume $M \notin \mathscr{P}(R)$. We show that

$$(*) \quad M = \bigcup_{P \subsetneq M, P \in \mathscr{P}(R)} P.$$

If this is not the case, then there exists an $a \in M$ such that a is not in the union. However, this implies that M is minimal over (a), forcing $M \in \mathscr{P}(R)$, and thus a contradiction. Hence, (*) holds and we may conclude, with the aid of the treed assumption, that $c(f) \subseteq P$ for some $P \in \mathscr{P}(R)$. Therefore,

$$f \in \bigcup_{P \in \mathscr{P}(R)} P[x],$$

and so $S = U$.

An interesting and useful fragment contained in the proof of Proposition 2.5 is that if $Q \in \mathrm{Spec}(R)$ and $Q \notin \mathscr{P}(R)$, then

$$Q = \bigcup_{P \subsetneq Q, P \in \mathscr{P}(R)} P.$$

In the following proposition we will indicate an instance when the converse is true, and we will also show, by way of an example, that some additional hypothesis is needed. Recall that a ring R is said to be *coherent* if each finitely generated ideal of R is finitely presented [**6**].

PROPOSITION 2.6. *Let R be coherent and treed. Then, $Q \notin \mathscr{P}(R)$ if and only if $Q = \bigcup_{P \subsetneq Q, P \in \mathscr{P}(R)} P$.*

Proof. We consider the "if" part. Suppose

$$Q = \bigcup_{P \subsetneq Q, P \in \mathscr{P}(R)} P \quad \text{and} \quad Q \in \mathscr{P}(R).$$

Thus, since R is coherent, Q is minimal over a finitely generated ideal of R [**6**, Theorem 2.2]. However, since R is treed, this leads to a contradiction.

Example 2.7. The following example demonstrates that Proposition 2.6 is not generally true in the class of treed domains. More specifically, we shall give an example of a local, integrally closed, treed domain not satisfying the "if" direction of the proposition.

Let k be a field and x an indeterminate over k. Let V be a valuation ring of the form $k(x) + M$ such that each nonmaximal prime ideal P of V has finite height, and $M = \cup_{P \subsetneq M} P$, where M is the maximal ideal of V. Note that each nonmaximal prime ideal P of V is in $\mathscr{P}(V)$.

Let $R = k + M$ and recall that $\operatorname{Spec}(R) = \operatorname{Spec}(V)$ [**7**, Exercise 11, p. 202]. Hence,

$$M = \cup_{P \subsetneq M, P \in \mathscr{P}(R)} P.$$

We will show that $M \in \mathscr{P}(R)$. Before doing so, notice that R is quasi-local, integrally closed and treed [**7**, Exercise 11, p. 202]. Also, by reasoning as in Proposition 2.6, it is easy to see that M is not finitely generated in V or R, and $M \notin \mathscr{P}(V)$.

By Remark 2.4 ((d) $\Rightarrow$ (b)), to show that $M \in \mathscr{P}(R)$, it suffices to prove that $M^{-1} \neq R$ (throughout this example M^{-1} means: $M^{-1} = \{x \in$ quotient field of $R: xM \subseteq R\}$). We claim that $M^{-1} = V$. Clearly $V \subseteq M^{-1}$, so we focus on the other inclusion. Let $u \in M^{-1}$ and suppose $u \notin V$. Then $u^{-1} \in M$, and so either $uM = M$ or $uM = R$. In the first case $1 \in M$ and in the second case M is finitely generated in R. These contradictions establish the claim. Hence $M^{-1} \neq R$, since $R \neq V$, and therefore $M \in \mathscr{P}(R)$.

Thus far we have provided two large classes of domains satisfying $S = U$. Two obvious classes where $S = U$ that we have not yet investigated are those rings R where (a): $\operatorname{Max}(R) \subseteq \mathscr{P}(R)$ or (b): $\operatorname{Spec}(R) = \mathscr{P}(R)$. (Recall that condition (a) figured into Proposition 2.2.) We shall concentrate on condition (b).

First observe that any 1-dimensional domain or any domain with a finite number of prime ideals satisfies (b), and hence $S = U$ for such domains. It is natural to ask, if $\mathscr{P}(R)$ is a finite set, must $S = U$. This is the content of our next proposition, but first we consider a useful elementary lemma.

LEMMA 2.8. *Let P, $Q \in \operatorname{Spec}(R)$ such that $P \subseteq Q$. Then, $P \in \mathscr{P}(R)$ if and only if $PR_Q \in \mathscr{P}(R_Q)$.*

Proof. ($\Rightarrow$): Assume P is minimal over $(a:b)$ for some $a, b \in R$. Now, since $R \subseteq R_Q$ is a flat extension, we know that

$$(a:b)R_Q = (aR_Q:bR_Q) \subseteq PR_Q$$

[**5**, (12) p. 25]. It is straightforward to see that PR_Q is minimal over $(aR_Q:bR_Q)$, and hence $PR_Q \in \mathscr{P}(R_Q)$.

($\Leftarrow$): Suppose that PR_Q is minimal over $(a/s:b/t)$ for some a/s, $b/t \in R_Q$. Since $(a/s:b/t) = (aR_Q:bR_Q) = (a:b)R_Q$, it is easy to verify that $P \in \mathscr{P}(R)$.

PROPOSITION 2.9. *If $\mathscr{P}(R)$ is a finite set, then* $\operatorname{Spec}(R) = \mathscr{P}(R)$.

Proof. It suffices to show that $\operatorname{Spec}(R)$ is a finite set. Let $\mathscr{P}(R) = \{P_1, \ldots, P_t\}$. By Theorem 2.0, $R = \bigcap_{i=1}^{t} R_{P_i}$, and so R is a quasi-semi-local domain [**13**, Theorem 105]. Thus, it is enough to show that the spectrum of each localization at each maximal ideal is a finite set. Hence, we may assume R is quasi-local with maximal ideal M, and by Lemma 2.8, $\mathscr{P}(R)$ is a finite set.

We claim that if $P \in \operatorname{Spec}(R)$ and P has finite height, then $P \in \mathscr{P}(R)$. Our proof is by induction on the height of P. If $\operatorname{ht}(P) = 1$, then P is certainly in $\mathscr{P}(R)$, so assume the claim is true for all prime ideals of height less than n. Let $\operatorname{ht}(P) = n$. Since $\mathscr{P}(R)$ is finite, P cannot be in the union of the prime ideals that are properly contained in P. Hence, P is minimal over a principal ideal, and so $P \in \mathscr{P}(R)$.

To complete the proof we need only show that the height of M is finite. Assume the contrary. Let $\{P_i\}_{i=1}^{t}$ be the complete set of prime ideals of R of finite height. (Possibly $\{P_i\}_{i=1}^{t} = \{0\}$). Let $I = \bigcup_{i=1}^{t} P_i$ and set $m = \max\{\operatorname{ht}(P_i)\}_{i=1}^{t}$. Choose $a_1 \in M \backslash I$ and let Q_1 be a prime ideal minimal over (a_1). Then Q_1 does not have finite height. Choose N_2 a prime ideal of R such that $N_2 \subsetneqq Q_1$ and $\operatorname{ht}(N_2) > m$. Now select $a_2 \in N_2 \backslash I$ and let Q_2 be a minimal prime ideal of (a_2). Again we see that Q_2 does not have finite height. Continuing this process we get an infinite sequence of prime ideals $Q_1 \supsetneqq Q_2 \supsetneqq Q_3 \supsetneqq \ldots$ such that each $Q_i \in \mathscr{P}(R)$. This contradicts the fact that $\mathscr{P}(R)$ is a finite set. Therefore, the height of M is finite, and the proof is complete.

We have seen (Example 2.7) that if R is a Prüfer domain it may be the case that $\mathscr{P}(R) \neq \operatorname{Spec}(R)$, even though $S = U$. Our next goal is to characterize those Prüfer domains R satisfying $\operatorname{Spec}(R) = \mathscr{P}(R)$.

LEMMA 2.10. *Let R be a coherent and treed domain. Then,* $\operatorname{Spec}(R) = \mathscr{P}(R)$ *if and only if R satisfies a.c.c. on prime ideals.*

Proof. ($\Rightarrow$): This direction does not require the treed assumption, and follows in a manner similar to the proof of Proposition 2.6.

($\Leftarrow$): This direction does not require the coherence assumption, and is also straightforward to verify.

PROPOSITION 2.11. *R is a Prüfer domain with a.c.c. on prime ideals if and only if R is coherent, integrally closed, and* $\mathscr{P}(R) = \operatorname{Spec}(R)$.

Proof. The "only if" part follows from Lemma 2.10. For the "if" part, it suffices by Lemma 2.10 to show that R is a Prüfer domain. By Lemma 2.8 and [**11**, Corollary 3.1], we may assume R is quasi-local with maximal ideal M. Let I be a finitely generated ideal of R and set $J = II^{-1}$. Consider a finite presentation of I:

$$R^m \to R^n \to I \to 0.$$

By applying $\mathrm{Hom}_R(-, R)$ to this exact sequence we obtain the following exact sequence:

$$0 \to \mathrm{Hom}_R(I, R) \to R^n \to R^m.$$

As $I^{-1} \cong \mathrm{Hom}_R(I, R)$ as R-modules and R is coherent, we can conclude that I^{-1} is finitely generated as an R-module [**5**, Exercise 11, pp. 43–44]. Hence $J^{-1} = R$ [**13**, Exercise 39, p. 45]. Thus by Theorem 2.0, $J \not\subseteq M$ and hence $J = R$. Therefore, I is invertible and R is a valuation domain.

Remark 2.12. It is worthwhile to mention that in Lemma 2.10, one cannot replace a.c.c. on prime ideals with a.c.c. on radical ideals. To see this, let R be a 1-dimension Bezout domain with infinitely many maximal ideals and such that R is also a G-domain [**9**, p. 279]. Clearly R is coherent and treed, but R does not satisfy a.c.c. on radical ideals ($\mathrm{Spec}(R)$ is not a Noetherian space [**5**, p. 97]).

We complete this section by considering how the condition $S = U$ behaves under a few different changes of rings.

Remark 2.13. If $S_R = U_R$, it is not the case in general that
(a) $S_{\bar{R}} = U_{\bar{R}}$ ($\bar{R}$ is the integral closure of R);
(b) $S_{R[x]} = U_{R[x]}$.

(a). Let R be a two dimensional Noetherian domain such that $\mathrm{depth}(R) = 1$. Thus by Proposition 2.2, $S_R = U_R$. However, $\bar{R}$ is Noetherian [**15**, Theorem 33.12], and hence $S_{\bar{R}} \neq U_{\bar{R}}$ by Corollary 2.3.

(b). Let R be any domain that is not a field such that $S_R = U_R$. Then since $\mathrm{depth}(R[x]) > 1$, Lemma 2.1 implies that $S_{R[x]} \neq U_{R[x]}$.

We conclude this remark by providing one change of rings that does preserve the condition $S = U$. Let D be a domain, and K any field containing D. Let V be a valuation ring of the form $K + M$, and let $R = D + M$. We claim that if $S_D = U_D$, then $S_R = U_R$. It suffices to show that $U_R \subseteq S_R$, so let $f \in U_R$. Write

$$f = (d_0 + m_0) + \ldots + (d_n + m_n)x^n,$$

where $d_i \in D$ and $m_i \in M$. Let

$$\bar{f} = d_0 + \ldots + d_n x^n.$$

It is straightforward to verify that $c_D(\bar{f})^{-1} = D$. Thus $c_D(\bar{f}) = D$, and this implies that $c_R(f) = R$. Therefore, $f \in S_R$ and so $S_R = U_R$.

3. When is $R[x]_U$ a Prüfer domain? In [**3**, Theorem 4], Arnold characterized when $R(x)$ is a Prüfer domain. More specifically he proved that the following are equivalent for an integrally closed domain R: (a) R is a Prüfer domain; (b) $R(x)$ is a Prüfer domain; (c) $R(x) = R^K$ (R^K denotes the Kronecker function ring of R); (d) R^K is a localization of $R[x]$; (e) Each prime ideal of $R(x)$ is the extension of a prime ideal of R. When R is integrally closed it is known that R^K is a Bezout domain [**7**, Theorem 32.7], hence one is able to conclude that when R is a Prüfer domain, $R(x)$ is a Bezout domain. It is our desire to find conditions when $R[x]_U$ is a Prüfer domain. In the situations we shall discuss, it will also turn out that $R[x]_U$ is a Bezout domain. We first present a lemma which is partially modelled after [**3**, Theorem 4].

LEMMA 3.0. *Let R be an integrally closed domain and let W be a multiplicative system in $R[x]$. If each prime ideal of $R[x]_W$ is extended from a prime ideal of R, then $R[x]_W$ is a Bezout domain.*

Proof. First we will show that $R[x]_W$ is a Prüfer domain. Let N be a prime ideal of $R[x]_W$. By assumption

$$N = PR[x]_W = P[x]R[x]_W,$$

where $P \in \operatorname{Spec}(R)$. By [**3**, Lemmas 1 and 2],

$$(\Delta) \quad (R[x]_W)_N = R[x]_{P[x]} = R_P[x]_{PR_P[x]}.$$

To show that $(R[x]_W)_N$ is a valuation ring, it suffices to show that R_P is a valuation ring [**7**, Proposition 18.7].

Let u be a nonzero element of the quotient field of R, and let $M = \ker(R[x] \rightarrow R[u])$, where the homomorphism is the evaluation map. It follows that $M \cap W \neq \emptyset$, and hence $M \not\subseteq P[x]$. Choose $f(x) \in M \backslash P[x]$. Whence, since $f(u) = 0$, an application of [**16**, p. 19] shows that either u or u^{-1} is in R_P. Therefore, R_P is a valuation domain, and hence $R[x]_W$ is a Prüfer domain.

We are now ready to show that $R[x]_W$ is a Bezout domain. Let (f, g) be an ideal of $R[x]$ such that $(f, g) \cap W = \emptyset$. It suffices to show that $(f, g)R[x]_W$ is a principal ideal. Choose $h \in (f, g)$ such that $c(h) = c(f) + c(g)$. We will show that

$$(f, g)R[x]_W = hR[x]_W.$$

It is enough to verify this equality locally. Let N be a proper prime ideal of $R[x]_W$, and let $P = N \cap R$. From (Δ) and what we have proved above, we may assume R is a valuation ring with maximal ideal P. We

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prove that

$$(f, g)R[x]_{P[x]} = hR[x]_{P[x]}.$$

Write $f = af_1$, and $g = bg_1$, where $c(f_1) = c(g_1) = R$ [**7**, Remark 17.2]. If $(c) = (a, b)$, then $h = ch_1$, where $c(h_1) = R$. It follows that

$$(f, g)R[x]_{P[x]} = (a, b)R[x]_{P[x]} = (c)R[x]_{P[x]} = hR[x]_{P[x]}.$$

This completes the proof.

Before we state our main result, recall that an element $f \in R[x]$ is called *primitive* if $c(f)$ is not contained in any proper principal ideal of R. Let V be the set of all primitive polynomials in $R[x]$, and note that $S \subseteq U \subseteq V$ [**17**, Theorem C].

THEOREM 3.1. *If R is either a*
(a) GCD-*domain, or an*
(b) *Integrally closed coherent domain, or a*
(c) *Krull domain, then*
$R[x]_U$ is a Bezout domain.

Proof. In each case we will show that Lemma 3.0 applies. Let $Q \in \operatorname{Spec}(R[x])$ such that $Q \cap U = \emptyset$. Set $P = Q \cap R$. We claim that in cases (a), (b) and (c), $Q = P[x]$.

(a): Let R be a GCD-domain and let $f \in Q$. We may write $f = df_1$, where d is the g.c.d. of the coefficients of f. Hence, $f_1 \in V$ and thus $f_1 \in U$, since U and V coincide in a GCD-domain [**17**, Theorem H]. Therefore, since $f_1 \notin Q$ we may conclude that $d \in Q \cap R = P$, and so $f \in P[x]$. This completes part (a) of the proof.

(b): Let R be an integrally closed coherent domain, and let $g \in Q\backslash P[x]$. Set $J = c(g)$ and notice that $J \not\subseteq Q$. From $JJ^{-1}g \subseteq Q$, it follows that $J^{-1}g \subseteq Q$. However, J^{-1} is finitely generated as an R-module (see proof of Proposition 2.11), and since

$$J^{-1}g \subseteq Q \subseteq \cup_{P \in \mathscr{P}(R)} P[x],$$

we see that $J^{-1}g \subseteq P'[x]$ for some $P' \in \mathscr{P}(R)$. Thus $c(J^{-1}g) \subseteq P'$. (Recall that if I is an ideal in $R[x]$, then $c(I)$ is the ideal of R generated by the coefficients of the polynomials in I.) However, since $(JJ^{-1})^{-1} = R$ [**13**, Exercise 39, p. 45], it follows that $[c(J^{-1}g)]^{-1} = R$, which contradicts Theorem 2.0. Therefore, $Q = P[x]$ and part (b) is complete.

(c): Let R be a Krull domain. We shall first prove that $\mathscr{P}(R) = \{P' : P'$ is a height 1 prime ideal of $R\}$. Certainly the right-hand side is contained in the left-hand side. Let M be a prime ideal of R of height greater than 1. Pick a nonzero element $a \in M$, and let $\{P_1, \ldots, P_t\}$ be the complete set of height 1 prime ideals of R that contain a. Choose

$$b \in M\backslash \cup_{i=1}^{t} P_i.$$

Then, as in the proof of Corollary 2.3 ((iii) $\Rightarrow$ (ii)), $\{a, b\}$ is an R-sequence in M. Therefore, $M \notin \mathscr{P}(R)$.

Secondly, we claim that $Q \cap R = P \in \mathscr{P}(R)$. Suppose that $P \notin \mathscr{P}(R)$, and thus $\operatorname{ht}(P) > 1$. As above, we see that P contains an R-sequence of length two, say $\{a, b\}$. Since

$$P[x] \subseteq Q \subseteq \cup_{P \in \mathscr{P}(R)} P[x],$$

we see that $(a, b) \subseteq P'$ for some $P' \in \mathscr{P}(R)$. This contradiction establishes that $P \in \mathscr{P}(R)$.

We are ready to prove that $Q = P[x]$. If there is an $h \in Q \backslash P[x]$, then there are only finitely many $P_i \in \mathscr{P}(R)$ such that $h \in P_i[x]$ $(1 \leqq i \leqq t)$. Choose

$$k \in P[x] \backslash \cup_{i=1}^{t} P_i[x],$$

and let $\deg(h) = s$. Then

$$h + x^{s+1}k \in Q \subseteq \cup_{P \in \mathscr{P}(R)} P[x],$$

and so $h, k \in P'[x]$ for some $P' \in \mathscr{P}(R)$. This is a contradiction, since $P'[x] = P_i[x]$ for some $1 \leqq i \leqq t$, and therefore $Q = P[x]$ to complete the proof.

As a corollary to our main theorem, new characterizations of Bezout, Prüfer, and Dedekind domains are given.

COROLLARY 3.2. (i) *R is a Bezout domain if and only if R is a* GCD-*domain and $S = U$.*

(ii) *R is a Prüfer domain if and only if R is an integrally closed coherent domain and $S = U$.*

(iii) *R is a Dedekind domain if and only if R is a Krull domain and $S = U$.*

Proof. Since the "only if" direction is clear in each case, we shall concentrate on the "if" part. Note that in each case we may apply Theorem 3.1 and Arnold's characterization of when $R(x)$ is a Prüfer domain [**3**, Theorem 6] to conclude that R is a Prüfer domain. In case (i), since R is a GCD-domain, R is a Bezout domain [**13**, Exercise 15, p. 42]. For case (iii), since R is a Krull domain, it follows that R is a Dedekind domain [**7**, Theorem 43.16].

Remark 3.3. It is now appropriate to comment on a few points related to Theorem 3.1. First, with Arnold's characterization in mind (see the introduction of this section), it is interesting to note that for an integrally closed domain R, R is a Prüfer domain if and only if $R[x]_U = R^K$.

In general there is no containment relation between the rings R^K and $R[x]_U$. Let $R = F[y_1, y_2]$ be the polynomial ring in two indeterminates

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over a field F. Let $f = y_1 + y_2x \in R[x]$. Then $c(f)^{-1} = R$, and thus $1/f \in R[x]_U \backslash R^K$.

On the other hand, consider the ring R in Remark 3.6 (d). Since $S = U = V$, and R is not a valuation ring, we may conclude that

$$R[x]_U = R(x) \subsetneq R^K.$$

Therefore $R^K \not\subseteq R[x]_U$.

Finally, it is worthwhile to point out that the proof of Theorem 3.1 (c) shows that if R is a Krull domain, then the prime ideals of $R[x]_U$ are the extensions of prime ideals in $\mathscr{P}(R)$. This need not be the case if R is a GCD-domain or if R is an integrally closed coherent domain. The valuation ring given in Example 2.7 displays this phenomenon aptly.

Remark 3.4. At this time we would like to mention a few open questions, but first we make an observation. Note that if $R[x]_U$ is a Prüfer domain, then R_P is a valuation domain for each $P \in \mathscr{P}(R)$, and hence R is integrally closed. To see this, let $P \in \mathscr{P}(R)$, let K denote the quotient field of R, and notice that

$$(R[x]_U)_{P[x]} \cap K = R[x]_{P[x]} \cap K = R_P,$$

which shows that R_P is a valuation domain [**7**, Theorem 19.16]. Since $R = \bigcap_{P \in \mathscr{P}(R)} R_P$ (Theorem 2.0), we have that R is integrally closed.

The following questions are open.

(a): If R_P is a valuation domain for each $P \in \mathscr{P}(R)$, is $R[x]_U$ a Prüfer domain?

(b): If $R[x]_U$ is a Prüfer (Bezout) domain, are the prime ideals of $R[x]_U$ extended from prime ideals of R?

(c): If $R[x]_U$ is a Prüfer domain, is it a Bezout domain?

With reference to question (b), we point out that since the the essential valuation rings of $R[x]$ are well understood [**7**, Exercise 12, p. 221], it is easy to establish that if $Q \in \operatorname{Spec}(R[x])$, and $Q \cap U = \emptyset$ as well as $Q \cap R \neq 0$, then $Q = P[x]$ for some $P \in \mathscr{P}(R)$. Hence, one may concentrate his efforts on the "uppers" of 0 (see [**14**]) that do not meet U. One possible way of showing that no such uppers exist, when $R[x]_U$ is a Prüfer domain, is to show that if Q is such an upper, then $Q \subseteq P[x]$ for some $P \in \mathscr{P}(R)$, and this would contradict [**7**, Theorem 19.15].

Remark 3.5. We now consider a point that is somewhat connected to the above comments made in the latter part of Remark 3.4. It will help illustrate how the prime ideal structure of $R[x]_U$ is related to the to structure of $\mathscr{P}(R[x])$.

We claim that if $Q \in \mathscr{P}(R[x])$ and $Q \cap R \neq 0$, then $Q \cap U = \emptyset$. Suppose this is not the case. Let $f \in Q \cap U$, and choose $0 \neq a \in Q \cap R$.

Note that $a, f \in P[x]$ for some $P \in \mathscr{P}(R)$, and also observe that $P[x] \in \mathscr{P}(R[x])$ by flatness [**5**, (12), p. 25]. Hence, since $\{a, f\}$ forms an $R[x]$-sequence in $P[x]$ [**10**, 3.2C], the desired contradiction is obtained.

We conclude this remark by noting that Theorem 3.1 implies that one cannot generally conclude that the "uppers" of 0 survive in $R[x]_U$, even though they all belong to $\mathscr{P}(R[x])$.

Remark 3.6. Recall from the paragraph preceding Theorem 3.1 that V denotes the set of all primitive polynomials, and $S \subseteq U \subseteq V$. In general, each of these inclusions may be strict inclusions, and in particular V need not even be multiplicatively closed. However, when $S = V$, V is certainly multiplicatively closed, and our results pertaining to $S = U$ apply. (Arnold and Sheldon in [**4**], study the situation when V is a multiplicative system and when $U = V$.)

We give a few indications of how much more restrictive the condition $S = V$ is in comparison to $S = U$.

(a): If P is a finitely generated proper prime ideal of R and if $S = V$, then P is a principal ideal of R.

Proof. We may assume $P \neq 0$. Since $P \neq R$ and P is finitely generated, there exists an $a \in R$ such that $P \subseteq (a) \subsetneqq R$. We claim $P = (a)$; for if not, then $P = Pa$, and thus $P = 0$, a contradiction.

(b): R is a PID if and only if R is a Noetherian domain and $S = V$.

This follows easily by combining (*a*) and [**13**, Exercise 10, 1.8].

(c): R is a valuation ring if and only if R is coherent, quasi-local, and $S = V$.

Proof. Since the "only if" direction is clear, we shall concentrate on the reverse direction. Let M be the maximal ideal of R. It suffices to show that M is flat as an R-module [**18**, Lemma 3.9]. Let I be a finitely generated proper ideal of R and note that $I \subseteq (a) \subseteq M$. Hence, the principal subideals of M are cofinal in the set of finitely generated subideals of M, and thus M, being the direct limit of principal ideals, is flat [**5**, Proposition 9, p. 20].

(d): The following example shows the need for the assumption of coherence in (b) above, i.e., we construct a quasi-local integrally closed, non-valuation domain satisfying $S = V$.

Let W be a valuation ring with principal maximal ideal, and denote the quotient field of W by K. Let W' be a valuation ring of the form $K(x) + M$, where x is an indeterminate over K, and M is the nonzero maximal ideal of W'. Set $R = W + M$, and note that R is quasi-local, integrally closed, and R is not a valuation ring [**7**, Exercise 11, p. 202]. To see that $S = V$, it is enough to observe that the maximal ideal of R is principal.

(e): In part (b) (respectively, part (c)) of this remark, if R satisfies $V = S$ and if R is Noetherian (respectively, quasi-local coherent), then R is integrally closed. The following example shows that this is not generally true. Let T be a rank one discrete valuation ring of the form $\mathbf{Q}(\sqrt{2}) + M$, where $\mathbf{Q}$ is the rational numbers. Then $R = Z_{(3)} + M$ is the required example.

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