The Geometry of Quadratic Differential Systems with a Weak Focus of Third Order

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Abstract. In this article we determine the global geometry of the planar quadratic differential systems with a weak focus of third order. This class plays a significant role in the context of Hilbert's 16-th problem. Indeed, all examples of quadratic differential systems with at least four limit cycles, were obtained by perturbing a system in this family. We use the algebro-geometric concepts of divisor and zero-cycle to encode global properties of the systems and to give structure to this class. We give a theorem of topological classification of such systems in terms of integer-valued affine invariants. According to the possible values taken by them in this family we obtain a total of 18 topologically distinct phase portraits. We show that inside the class of all quadratic systems with the topology of the coefficients, there exists a neighborhood of the family of quadratic systems with a weak focus of third order and which may have graphics but no polycycle in the sense of [15] and no limit cycle, such that any quadratic system in this neighborhood has at most four limit cycles.

1 Introduction, Brief Review of the Literature and Informal Outline of Results

The complete characterization of the phase portraits for real planar quadratic vector fields is not known, and attempting to classify these systems, which occur rather often in applications, is quite a complex task. This family of systems depends on twelve parameters, but due to the group action of real affine transformations and positive time rescaling, the class ultimately depends on five parameters. Bifurcation diagrams were constructed for some algebraic and semialgebraic subsets of this class, see for example [4, 5, 8, 10, 25, 33, 41, 39, 50, 54]. With the exception of some articles (for example [50]) the classifications of systems were done in terms of local charts and inequalities on the coefficients of the systems written in these charts. This line of work follows the program stated by Coppel in his nice short article which appeared in 1966 (*cf.* [11]). Coppel thought that the phase portraits of quadratic systems could be characterized by means of algebraic inequalities on the coefficients. We now know that algebraic inequalities would not be sufficient; analytic as well as non-analytic ones (*cf.* references [37] and [14, pp. 118–119]) need to be taken into consideration. We would also want to see that classifications be done not in terms of coordinate charts as in previous works but in more intrinsic terms, to reveal the geometry of the

310

systems. Since it is expected that quadratic differential systems will yield more than two thousand phase portraits, tools for organizing this maze of phase portraits need to be introduced.

The phase portraits are classified by using the topological equivalence [48]. Just as homotopy groups could be used to distinguish topological spaces, we need to have invariants to distinguish between phase portraits. A complete set of invariants was given by Markus for C^1 systems on the plane (*cf.* [28], also [31]). He associated to every C^1 system its separatrix configuration and he showed that two C^1 systems are topologically equivalent if and only if their corresponding separatrix configurations are topologically equivalent. The separatrix configuration is formed by all the separatrices of the system, together with an orbit for each open connected component of the complement in the plane of the union of all separatrices. Each of these components is called a *canonical region*. The separatrix configurations could still be rather complex, and criteria to distinguish them are needed.

Global concepts introduced in [34, 42] are used here for classifying the family of quadratic differential systems we study. These geometric concepts are: zero-cycles of the affine or projective plane, real or complex, and divisors corresponding to the line at infinity. We associate to these, integer-valued global affine invariants which help us distinguish (and sometimes even identify) separatrix configurations. We apply these ideas to the particular class QW3 of all quadratic systems with a weak focus of third order. In the end we obtain phase portraits of this family and their classification up to topological equivalence. For a large part of the parameter space this classification is intrinsic, *i.e.*, it is independent of the normal form for the systems considered since it only depends on integer-valued affine invariants which could be calculated, as we indicate in Section 10, IV, for any presentation of the systems.

Since a weak focus may produce up to three limit cycles in a quadratic perturbation of the system, the study of this class is of interest in itself in the context of Hilbert's 16-th problem. This problem asks for the maximum $H(m)$ (the Hilbert number) of the numbers of limit cycles which real polynomial vector fields defined by polynomials of maximum degree m could have [21]. The nature of the problem is global in two ways: we are interested in the whole class of polynomial vector fields of a maximum fixed degree m and at the same time in the behavior on the whole plane of phase curves of each individual member of this family. The problem is still unsolved, even for the case $m = 2$. The most important result of a global nature is the individual finiteness theorem proved independently in the late 1980's by Ecalle [17] and Il'yashenko [22]. This theorem says that each fixed polynomial vector field has a finite number of limit cycles. This result was stated as a theorem by Dulac [13] in 1923 but the proof Dulac gave was flawed. If we consider the family of all quadratic vector fields, it is not even proved that $H(2)$ is a finite number. There is a program under way (*cf.* [15]) to prove this. The program proposes to show that all the 121 graphics listed in [15], which intervene in the problem of proving the finiteness of H(2), have finite cyclicity (*cf.* [36]). Because in over one hundred years since Hilbert stated the problem, no example was found where we can prove that more than four limit cycles exist, it is not only conjectured that $H(2)$ is finite but also that $H(2) = 4$.

All known examples of quadratic vector fields having at least four limit cycles were obtained by perturbing a quadratic vector field with a weak focus of third order, a

good motivation for understanding this family.

Assuming the proof of the finiteness of $H(2)$ were done, we would still know very little about the geometry of the systems in this class or of the class itself. For this reason along with a search for a proof of the finiteness part it is necessary to advance the knowledge on this class, and the present work is a step in this direction. For a general framework of study of the class of all quadratic differential systems we refer to the article of Roussarie and Schlomiuk [38].

The work is organized as follows. In Section 2 we describe the normal form for the family of systems we consider. Although they are an essential tool for performing calculations, it is necessary to break away from specific normal forms and as much as possible, state results which are applicable to any specific presentation of the systems. As we explain in later sections, for most phase portraits of the classification, this is possible.

For the study of real planar polynomial vector fields two compactifications are used. In Section 3 we describe very briefly the Poincaré compactification on the 2-dimensional sphere. We also introduce two foliations with singularities on the real and complex projective planes, associated to the real polynomial vector fields. Although a different compactification, the real foliation with singularities on the real projective plane is closely related to the Poincaré compactification on the sphere. The two polynomials defining a real planar polynomial vector field also define a complex vector field when the variables range over $\mathbb{C}$. To this complex vector field on $\mathbb{C}^2$ we can associate a compactification by passing to the foliation with singularities on the complex projective plane $\mathbb{CP}^2$. This foliation turns out to be very important for the study of the real vector field.

In Section 4, intersection numbers are attached to the singularities of the complex foliation with singularities associated to a real vector field.

In Section 5, we introduce some concepts which encode globally the information about singularities. They are *zero-cycles* of the plane and *divisors* (*cf.* [18]) on the line at infinity and they are applied in Section 6 to the specific study of the class QW3 of quadratic systems with a weak focus of third order. Some singularities are composite or multiple, arising from collision of neighboring points which are singularities in a perturbation of the system. We use the *zero-cycles* on both the real and the complex projective planes and *divisors* on the line at infinity to encode the multiplicities of all the singular points, finite or infinite. We also need to encode globally the information on the topology of the phase portrait around each one of the singularities. This is done by using in Proposition 10 of Section 6, the zero-cycle *DI* encoding globally the indices of singularities in the real projective plane of the compactified systems. The *types* of these divisors and zero-cycles and their *degrees* (as defined in Section 5) are affinely invariant.

In Section 7 we list the basic theorems from the literature, and in particular the properties of quadratic systems or of QW3, which are used in Section 8. We thus mention two important results: the nonexistence of a limit cycle surrounding a weak focus of third order (*cf.* [24]) and a recently obtained result affirming that if a quadratic system has limit cycles around two foci, then around one of them we cannot have more than just one limit cycle (*cf.* [52, 53]). The main result of Section 7 is Theorem 12 about the bifurcation diagram for the class QW3. To facilitate future

studies, its proof is here made much more explicit. An important fact coming out of this proof is that roughly speaking the knowledge of portraits of nongeneric systems, the systems with center (they are integrable in the quadratic case), yields in fact this bifurcation diagram. This is an improvement over [2], [3] and [5]. Firstly because of the use here of the projective plane as the parameter space, which allowed us to gather the pieces of the bifurcation diagram in just one picture, in view of a symmetry possessed by the systems a half-disc. In each one of the previously mentioned articles these pieces appear separately drawn. Viewing the changes in the phase portraits as parameters vary facilitates understanding global properties of this class. The integer-valued invariants as defined by us in Section 7 pointed out to us the errors in the previous literature. In particular they allowed us to correct statements in [5], [2] and [3] as we indicate in Sections 9 and 10.

In Section 8 we introduce a global invariant denoted by $\mathcal{I}$, which classifies the phase portraits obtained for the systems in the class QW3 up to topological equivalence. Theorem 14 shows clearly that for a large part of the parameter space, call it here *A*, these phase portraits are uniquely determined (up to topological equivalence) by the values of an invariant, $\mathcal{J}$, depending only on the values of the above divisors and zero-cycles. In this region *A* the bifurcation diagram is shown to be part of a real algebraic set. For any value in *A* we have a phase portrait which has no limit cycle and no graphic (*cf.* [15]). In the complement *B* of *A* in the parameter space we have limit cycles or graphics. In *B*, $\mathcal{J}$ takes only two values and it cannot distinguish all the phase portraits. New concepts are thus needed and we introduce them, yielding the affine invariant $\mathcal{I}$. It is in *B* that a connected bifurcation curve $\mathcal{G}$, part analytic, part algebraic, occurs on the diagram of Figure 3. On $\mathcal{G}$ the systems do not have limit cycles. $\mathcal{G}$ is the only subset in the parameter space where graphics occur; more precisely, for all points in $\mathcal{G}$ the systems have a unique graphic (which in two of the three possible cases is a polycycle (*cf.* [15])) with two singularities, both at infinity and with two path curves, one of them part of the line at infinity. We prove the existence and analyticity of the curve $\mathcal{G}$, and numerical computations determine the position of its non-algebraic part with as high accuracy as we wish.

The region with limit cycles, where three distinct phase portraits occur, is bounded by the curve $\mathcal{G}$ and by a line ($a = 0$, *i.e.*, the vertical line of the diagram in Figure 2, also in Figure 3) on which we have symmetric systems with two centers.

There is a difference between defining invariants and actually being able to compute them for any given presentation of the systems. We indicate in Section 10 how part of the integer-valued invariants defined here relate to the algebraic invariants which in turn can be computed for any given particular chart in which the systems may be presented.

2 Quadratic Vector Fields With a Weak Focus of Third Order

A singular point *p* of a planar vector field *X* in $\mathbb{R}^2$ is a *linear center* if the eigenvalues of its linear part, $DX(p)$, are imaginary numbers, *i.e.*, $\pm\beta i$ with $\beta \in \mathbb{R}\setminus\{0\}$. We say that *p* is a *center* of *X* if *p* is an isolated singularity such that there exists a neighborhood *U* of *p* where all non-trivial orbits of *X* are periodic. It is known that a linear center *p* is either a center, or a focus. In this last case *p* is called a *weak focus*. We recall that *p* is

a *strong focus* if the eigenvalues of its linear part are of the form $\alpha \pm \beta i$ with $\alpha\beta \neq 0$.

The next result is due to Shi [45, 46]. This lemma is better stated in [40] where the rings involved are explicitly written, and where a sketch of the proof is also given. For the sake of completeness we give below the statement of this lemma.

Lemma 1   *Consider the polynomial system:*

(1)
$$\dot{x} = p(x, y) = -y + p_2(x, y) + \cdots + p_m(x, y),$$
$$\dot{y} = q(x, y) = x + q_2(x, y) + \cdots + q_m(x, y),$$

where

$$p_i(x, y) = \sum_{j=0}^{i} a_{ij} x^{i-j} y^j, \quad q_i(x, y) = \sum_{j=0}^{i} b_{ij} x^{i-j} y^j.$$

Then there exists a formal power series $F \in \mathbb{Q}[a_{20}, \ldots, b_{0m}][[x, y]]$,

$$F = \frac{1}{2}(x^2 + y^2) + F_3(x, y) + F_4(x, y) + \cdots,$$

and there exists polynomials $V_1, \ldots, V_i, \ldots \in \mathbb{Q}[a_{20}, \ldots, b_{0m}]$ such that

$$\frac{dF}{dt} = \frac{\partial F}{\partial x} p + \frac{\partial F}{\partial y} q = \sum_{i=1}^{\infty} V_i (x^2 + y^2)^{i+1}.$$

The quantities V_i are not uniquely determined. For each i there is an infinite number of possibilities for a V_i. But according to a result also proven by Shi, all such V_is are in the same coset modulo the ideal generated by $V_1, \ldots, V_{i-1}$ in the ring $\mathbb{Q}[a_{20}, \ldots, b_{0m}]$. From the work of Poincaré [35] it follows that system (1) has a center at the origin if and only if $V_i = 0$ for all i. By Hilbert's basis theorem, the ideal $I = \langle V_1, \ldots, V_i, \ldots \rangle$ has a finite basis. It follows from the work of Bautin [6] that for quadratic systems ($m = 2$) this ideal is determined by the values of V_i with $i \leq 3$. The above result implies that $V_1 = V_2 = V_3 = 0$ if and only if $V_i = 0$ for all i and the origin is a center. We say that the origin of a quadratic system is
   (i) a *weak focus of first order* if $V_1 \neq 0$;
   (ii) a *weak focus of second order* if $V_1 = 0$ and $V_2 \neq 0$; and
   (iii) a *weak focus of third order* if $V_1 = V_2 = 0$ and $V_3 \neq 0$.

A quadratic system with a focus or a center at the origin can always be written in the form

(2) $\dot{x} = ax - by + a_{20}x^2 + a_{11}xy + a_{02}y^2, \quad \dot{y} = bx + ay + b_{20}x^2 + b_{11}xy + b_{02}y^2,$

with $b \neq 0$. Moreover, since the focus is weak we can assume that $a = 0$. We remark that the change of variables for writing an arbitrary quadratic system with a weak focus into the form (2) with $a = 0$ is continuous. Doing a rescaling of the independent variable we can assume without loss of generality that $b = 1$.

Proposition 2 *Any quadratic system with a weak focus at the origin can be written into the normal form*

$$(3) \qquad \dot{x} = -y + lx^2 + rxy + ny^2, \quad \dot{y} = x + ax^2 + bxy,$$

by a linear change of variables and a rescaling of the independent variable. Furthermore, the transformation into this normal form depends continuously on the parameters.

Proof For the system (2) with $a = 0$ and $b = 1$, we perform the linear change of variables

$$(4) \qquad x = X\cos\theta + Y\sin\theta \quad y = -X\sin\theta + Y\cos\theta.$$

Then the quadratic system (2) becomes

$$\dot{X} = -Y + a'_{20}X^2 + a'_{11}XY + a'_{02}Y^2, \quad \dot{Y} = X + b'_{20}X^2 + b'_{11}XY + b'_{02}Y^2.$$

with

$$(5) \quad b'_{02} = b_{02}\cos^3\theta + (a_{02} + b_{11})\cos^2\theta\sin\theta + (a_{11} + b_{20})\cos\theta\sin^2\theta + a_{20}\sin^3\theta.$$

We note that this cubic trigonometric polynomial cannot be identically zero. Indeed, the case $b_{20} = a_{20} = a_{02} + b_{11} = a_{11} + b_{20} = 0$ produces the quadratic system

$$\dot{x} = -y(1 - a_{11}x - a_{02}y), \quad \dot{y} = x(1 - a_{11}x - a_{02}y);$$

and clearly this system has a center at the origin instead of a focus.

Since the cubic trigonometric polynomial (5) is not identically zero, it always has a real solution θ_0 depending continuously on the coefficients. For instance, when $a_{20} \neq 0$ then the cubic (5) has always a real root in $\tan\theta$ which goes to infinity when a_{20} tends to zero, and we can take $\theta_0 = \pi/2$ for $a_{20} = 0$. The process of associating to any system (2) with $a = 0$, $b = 1$, the angle θ_0 is continuous and the coefficients a'_{ij} and b'_{ij} are obtained from a_{ij} and b_{ij} by continuous operations, and doing the change of variables (4) with $\theta = \theta_0$, the proposition is proved. ∎

Looking at the proof of this proposition and taking into account the remark made just before the statement of the previous proposition, it follows that we can pass any quadratic system with a weak focus to the normal form (3) in a continuous way.

Calculations of Li [23] yield that $V_1 = L_1$, $V_2 = L_2 \pmod{V_1}$, $V_3 = L_3 \pmod{V_1, V_2}$ where

$$L_1 = r(l + n) - a(b + 2l),$$

$$L_2 = ra(5a - r)[(l + n)^2(n + b) - a^2(b + 2l + n)],$$

$$L_3 = ra^2[2a^2 + n(l + 2n)][(l + n)^2(n + b) - a^2(b + 2l + n)].$$

Using the above proposition and $L_1 = 0$, $L_2 = 0$, $L_3 \neq 0$ we obtain:

Corollary 3 *A planar quadratic vector field with a weak focus of third order may be written in the following form where the weak focus is placed at the origin:*

$$(6) \quad \dot{x} = p(x, y) = -y + lx^2 + 5axy + ny^2, \quad \dot{y} = q(x, y) = x + ax^2 + (3l + 5n)xy,$$

with $L_3 = 5a^3[2a^2 + n(l + 2n)][3(l + n)^2(l + 2n) - a^2(5l + 6n)] \neq 0.$

Systems (6) depend on the parameter $\lambda = (a, l, n) \in \mathbb{R}^3$. We consider systems (6) which are nonlinear, *i.e.*, $\lambda = (a, l, n) \neq 0$. In this case a system (6) can be rescaled, therefore the parameter space needed is actually the real projective plane $\mathbb{RP}(2)$ and not $\mathbb{R}^3$.

3 The Poincaré Compactification and the Complex (Real) Foliation with Singularities on $\mathbb{CP}^2$ ($\mathbb{RP}^2$) Associated to a Real Planar Polynomial Vector Field

A real planar polynomial vector field ξ can be compactified on the sphere as follows: Consider the x, y plane as being the plane $Z = 1$ in the space $\mathbb{R}^3$ with coordinates X, Y, Z. The central projection of the vector field ξ on the sphere of radius one yields a diffeomorphic vector field on the upper hemisphere and also another vector field on the lower hemisphere. There exists (for a proof *cf.* [19]) an analytic vector field $p(\xi)$ on the whole sphere such that its restriction on the upper hemisphere has the same phase curves as the one constructed above from the polynomial vector field. The projection of the closed northern hemisphere H^+ of $\mathbb{S}^2$ on $Z = 0$ under $(X, Y, Z) \rightarrow (X, Y)$ is called *the Poincaré disc*. A singular point q of $p(\xi)$ is called an *infinite* (respectively *finite*) singular point if $q \in \mathbb{S}^1$ (respectively $q \in \mathbb{S}^2 \setminus \mathbb{S}^1$). By the *Poincaré compactification of a polynomial vector field* we mean the vector field $p(\xi)$ restricted to the upper hemisphere completed with the equator.

Ideas in the remaining part of this section go back to Darboux's work [12]. Let $p(x, y)$ and $q(x, y)$ be polynomials with real coefficients. For the vector field

$$(7) \qquad\qquad p\frac{\partial}{\partial x} + q\frac{\partial}{\partial y},$$

or equivalently for the differential system

$$(8) \qquad\qquad \dot{x} = p(x, y), \quad \dot{y} = q(x, y),$$

we consider the associated differential 1-form $\omega_1 = q(x, y)dx - p(x, y)dy$, and the differential equation

$$(9) \qquad\qquad \omega_1 = 0.$$

Clearly, equation (9) defines a foliation with singularities on $\mathbb{C}^2$. The affine plane $\mathbb{C}^2$ is compactified on the complex projective space $\mathbb{CP}^2 = (\mathbb{C}^3 \setminus \{0\})/\sim$, where $(X, Y, Z) \sim (X', Y', Z')$ if and only if $(X, Y, Z) = \lambda(X', Y', Z')$ for some complex $\lambda \neq 0$. The equivalence class of (X, Y, Z) will be denoted by $[X:Y:Z]$.

The foliation defined by equation (9) on $\mathbb{C}^2$ can be extended to a singular foliation on $\mathbb{CP}^2$ and the 1-form ω_1 can be extended to a meromorphic 1-form ω on $\mathbb{CP}^2$ which yields an equation $\omega = 0$, *i.e.*,

$$(10) \qquad A(X, Y, Z)dX + B(X, Y, Z)dY + C(X, Y, Z)dZ = 0,$$

whose coefficients A, B, C are homogeneous polynomials and satisfy the relation:

$$(11) \qquad A(X, Y, Z)X + B(X, Y, Z)Y + C(X, Y, Z)Z = 0,$$

Indeed, consider the map $i \colon \mathbb{C}^3 \backslash \{Z = 0\} \to \mathbb{C}^2$, given by $i(X, Y, Z) = (X/Z, Y/Z) = (x, y)$ and suppose that $\max\{\deg(p), \deg(q)\} = m > 0$. Since $x = X/Z$ and $y = Y/Z$ we have:

$$dx = (ZdX - XdZ)/Z^2, \quad dy = (ZdY - YdZ)/Z^2,$$

the pull-back form $i^*(\omega_1)$ has poles at $Z = 0$ and the equation (9) can be written as

$$i^*(\omega_1) = q(X/Z, Y/Z)(ZdX - XdZ)/Z^2 - p(X/Z, Y/Z)(ZdY - YdZ)/Z^2 = 0.$$

Then the 1-form $\omega = Z^{m+2}i^*(\omega_1)$ in $\mathbb{C}^3 \setminus \{Z \neq 0\}$ has homogeneous polynomial coefficients of degree $m + 1$, and for $Z \neq 0$ the equations $\omega = 0$ and $i^*(\omega_1) = 0$ have the same solutions. Therefore the differential equation $\omega = 0$ can be written as (10) where

$$\begin{aligned}
A(X, Y, Z) &= ZQ(X, Y, Z) = Z^{m+1}q(X/Z, Y/Z), \\
(12) \qquad B(X, Y, Z) &= -ZP(X, Y, Z) = -Z^{m+1}p(X/Z, Y/Z), \\
C(X, Y, Z) &= YP(X, Y, Z) - XQ(X, Y, Z).
\end{aligned}$$

Clearly A, B and C are homogeneous polynomials of degree $m + 1$ satisfying (11).

To study the foliation with singularities defined by the differential equation (10) subject to (11) with A, B, C satisfying the above conditions in the neighborhood of the line $Z = 0$, we consider the two charts of $\mathbb{CP}^2$: $(u, z) = (Y/X, Z/X)$, $X \neq 0$, and $(v, w) = (X/Y, Z/Y)$, $Y \neq 0$, covering this line. We note that in the intersection of the charts $(x, y) = (X/Z, Y/Z)$ and (u, z) (respectively (v, w)) we have the change of coordinates $x = 1/z$, $y = u/z$ (respectively $x = v/w$, $y = 1/w$). Except for the point $[0{:}1{:}0]$, the foliation defined by equations (10),(11) with A, B, C as in (12) yields, in the neighborhood of the line $Z = 0$, the foliation defined by the phase curves of the vector field associated with the system

$$(13) \qquad \dot{u} = uP(1, u, z) - Q(1, u, z) = C(1, u, z), \quad \dot{z} = zP(1, u, z);$$

and except for the point $[1{:}0{:}0]$ it yields, in the neighborhood of the line $Z = 0$, the foliation defined in this neighborhood by the integral curves of the vector field associated with the system

$$(14) \qquad \dot{v} = vQ(v, 1, w) - P(v, 1, w) = -C(v, 1, w), \quad \dot{w} = wP(v, 1, w).$$

Similarly way we can associate a real foliation with singularities on $\mathbb{RP}^2$ to a real planar polynomial vector field.

4   Intersection Numbers

In this section we briefly recall the notion of intersection number of two algebraic curves at a point (*cf.* [18]).

The intersection number of two affine algebraic curves $C\colon f(x, y) = 0$ and $C'\colon g(x, y) = 0$ over $\mathbb{C}$ at a point in $\mathbb{C}^2$ is the number

$$I_a(f, g) = \dim_{\mathbb{C}} \mathbf{O}_a / (f, g),$$

where $\mathbf{O}_a$ is the local ring of the affine complex plane $\mathbf{A}^2(\mathbb{C}) = \mathbb{C}^2$ at a; *i.e.*, $\mathbf{O}_a$ is the ring of rational functions $r(x, y)/s(x, y)$ which are defined at a, *i.e.*, $s(a) \neq 0$.

In our case, since the polynomial differential systems are quadratic, the intersection numbers $I_a(p, q)$ for p, q as in (2), at the singular points a in $\mathbb{C}^2$ can be computed easily by using the axioms (*cf.* [18]). For two projective curves in $\mathbb{CP}^2$, $F(X, Y, Z) = 0$ and $G(X, Y, Z) = 0$, where F and G are homogeneous polynomials in the variables X, Y and Z over $\mathbb{C}$, assuming for instance $Z \neq 0$, we define $I_W(F, G) = I_w(f, g)$ where $f = F(x, y, 1)$, $g = G(x, y, 1)$ and $w = (X/Z, Y/Z)$. It is known that $I_W(F, G)$ is independent of the choice of a local chart, and of a projective change of variables, see again [18].

Clearly the above concept of intersection multiplicity extends to that of intersection multiplicity of several curves at a point of the projective plane. In particular we will be interested in the way the projective curves $A = 0$, $B = 0$ and $C = 0$ intersect and hence in the values of

$$I_a(A, B, C) = \dim_{\mathbb{C}} \mathbf{O}_a / (A, B, C).$$

Here $\mathbf{O}_a$ is the local ring at a of the complex projective plane (for more information see [18]) and (A, B, C) is the homogeneous ideal generated by these three polynomials.

If a is a finite or infinite singular point of system (6) and A, B and C are defined as in (12), then we have that $I_a(P, Q)$, $I_a(C, Z)$ and $I_a(A, B, C)$ are invariant with respect to affine transformations ([34]) and

$$(15) \qquad I_a(A, B, C) = \begin{cases} I_a(P, Q) = I_a(p, q) & \text{if } a \text{ is finite,} \\ I_a(P, Q) + I_a(C, Z) & \text{if } a \text{ is infinite.} \end{cases}$$

5   Zero-Cycles and Divisors

In this section we shall use the algebro-geometric notions of zero-cycle and divisor for the purpose of classifying systems (6). We briefly recall the definitions of these notions.

Let V be an irreducible algebraic variety over a field K. A *cycle of dimension r or r-cycle* on V is a formal sum $\sum_W n_W W$, where W is a subvariety of V of dimension r which is not contained in the singular locus of V, $n_W \in \mathbb{Z}$, and only a finite number of n_Ws are non-zero. The *support* of a cycle C is the set $\mathrm{Supp}(C) = \{W | n_W \neq 0\}$. We denote by $\mathrm{Max}(C)$ the maximum value of the coefficients n_W in C. For every $m \leq$

Max(C) let $s(m)$ be the number of the coefficients n_W in C which are equal to m. We call *type* of the cycle C the set of ordered couples $\big(s(m), m\big)$ where $1 \leq m \leq \text{Max}(C)$. The *degree* of a cycle of $\mathbb{C}^2$, (respectively $\mathbb{R}^2$, $\mathbb{CP}^2$, $\mathbb{RP}^2$) is the sum of its coefficients n_W. If J is a cycle we denote by $\deg(J)$, its degree.

An $(n-1)$-*cycle* is called a *divisor*. These notions which occur frequently in algebraic geometry [20], were used for classification purposes of planar quadratic differential systems by Pal and Schlomiuk [34, 42]. They are also helpful here as we indicate below.

Using Section 3, we can associate to the differential system (6) a singular foliation on $\mathbb{CP}^2$ described by the equation (10) satisfying (11), where

$$
\begin{aligned}
A(X, Y, Z) &= ZQ(X, Y, Z) = Z(XZ + aX^2 + (3l + 5n)XY), \\
B(X, Y, Z) &= -ZP(X, Y, Z) = -Z(-YZ + lX^2 + 5aXY + nY^2), \\
C(X, Y, Z) &= YP(X, Y, Z) - XQ(X, Y, Z) \\
&= -aX^3 - (2l + 5n)X^2Y - X^2Z + 5aXY^2 - Y^2Z + nY^3.
\end{aligned}
$$

(16)

We note that the straight line $Z = 0$ is always an algebraic invariant curve of this foliation and that its singular points are the solutions of the system $A(X, Y, Z) = B(X, Y, Z) = C(X, Y, Z) = 0$.

We consider a real polynomial differential system

$$
\dot{x} = f(x, y), \quad \dot{y} = g(x, y), \tag{17}
$$

with f and g relatively prime polynomials in $\mathbb{C}[x, y]$ with $\max\big(\deg(f), \deg(g)\big) = m$. To this system we can associate several zero-cycles and divisors:

(i) Two zero-cycles which encode the information regarding the intersection numbers of the real projective curves $F = 0$ and $G = 0$ in $\mathbb{CP}^2$:

$$
D_{\mathbb{K}}(F, G) = \sum_{W \in \mathbb{KP}^2} I_W(F, G)W,
$$

with $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, where

$$
F(X, Y, Z) = Z^m f(X/Z, Y/Z), \quad G(X, Y, Z) = Z^m g(X/Z, Y/Z).
$$

We note that when $W \in \mathbb{RP}^2$ the sum runs only on the real intersection points of $F = 0$ and $G = 0$. Always $I_W(F, G)$ is computed over $\mathbb{C}$.

(ii) Two divisors on the line at infinity $Z = 0$ which encode the multiplicities of the intersection points of $F = 0$ with $G = 0$ in $\mathbb{KP}^2$ which lie on $Z = 0$:

$$
D_{\mathbb{K}}(F, G; Z) = \sum_{W \in \{Z=0\} \cap \mathbb{KP}^2} I_W(F, G)W.
$$

(iii) Two zero-cycles which encode the information regarding the intersection numbers of the affine algebraic curves $f = 0$ and $g = 0$ in $\mathbb{K}^2$:

$$
D_{\mathbb{K}}(f, g) = D_{\mathbb{K}}(F, G) - D_{\mathbb{K}}(F, G; Z).
$$

(iv) A divisor which encodes the multiplicities of complex (respectively real) singular points at infinity which count how many complex real or complex singular points at infinity will bifurcate at infinity in a complex (respectively real) perturbation of the system:

$$D_{\mathbb{C}}(C, Z) = \sum_{W \in \{Z=0\} \cap \mathbb{KP}^2} I_W(C, Z)W,$$

where $C(X, Y, Z) = YF(X, Y, Z) - XG(X, Y, Z)$ and Z does not divide C.

(v) A zero-cycle which encodes the information regarding the intersection numbers of all the real or complex singularities W of the foliation in $\mathbb{CP}^2$ (respectively $\mathbb{RP}^2$) associated to a system (17):

$$D_{\mathbb{C}} = D_{\mathbb{C}}(f, g) + D_{\mathbb{C}}(F, G; Z) + D_{\mathbb{C}}(C, Z) = \sum_{W \in \mathbb{CP}^2} I_W(A, B, C)W,$$

where Z does not divide C. In this last equality we have used (15).

The foliation with singularities in $\mathbb{RP}^2$ associated to the system (17) can be obtained by identifying the diametrically opposite trajectories of the Poincaré compactification of system (17) and disregarding the orientation on the orbits. Then we can define for every isolated singular point W of the foliation in $\mathbb{RP}^2$ its *topological index*, $i(W)$, as the topological index of one of the two diametrically opposed singular points of the Poincaré compactification of system (17) which after the identification give the point W. A singular point is called *elementary* if at least one of its eigenvalues is not zero.

(vi) We introduce two new zero-cycles DI and $D_C I$ which will encode the topological indices of all the isolated singular points W of the foliation in $\mathbb{RP}^2$ associated to system (17) as follows:

$$DI = \sum_{W \in \mathbb{RP}^2} i(W)W, \quad D_C I = \sum_{W \in \mathbb{CP}^2} j(W)W.$$

where $i(W)$ is as defined above and $j(W) = i(W)$ for points W in $\mathbb{RP}^2$ and it is zero elsewhere in $\mathbb{CP}^2$.

6   Zero-Cycles and Divisors for Systems (6) and Their Bifurcation Curves

We apply the notions of the preceding section to our family of real differential systems (6). Our final goal is to give a clear, geometrical classification of this class.

For the real differential systems (6) with nonzero L_3 we associate the families of zero-cycles or divisors on $Z = 0$ introduced in Section 5 indexed by λ where $\lambda = [a{:}l{:}n] \in \mathbb{RP}(2)$.

Remark 4 Although the expressions of the zero-cycles and divisors in the following propositions depend on the specific canonical form (6), their *degrees* and their *types* as defined in the preceding section are affine invariant.

Indeed, these *degrees* and *types* depend only on the coefficients of these cycles which are affine invariant being intersection multiplicities. Furthermore, as we explain in our last section, these geometric invariants could be expressed in terms of algebraic invariants which in turn can be computed for any given normal form of the systems, other than (6), and the computation implemented on a computer.

To compute the cycles and obtain the bifurcation diagram, we first introduce some notations and perform some calculations. Not to repeat unnecessary computations we first note that the systems (6) have a symmetry. Indeed, a system (6) corresponding to the parameter $\lambda = (a, l, n)$ with $a < 0$ is topologically equivalent to system (6) corresponding to the parameter $\lambda' = (-a, l, n)$ through the symmetry $(x, y, t) \rightarrow (-x, y, -t)$. Due to this, we only consider the parameters $[a{:}l{:}n] \in \mathbb{RP}^2$ with $a \geq 0$ and $n \geq 0$. We can identify this subset of the parameter space with the quarter of the 2-dimensional sphere $a^2 + l^2 + n^2 = 1$ of $\mathbb{R}^3 = \{(a, l, n) : a, b, c \in \mathbb{R}\}$ having $a \geq 0$ and $n \geq 0$. We can view the bifurcation diagram in the disc $\{(a, l) : a^2 + l^2 \leq 1\}$ via vertical projection. However to obtain a better picture, we prefer to project this quarter of sphere on the plane $n = 0$ as follows. To every point p on this quarter of sphere we associate the intersection point p' of the plane $n = 0$ with the line joining p with the point $(0, 0, -3/4)$. In this way the parameter space of points $[a{:}l{:}n] \in \mathbb{RP}^2$, images of non-zero (a, l, n) in R^3 with $a \geq 0$, $n \geq 0$, can be identified with the half-disc $D = \{(a, l) : a^2 + l^2 \leq 1 \text{ and } a \geq 0\}$.

$D_{\lambda,\mathbb{C}}(P, Q; Z)$ involves the common roots of $p_2(x, y)$ and $q_2(x, y)$. We define

$$\Omega = \text{Resultant}\big(p_2(x, 1), q_2(x, 1), x \big) = n[l(3l + 5n)^2 - 3a^2(5l + 8n)] = n\bar{\Omega}.$$

We remark that for a weak focus of third order, it is not possible to have both $n = 0$ and $\bar{\Omega} = 0$ as in this case we would also have $L_3 = 0$ which yields a center at the origin.

Both $D_{\lambda,\mathbb{C}}(p, q)$ and $D_{\lambda,\mathbb{C}}(P, Q; Z)$ involve the intersection points of $P = 0$ with $Q = 0$. These are obtained by intersecting each of the lines $X = 0$ and $Z + aX + (3l+5n)Y = 0$ with $P = 0$. In the first case we obtain the points $[0{:}0{:}1]$ and $[0{:}1{:}n]$. In general, in the second case we obtain two points q_3 and q_4 which if $\bar{\Omega} \neq 0$ are given by

$$[3a(2l + 3n) + (3l + 5n)\sqrt{3\delta}{:}3a^2 - l(3l + 5n) - a\sqrt{3\delta}{:}\bar{\Omega}],$$

$$[3a(2l + 3n) - (3l + 5n)\sqrt{3\delta}{:}3a^2 - l(3l + 5n) + a\sqrt{3\delta}{:}\bar{\Omega}],$$

where $\delta = 3a^2 - l(l + 2n)$. They are real if $\delta \geq 0$, or complex if $\delta < 0$ and, $q_3 = q_4$ if $\delta = 0$. The real projective curve $\delta = 0$ restricted to $\mathbb{R}^2$ has two branches which we denote by δ_i, see Figure 1. If we denote by $R(y)$ the resultant of $p(x, y)$ and $q(x, y)$ with respect to x, we note that δ is a factor of the resultant of $R(y)$ and $R'(y)$ with respect to y. If $\bar{\Omega} = 0$, then one of the singularities q_3 and q_4 is finite and the other $[3l + 5n{:} - a{:}0]$ is infinite. When $\bar{\Omega} = 0$ and $L_3 \neq 0$ the finite singularity is $[6a^2 - 3l^2 - 5ln{:}-al{:}a(2l(3l + 5n) - 6a^2)]$. When it is on $l + 2n = 0$ the point $[0{:}1{:}n]$ satisfies both $Z + aX + (3l + 5n)Y = 0$ and $P = 0$. In this case q_2 coincides with one of the two points q_3, q_4.

Viewed in the half disc D, the real algebraic curve $\bar{\Omega} = 0$ restricted to D has three parts which we denote by $\bar{\Omega}_i$, $i = 1, 2, 3$, and $\bar{\Omega}_3$ is described below; see Figure 1. The line $l + 2n = 0$ cuts $\bar{\Omega}_3$, and determines on $\bar{\Omega}_3$ two open components $\bar{\Omega}_{31}$ and $\bar{\Omega}_{32}$, as indicated in Figure 1. The curve $\bar{\Omega} = 0$ determines on the straight line $l + 2n = 0$ viewed on D two segments $\ln_1$ and $\ln_2$, as indicated in Figure 1.

For the divisor $D_{\lambda,\mathbb{C}}(C; Z)$, we need the common points of $C = 0$ and $Z = 0$. We denote by $c(x) = C(x, 1, 0) = p_2(x, 1) - x q_2(x, 1)$, where p_2 and q_2 are the degree 2 homogeneous parts of p, respectively q. Let Δ be the discriminant of the cubic polynomial $c(x)$. We have:

$$\Delta = 125a^4 + a^2(25l^2 + 170ln + 262n^2) + n(2l + 5n)^3 = \text{Resultant}(c(x), c'(x), x)/(4a).$$

It is only necessary to do the resultant with respect to x, because when a is not zero the system (6) has no infinite singular points $[X:Y:Z]$ with $Y = 0$. This also implies that all common points of $C = 0$ and $Z = 0$ must be of the form $p = [r:1:0]$. We remark that $c(x)$ cannot have a triple root in our family, for this would imply that $c(x)$, $c'(x)$ and $c''(x)$ have a common root, which yields $a = 0$, impossible in our case. The real roots of $c(x)$ denoted in increasing values by r_i, give infinite points $[r_i:1:0]$ denoted by p_i.

The real algebraic curve $n\bar{\Omega}\delta(l + 2n)\Delta = 0$ divides the part corresponding to the half-disc D into 9 open connected components R_i, see Figure 1. In the following discussion we omit from these components the points of D such that $L_3 = 0$. We denote by ∂R_i the boundary of R_i in D. The interior of $\partial R_i \cap \{n = 0\}$ with the topology of the straight line $n = 0$ viewed in D is denoted by n_i; consequently $n_i \subset \partial R_i$, see Figure 1. In this figure the curve $\Delta = 0$ is denoted simply by Δ.

Convention   In order not to have two separate tables for real and complex divisors, we make the convention to use the symbol p^c (respectively q^c) to specify that a complex point p (respectively q) is not real. In this way the real divisors could be read directly from the table of complex divisors.

Notation   We denote by $\mathbb{N}_{\mathbb{C}}(S)$ the number of distinct complex singular points of the complex foliation with singularities (10) with (16), associated to a real planar quadratic system S. We denote by $\mathbb{N}_{\mathbb{R},f}(S)$, $\mathbb{N}_{\mathbb{R},\infty}(S)$ the number of finite, respectively infinite, distinct real singular points of the system S, by DI_f the finite part of the zero-cycle DI.

Proposition 5   *For all values of the parameter* $\lambda \in \mathbb{RP}^2$, *the divisors* $D_{\lambda,\mathbb{C}}(P, Q; Z)$, $D_{\lambda,\mathbb{C}}(C, Z)$ *and the zero-cycles* $D_{\lambda,\mathbb{C}}(p, q)$ *and* $D_{\lambda,\mathbb{C}}$ *for systems* (6) *with* $L_3 \neq 0$ *are well defined and we have*

(a) $\deg\big(D_{\lambda,\mathbb{C}}(P, Q; Z)\big) \leq 1$. *Moreover,*

$$D_{\lambda,\mathbb{C}}(P, Q; Z) = \begin{cases} 0 & \text{if } \Omega(\lambda) \neq 0, \\ [3l + 5n : -a : 0] & \text{if } \bar{\Omega}(\lambda) = 0, \\ [0:1:0] & \text{if } n = 0. \end{cases}$$
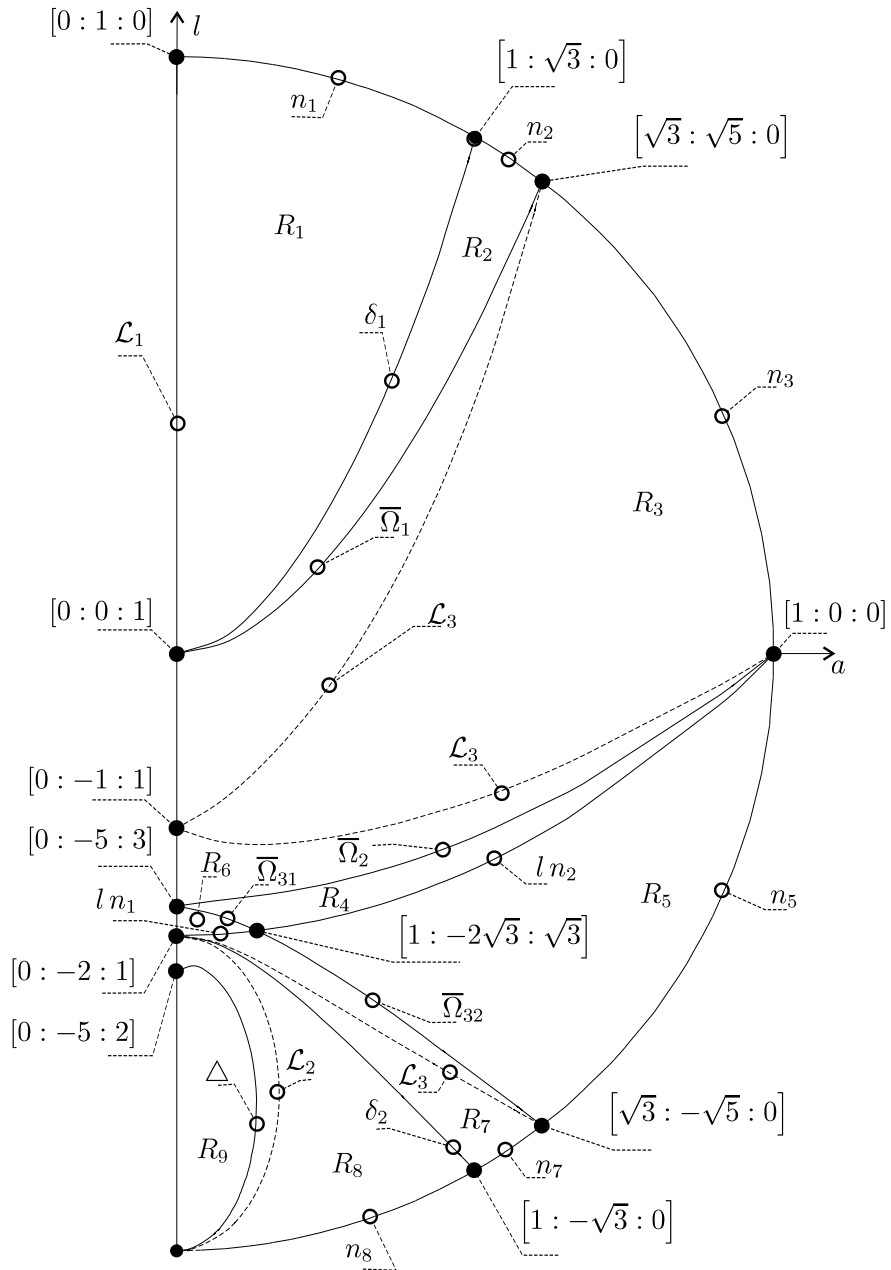
Figure 1: Bifurcation curves.

Hence the bifurcation curve for this divisor is $\Omega(\lambda) = n\bar{\Omega}(\lambda) = 0$.

(b) *The zero-cycles $D_{\lambda,\mathbb{C}}(p,q)$ for systems (6) with $L_3 \neq 0$ are given by:*

$$D_{\lambda,\mathbb{C}}(p,q) = \begin{cases} q_1 + q_2 + q_3 + q_4 & \textit{if } \delta > 0 \textit{ and } l + 2n \neq 0, \\ q_1 + q_2 + q_3^c + q_4^c & \textit{if } \delta < 0 \textit{ and } l + 2n \neq 0, \\ q_1 + q_2 + 2q_3 & \textit{if } \delta = 0, \\ q_1 + 2q_2 + q_3 & \textit{if } l + 2n = 0, \\ q_1 + q_2^c + q_3^c & \textit{in } n_1 \cup n_8. \end{cases}$$

Thus the bifurcation curve for this zero-cycle is $\delta(l + 2n) = 0$.

(c) *The divisor $D_{\lambda,\mathbb{C}}(C,Z)$ for systems (6) with $L_3 \neq 0$ are given by:*

$$D_{\lambda,\mathbb{C}}(C,Z) = \begin{cases} p_1 + p_2 + p_3 & \textit{if } \Delta > 0, \\ p_1 + p_2^c + p_3^c & \textit{if } \Delta < 0, \\ 2p_1 + p_2 & \textit{if } \Delta = 0. \end{cases}$$

Thus the bifurcation curve for this divisor is $\Delta = 0$.

(d) *For $\lambda \in \mathbb{RP}^2$ the zero-cycle $D_{\lambda,\mathbb{C}}$ for a system (6) with $L_3 \neq 0$ is well defined and for points corresponding to D it is given below, listed according to decreasing values for $\mathbb{N}_{\mathbb{C}}$.*

$\mathbb{N}_{\mathbb{C}} = 7$	$q_1 + q_2 + q_3^c + q_4^c + p_1 + p_2 + p_3$	*if* $\lambda \in R_1 \cup R_8 \cup R_9$,
	$q_1 + q_2 + q_3 + q_4 + p_1 + p_2 + p_3$	*if* $\lambda \in R_2 \cup \cdots \cup R_7$,
$\mathbb{N}_{\mathbb{C}} = 6$	$q_1 + q_3^c + q_4^c + 2p_1 + p_2 + p_3$	*if* $\lambda \in n_1 \cup n_8, (q_2 = p_1)$,
	$q_1 + q_3 + q_4 + 2p_1 + p_2 + p_3$	*if* $\lambda \in n_2 \cup n_3 \cup n_5 \cup n_7, (q_2 = p_1)$,
	$q_1 + q_2 + 2q_3 + p_1 + p_2 + p_3$	*if* $\lambda \in \delta_1 \cup \delta_2$,
	$q_1 + q_2 + q_4 + p_1 + 2p_2 + p_3$	*if* $\lambda \in \bar{\Omega}_1, (q_3 = p_2)$,
	$q_1 + q_2 + q_3 + 2p_1 + p_2 + p_3$	*if* $\lambda \in \bar{\Omega}_2, (q_4 = p_1)$,
	$q_1 + q_2 + q_4 + p_1 + p_2 + 2p_3$	*if* $\lambda \in \bar{\Omega}_{31} \cup \bar{\Omega}_{32}, (q_3 = p_3)$,
	$q_1 + 2q_2 + q_3 + p_1 + p_2 + p_3$	*if* $\lambda \in \ln_1 \cup \ln_2$,
	$q_1 + q_2 + q_3^c + q_4^c + 2p_1 + p_3$	*if* $\lambda \in \Delta$,
$\mathbb{N}_{\mathbb{C}} = 5$	$q_1 + 2q_3 + 2p_1 + p_2 + p_3$	*if* $\lambda = [1 : \pm\sqrt{3} : 0], (q_2 = p_1)$,
	$q_1 + 2q_2 + p_1 + p_2 + 2p_3$	*if* $\lambda \in [1 : -2\sqrt{3} : \sqrt{3}], (q_3 = p_3)$.

Thus the bifurcation curve for this zero-cycle is the real curve $\Delta\Omega\delta(l + n) = 0$.

Proof (a) and (b) of the proposition follow easily from the expression of q_i for $i = 1, \ldots, 4$, and the definition of intersection numbers. (c) follows from the fact that $c(x)$ is a cubic polynomial, which for $a \neq 0$ cannot have a triple root and Δ is the discriminant of $c(x)$. (d) sums up the results concerning multiplicities. ∎

We note that in Proposition 5 if $n = 0$, we must have $\bar{\Omega} \neq 0$ since otherwise $L_3 = 0$. We denote by $\mathcal{L}$ the algebraic curve in $\mathbb{RP}^2$ formed by the points λ such that $L_3 = 0$.

Proposition 6 *The real curve $\mathcal{C} = \{\lambda \in \mathbb{RP}^2 : n\bar{\Omega}\delta(l + 2n)\Delta = 0\}$ is the saddle-node bifurcation curve for the systems (6) with weak focus of third order at the origin (see Figure 1). Furthermore, (a) on the curve $n\bar{\Omega} = 0$ infinite saddle-nodes arise from the collision of one finite singularity with an infinite one; (b) on the curve $\delta(l + 2n) = 0$ finite saddle-nodes arise from the collision of two finite singularities; (c) on the curve $\Delta = 0$ infinite saddle-nodes arise from the collision of two infinite singularities.*

Proof (a) By using the classification of the planar singular points which have exactly one zero eigenvalue, it follows easily that the singular point q_3 when $n \neq 0$ and $\bar{\Omega} = 0$, or the singular point q_2 when $n = 0$, is a saddle-node. From the definition of the points q_i's, it is clear that the above two points are finite points on $\Omega \neq 0$ that reach the infinity on $\Omega = 0$.

(b) Using the classification of the planar singular points which have exactly one zero eigenvalue (see for instance [1]), it follows easily that the singular point q_3 when $\delta = 0$, or the singular point q_2 when $l + 2n = 0$, is a saddle-node.

(c) The singular point p_1 when $\Delta = 0$ is a saddle-node. From the definition of the points p_i's, it is clear that the infinite singular points p_1 and p_2 collide when $\Delta = 0$. ∎

The only real singularities of the curve $\mathcal{C}$ in $D \setminus \mathcal{L}$ are the three points: $\{[1 : \pm\sqrt{3} : 0]\} = \{\delta = 0\} \cap \{n = 0\}$ and $[1 : -2\sqrt{3} : \sqrt{3}] = \{\bar{\Omega} = 0\} \cap \{l + 2n = 0\}$. We denote by $\text{Sing}(\mathcal{C})$, the set formed by these three points.

Corollary 7 *For the normal form (6) and points corresponding to D we have $\mathbb{N}_{\mathbb{C}}(\lambda)$ equal to 7, 6 or 5 if and only if $\lambda \notin \mathcal{C}$, $\lambda \in \mathcal{C} \setminus \text{Sing}(\mathcal{C})$ or $\lambda \in \text{Sing}(\mathcal{C})$, respectively.*

The next result follows easily from Proposition 5.

Corollary 8 *For systems S in the class of quadratic systems with a weak focus of third order, $\mathbb{N}_{\mathbb{R},f}(S)$ takes all the values from 1 to 4 and $\mathbb{N}_{\mathbb{R},\infty}(S)$ takes all the values from 1 to 3. For the normal form (6) and for points corresponding to D we have:*

$$
\mathbb{N}_{\mathbb{R},f}(\lambda) = \begin{cases}
4 & \text{iff } \lambda \in R_2 \cup \cdots \cup R_7, \\
3 & \text{iff } \lambda \in n_2 \cup n_3 \cup n_5 \cup n_7 \cup \delta_1 \cup \delta_2 \cup \bar{\Omega}_1 \\
& \quad \cup \bar{\Omega}_2 \cup \bar{\Omega}_{31} \cup \bar{\Omega}_{32} \cup \ln_1 \cup \ln_2, \\
2 & \text{iff } \lambda \in R_1 \cup R_8 \cup R_9 \cup \Delta \cup \text{Sing}(\mathcal{L}), \\
1 & \text{iff } \lambda \in n_1 \cup n_8;
\end{cases}
$$

and $\mathbb{N}_{\mathbb{R},\infty}(\lambda)$ is 1, 2 or 3 if and only if $\lambda \in R_9$, $\lambda \in \Delta$ or otherwise, respectively.

Proposition 9 *All finite and infinite singular points of a quadratic system with a weak focus of third order are elementary.*

Proof From the normal form (6) it is easy to check that the linear part of the finite and infinite singular points have either determinant different from zero, or trace different from zero. So the proposition follows. ∎

We recall that the local phase portraits of elementary singular points are well known and consequently also their topological indices which could only be -1, 0 or 1, see for instance [1].

Proposition 10   *For all values of the parameter $\lambda \in \mathbb{RP}^2$ the divisor DI_λ for a system (6) with $L_3 \neq 0$ is well defined and for points corresponding to D it is equal to*

$$\mathbb{N}_{\mathbb{C}} = 7 \quad 1q_1 - 1q_2 + 0q_3^c + 0q_4^c + 1p_1 - 1p_2 + 1p_3 \quad \text{if } \lambda \in R_1,$$

$$1q_1 - 1q_2 + 1q_3 - 1q_4 + 1p_1 - 1p_2 + 1p_3 \quad \text{if } \lambda \in R_2,$$

$$1q_1 - 1q_2 - 1q_3 - 1q_4 + 1p_1 + 1p_2 + 1p_3 \quad \text{if } \lambda \in R_3,$$

$$1q_1 - 1q_2 - 1q_3 + 1q_4 - 1p_1 + 1p_2 + 1p_3 \quad \text{if } \lambda \in R_4,$$

$$1q_1 + 1q_2 - 1q_3 - 1q_4 - 1p_1 + 1p_2 + 1p_3 \quad \text{if } \lambda \in R_5,$$

$$1q_1 - 1q_2 + 1q_3 + 1q_4 - 1p_1 + 1p_2 - 1p_3 \quad \text{if } \lambda \in R_6,$$

$$1q_1 + 1q_2 + 1q_3 - 1q_4 - 1p_1 + 1p_2 - 1p_3 \quad \text{if } \lambda \in R_7,$$

$$1q_1 + 1q_2 + 0q_3^c + 0q_4^c - 1p_1 + 1p_2 - 1p_3 \quad \text{if } \lambda \in R_8,$$

$$1q_1 + 1q_2 + 0q_3^c + 0q_4^c - 1p_1 + 0p_2^c + 0p_3^c \quad \text{if } \lambda \in R_9,$$

$$\mathbb{N}_{\mathbb{C}} = 6 \quad 1q_1 + 0q_3^c + 0q_4^c + 0p_1 - 1p_2 + 1p_3 \quad \text{if } \lambda \in n_1, q_2 = p_1,$$

$$1q_1 + 1q_3 - 1q_4 + 0p_1 - 1p_2 + 1p_3 \quad \text{if } \lambda \in n_2 \cup n_7, q_2 = p_1,$$

$$1q_1 - 1q_3 - 1q_4 + 0p_1 + 1p_2 + 1p_3 \quad \text{if } \lambda \in n_3 \cup n_5, q_2 = p_1,$$

$$1q_1 + 0q_3^c + 0q_4^c + 0p_1 + 1p_2 - 1p_3 \quad \text{if } \lambda \in n_8, q_2 = p_1,$$

$$1q_1 - 1q_2 + 0q_3 + 1p_1 - 1p_2 + 1p_3 \quad \text{if } \lambda \in \delta_1,$$

$$1q_1 + 1q_2 + 0q_3 - 1p_1 + 1p_2 - 1p_3 \quad \text{if } \lambda \in \delta_2,$$

$$1q_1 - 1q_2 - 1q_4 + 1p_1 + 0p_2 + 1p_3 \quad \text{if } \lambda \in \bar{\Omega}_1, q_3 = p_2,$$

$$1q_1 - 1q_2 - 1q_3 + 0p_1 + 1p_2 + 1p_3 \quad \text{if } \lambda \in \bar{\Omega}_2, q_4 = p_1,$$

$$1q_1 - 1q_2 + 1q_4 - 1p_1 + 1p_2 + 0p_3 \quad \text{if } \lambda \in \bar{\Omega}_{31}, q_3 = p_3,$$

$$1q_1 + 1q_2 - 1q_4 - 1p_1 + 1p_2 + 0p_3 \quad \text{if } \lambda \in \bar{\Omega}_{32}, q_3 = p_3,$$

$$1q_1 + 0q_2 + 1q_3 - 1p_1 + 1p_2 - 1p_3 \quad \text{if } \lambda \in \ln_1,$$

$$1q_1 + 0q_2 - 1q_3 - 1p_1 + 1p_2 + 1p_3 \quad \text{if } \lambda \in \ln_2,$$

$$1q_1 + 1q_2 + 0q_3^c + 0q_4^c + 0p_1 - 1p_2 \quad \text{if } \lambda \in \Delta,$$

$$\mathbb{N}_{\mathbb{C}} = 5 \qquad 1q_1 + 0q_3 + 0p_1 - 1p_2 + 1p_3 \qquad \text{if } \lambda \in [1: \pm \sqrt{3}:0], q_2 = p_1,$$

$$1q_1 + 0q_2 - 1p_1 + 1p_2 + 0p_3 \qquad \text{if } \lambda \in [1: -2\sqrt{3}:\sqrt{3}], q_3 = p_3.$$

To have all real singularities displayed in the statement of the Proposition 10 we included the saddle-nodes (real) points q_i or p_i, whose indices are zero and we wrote $0q_i$ or $0p_i$ along with $0q_i^c$ or $0p_i^c$. Although the above values for DI_λ are computed for the canonical form (2), the *types* of these divisors are affine invariants.

According to the Poincaré-Hopf Theorem (see for instance [30]) the sum of the indices of all singularities of the foliation in $\mathbb{RP}^2$ associated to system (6) is $\deg(DI) = 1$. Then, the next result follows from Proposition 10.

Corollary 11 *For systems S in the class of quadratic systems with a weak focus of third order the values taken by the function* $\deg(DI_f)(S)$ *are* -2, -1, 0, 1 *and* 2. *For the normal form (6) and for points corresponding to D we have:*

$$\deg[(DI_f)(\lambda)] = \begin{cases} -2 & \text{iff } \lambda \in R_3, \\ -1 & \text{iff } \lambda \in n_3 \cup n_5 \cup \bar{\Omega}_1 \cup \bar{\Omega}_2, \\ 0 & \text{iff } \lambda \in R_1 \cup R_2 \cup R_4 \cup R_5 \cup \delta_1 \cup \ln_2, \\ 1 & \text{iff } \lambda \in n_1 \cup n_2 \cup n_7 \cup n_8 \cup \bar{\Omega}_{31} \cup \bar{\Omega}_{32} \cup \mathrm{Sing}(\mathcal{L}), \\ 2 & \text{iff } \lambda \in R_6 \cup R_7 \cup R_8 \cup R_9 \cup \delta_2 \cup \ln_1 \cup \Delta. \end{cases}$$

The $\deg[(DI_f)(\lambda)]$ is a global topological index measuring the relative number of finite saddles versus antisaddles in the finite plane.

7 The Bifurcation Diagram of the Systems With a Weak Focus of Third Order

7.1 Basic Properties of Quadratic Systems and Specific Properties of Systems in QW3

We list below results which play a role in the study of the global phase portraits of real planar quadratic systems (6) having a weak focus of third order.

The following results hold for any quadratic system:

(i) A straight line either has at most two (finite) contact points with a quadratic system (which include the singular points), or it is formed by trajectories of the system; see Lemma 11.1 of [51]. We recall that by definition a *contact point* of a straight line L is a point of L where the vector field has the same direction as L, or it is zero.

(ii) If a straight line passing through two real finite singular points q_1 and q_2 of a quadratic system is not formed by trajectories, then it is divided by these two singular points in three segments $\overline{\infty q_1}$, $\overline{q_1 q_2}$ and $\overline{q_2 \infty}$ such that the trajectories cross $\overline{\infty q_1}$ and $\overline{q_2 \infty}$ in one direction, and they cross $\overline{q_1 q_2}$ in the opposite direction; see Lemma 11.4 of [51].

(iii) The straight line connecting a real finite singular point and a pair of real opposite infinite singular points in the Poincaré compactification of a system (6) is

either formed by trajectories or is a straight line without (finite) contact points except at that finite singular point; see Lemma 11.5 of [51].

(iv) If a quadratic system has a limit cycle, then it surrounds a unique singular point, and this point is a focus; see [11].

(v) If in a quadratic system the separatrix of an infinite saddle connects with the separatrix of the diametrically opposite infinite saddle, then this separatrix is an invariant straight line; see [49].

(vi) If a quadratic system has a center, then it is integrable; *i.e.*, there exists a nonconstant analytic first integral defined in the whole real plane except perhaps on some invariant algebraic curve; see [50], [27] and [39].

Finally we state here an important result on general quadratic systems obtained by Zhang Pingguang [52], [53].

(vii) If there are limit cycles surrounding two foci of a quadratic system, then around one of the foci there is at most one limit cycle.

The proof of this result is based on several lemmas as well as on previous articles. The main tools used in this proof are: the transformation of the systems into Lienard form and previous results in the literature on uniqueness of limit cycles in planar vector fields. For this see the English version [53] of [52].

We shall also need the following specific result for quadratic systems having a weak focus of third order at the origin:

(viii) There are no limit cycles of systems (6) surrounding the weak focus of third order, see [24].

We shall now use these properties for the study of the class QW3.

From (iv) it follows that the limit cycles of a system (6) may only be around the singular points $(0, 0)$ and $(0, 1/n)$ because they are the only possible foci of system (6) for convenient values of the parameters. Therefore, from (viii), the limit cycles of a system (6) can only be around the singular point $(0, 1/n)$ in case they exist, and only when this singular point is a focus. Now, from (vii), since by perturbing conveniently a weak focus of third order it is possible to produce three limit cycles surrounding the origin, it follows that for those systems (6) which have two foci, if they have limit cycles around $(0, 1/n)$, then they must have exactly one limit cycle.

Except for eliminating the possibility of existence of more than one limit cycle surrounding a focus at $(0, 1/n)$, the phase portraits of quadratic systems (6) were obtained by Artés in his 1984 Master's Thesis [3], supervised by Llibre and published by the Universitat Autònoma de Barcelona. At the time of the publication in 1984, the result (vii) was not yet proved. Interest shown by some mathematicians in the work [3] inspired the authors to write a new version and to publish it in a journal (*cf.* [5]) in order to facilitate wider access, but again this was prior to the result (vii). Furthermore the phase portraits in [5] are classified in terms of inequalities on the coefficients appearing in the normal form (6) and no concern was shown for obtaining intrinsic results. Here we checked the results and furthermore we introduced integer-valued invariants. These led us to detect that two phase portraits which were counted as distinct in [5] turned out to be topologically equivalent. As it is expected that the class of quadratic systems will yield more than two thousand phase portraits, global tools to distinguish or identify them are essential. Another aspect is the applicability of results to any given situation irrespective of particular presentation of the

systems, a problem which we discuss in our last section.

7.2   Phase Portraits of QW3 and Basic Features of the Bifurcation Diagram

The basic ingredient for obtaining the phase portraits of QW3 is the knowledge we have of the global phase portraits of the quadratic systems (6) with a center (*cf.* [39, 50]), which appear in our diagram on the components of the curve $\mathcal{L}$ or $L_3 = 0$, *i.e.*, the line $\mathcal{L}_1$ defined by $a = 0$ (the diameter in Figure 2), the conic $\mathcal{L}_2$ (a parabola in the affine part $n \neq 0$) defined by $2a^2 + n(l + 2n) = 0$, and the singular cubic $\mathcal{L}_3$ defined by $3(l + n)^2(l + 2n) - a^2(5l + 6n) = 0$ with a nodal singularity at the point $[0: -1:1]$. Perturbing these systems with center we obtain the phase portraits near by. Then using these as well as further analysis based on knowledge of singularities yields most of the results. A special discussion is needed for the region R_8 and for part of its border: $\Delta = 0$. We describe below the arguments needed to obtain the phase portraits on R_1 and on its boundary. Then, with the exception of the region R_8 and its partial boundary $\Delta = 0$, where more arguments will be needed, the phase portraits in all the remaining regions could be obtained in a similar way.

We highlight below some of the arguments. Although we picture our diagram on the half-disc D, we shall mark the points on D by their corresponding points in $\mathbb{RP}^2$ given in homogeneous coordinates, since for us the important object is the parameter space $\mathbb{RP}^2$. Thus, for instance, the point (a, l) of D will be denoted by $[a:l:\sqrt{1 - a^2 - l^2}]$.

The systems (6) are of two types: centers, which correspond to values of $\lambda = [a:l:n]$ for which $L_3 = 0$; or weak foci, which correspond to values of λ for which $L_3 \neq 0$.

The phase portraits of quadratic systems with centers are known (*cf.* [39, 50]). We place these on the bifurcation curves of centers. These are the components of the curve $\mathcal{L}$ or $L_3 = 0$, *i.e.*, the line $\mathcal{L}_1$ defined by $a = 0$ (the diameter in Figure 2), the conic $\mathcal{L}_2$ (a parabola in the affine part $n \neq 0$) defined by $2a^2 + n(l + 2n) = 0$, and the singular cubic $\mathcal{L}_3$ defined by $3(l + n)^2(l + 2n) - a^2(5l + 6n) = 0$ with a node at $[0: -1:1]$.

Systems (6) with $a = 0$ are given by

$$\dot{x} = -y + lx^2 + ny^2, \quad \dot{y} = x + (3l + 5n)xy.$$

These are the systems (4.10) in [39] with $b = -l$, $d = -n$ and $A = 3l + 5n = -3b - 5d$. So the line $a = 0$ is the line $A + 3b + 5d = 0$ which could be traced in Figure 5 of [39] with affine part ($d \neq 0$) in Figure 2 of [39]. We place on our Figure 2, and on $\mathcal{L}_1$ ($a = 0$) the corresponding phase portraits. We remark that the changes in phase portraits on $a = 0$ are given by the changes in the invariant algebraic curves which such systems have; namely, the straight line $(3l + 5n)y + 1 = 0$ and the conic $-l(2l + 5n)(l + 5n)x^2 - 2(l + 3n) - 4l(l + 3n)y + ln(l + 5n)y^2 = 0$. In a similar way we put on Figures 2 and 3 all the phase portraits having a center.

The phase portrait W_1 on R_1 follows easily from the phase portrait corresponding to the $\partial R_1 \cap \{a = 0\}$ which has a center. To obtain W_1 we use the behavior of the vector field on the explicit expression of the straight invariant line connecting two diametrically opposite infinite saddles of the system with center.
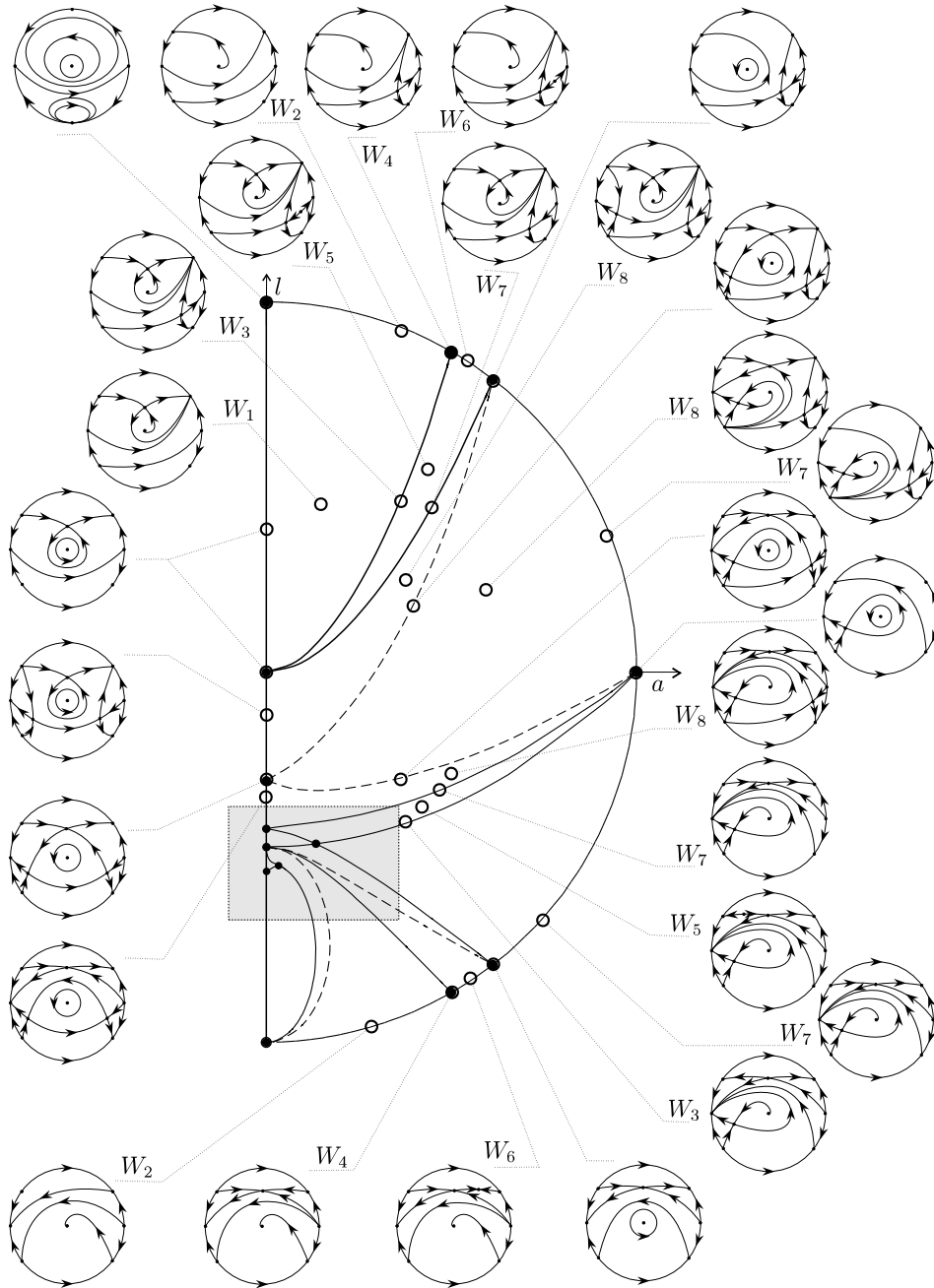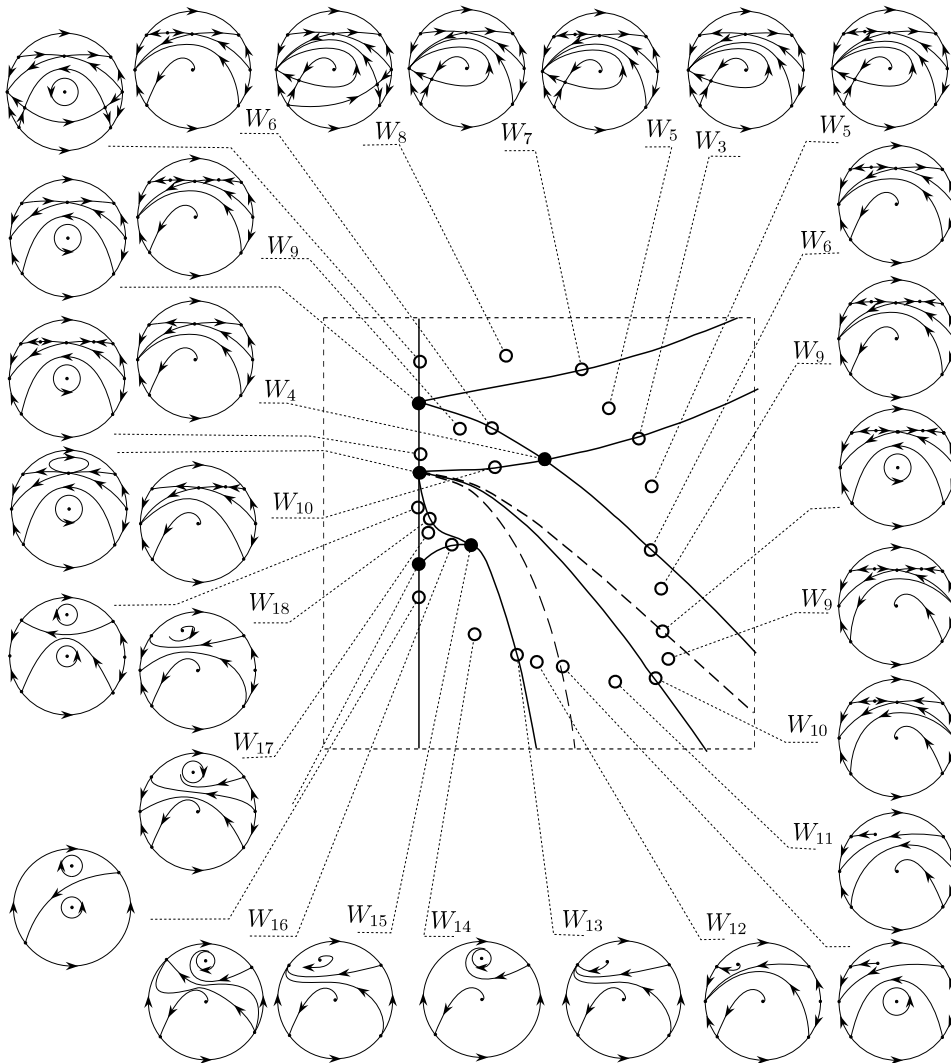
Figure 2: Phase portraits.

Figure 3: Phase portraits.

The phase portrait W_1 actually holds in the whole region R_1. Indeed, we rule out the two other global phase portraits which might occur, since they would imply the existence, for some parameter point, of a straight invariant line connecting two opposite points at infinity, a fact which cannot occur in a system in QW3 according to results in [49].

The phase portrait W_{14} in R_9 follows easily from the phase portrait on the border of R_9 which is on $a = 0$, which has a center. Using arguments analogous to the ones for W_1 we see that W_{14} holds in the whole region.

The phase portrait W_2 on n_1 is obtained from the phase portrait W_1 on R_1 taking into account that on n_1 the saddle $[0:1:n]$ of W_2 collides with an infinite node of W_2, creating a saddle-node at infinity.

The phase portrait W_3 on δ_1 follows from the phase portrait W_1 on R_1 knowing that on δ_1 there appears an additional singularity which is a finite saddle-node.

The phase portrait W_4 at $[1:\sqrt{3}:0]$ can be obtained easily from the phase portraits W_2 on n_1 and W_3 on δ_1. So we have described all the phase portraits on the region R_1 and its boundary points not on $a = 0$.

The detailed discussion for the regions $R_2, \ldots, R_7$, will appear as a technical report elsewhere.

We shall now describe the phase portraits in the region R_8 and on $\Delta = 0$. Here again we start by perturbing a system with center.

We start by using the curve $\mathcal{L}_2$ of centers. These centers are algebraically integrable systems which in [39] are of Class I. This curve enables us to find the phase portraits corresponding to any λ sufficiently close to $\mathcal{L}_2$. For such λs we obtain two distinct phase portraits, one for λ such that $2a^2+n(l+2n) < 0$ and another one for λ such that $2a^2+n(l+2n) > 0$. These phase portraits are respectively W_{11} and W_{12}. For obtaining them we use the behavior of the vector field on the algebraic curve which bounds the period annulus around the center of the phase portrait associated to $\mathcal{L}_2$ and look at how the local separatrices behave with respect to this curve in the perturbations.

In the subregion of R_8 where $2a^2 + n(l + 2n) > 0$ we have the curve $\Gamma = 25a^2 + 12(l+2n)n = 0$ (or simply Γ) on which the singular point $(0, 1/n) = [0:1:n]$ passes from being a node to a focus. This curve Γ has endpoints $[0:-2:1]$ and $[0:-1:0]$ and it is contained inside the part of R_8 which is limited by $\mathcal{L}_2$ and $\Delta = 0$; it does not intersect these last two curves except at its endpoints. We claim that on Γ and close to it there is no topological change in the phase portrait but only a C^∞ change when crossing Γ. Therefore the phase portrait in Γ and also close around it, is thus W_{12}.

Now we prove the claim. By Zhang Pingguang's result [52, 53], crossing Γ from a node to a focus there could appear at most one limit cycle surrounding the focus, which must be semistable near the bifurcation curve due to the fact that the focus or node is unstable and there is a stable separatrix coming from infinity surrounding it (see phase portrait W_{12}). But if for a system X in QW3 such a semistable limit cycle exists, then close to it there would be two limit cycles surrounding the focus, one stable and the other unstable, (because the QW3 systems form a rotated family with respect to the parameter a in an open neighborhood of Γ; see [51] for a definition and properties), and this is in contradiction with Zhang Pingguang's result. So the claim is proved.

We now discuss the behavior of the systems on $\Delta = 0$, or simply Δ. In the region

R_9 the phase portrait is W_{14}, and in the region R_8 near Δ and close to $[0\!:\!1\!:\!0]$ it is W_{12}. Then, moving just above Δ from R_8 to R_9 on a small segment with $l = $ constant which intersects Δ, an infinite saddle and an infinite node of W_{12} collide, and by continuity we obtain on Δ close to $[0\!:\!1\!:\!0]$ the phase portrait W_{13}. If we continue moving in the same direction, on the same segment, we get the phase portrait W_{14} on R_9.

The phase portraits in R_8 close to the open segment γ on $a = 0$ having endpoints $[0\!: -5\!:\!2]$ and $[0\!: -2\!:\!1]$ are again obtained from portraits with center, *i.e.*, from those on γ which have two centers. So, studying the behavior of the vector field on the branches of the hyperbola that limit the annular regions around the two centers, we obtain the phase portrait W_{17} on the points of R_8 which are sufficiently near to γ.

7.3 The Saddle to Saddle Connection Bifurcation Curve $\mathcal{G}_3$

We complete here the construction of the phase portraits in R_8. Take a point q on γ close to $[0\!: -2\!:\!1]$ (actually q could be taken as close as we wish to this point). For any point p of γ we consider an analytic arc σ_p inside the region R_8 connecting p with q. On σ_p close to p we have the phase portrait W_{17} and close to q we have W_{12}. Therefore, in between, there must exist at least a bifurcation point. We now show that on σ_p we have a bifurcation point for which the phase portrait has a saddle to saddle connection. We first note that in the whole of R_8 we have the same kinds of singularities, finite or infinite. Furthermore, in the lower part of the x,y plane the phase portraits have the same aspect: a canonical region bounded by a global separatrix connecting an infinite saddle, say S_1 with an infinite node say N_1, by an arc segment at infinity and by a global separatrix with one end at the weak focus and oriented towards an infinite saddle, say S_2. It is in the upper part that the phase portraits could look different. At infinity we have two more saddles: S_1' opposite S_1 having a local stable manifold $W_{loc}^s(S_1'(\lambda))$ and S_2' opposite S_2 having an unstable manifold $W_{loc}^u(S_2'(\lambda))$. Due to the analyticity of the systems in both x, y and in λ, the local invariant manifolds have local equations depending analytically on x, y, λ, (see [7], [29, p. 518 Appendice II] for proofs of the analyticity of the local invariant manifolds which also work with parameters).[1]

By analytic continuation on solution curves, the local invariant manifolds are continued globally to $W^s\big(S_1'(\lambda)\big)$, $W^u\big(S_2'(\lambda)\big)$ and have local equations $F(x, y, \lambda) = 0$, $G(x, y, \lambda) = 0$ around any point (x_0, y_0, λ_0), with F, G analytic in x, y, λ. Due to the transversality of the flow on $x = 0$ and due to the configuration of the singularities at infinity, both $W^s\big(S_1'(\lambda)\big)$, $W^u\big(S_2'(\lambda)\big)$ cut the y-axis. Let $y_1(\lambda)$, $y_2(\lambda)$ be the first points where the two global invariant manifolds cut the y-axis. Due to the analytic version of the Implicit Function Theorem (IFT), applied respectively to the local equations $F(x, y, \lambda) = 0$ and $G(x, y, \lambda) = 0$ of the two invariant manifolds around the points $\big(0, y_1(\lambda)\big)$, $\big(0, y_2(\lambda)\big)$, the functions $y_i(\lambda)$ of λ are analytic. The hypothesis of (IFT) is verified due to the analyticity of F and G and due to the transversality of the flow to $x = 0$. Hence the function $f(\lambda) = y_1(\lambda) - y_2(\lambda)$ is analytic in λ and defined for all points in R_8 belonging to a neighborhood N_p of σ_p. In N_p and

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[1] We thank C. Rousseau for checking this and mentioning it to us.

close to Γ, $f(\lambda)$ is positive (see phase portrait W_{18}) and close to $a = 0$ it is negative (see W_{12}). Hence by continuity, on σ_p there exists a point λ_0 such that $f(\lambda_0) = 0$, which means that for λ_0 we have a connection and the corresponding phase portrait is W_{18}. Let $\mathcal{G}_3$ be the set of all λ such that $f(\lambda) = 0$ when p runs over γ. Due to the analyticity of f in λ, the set $\mathcal{G}_3$ is analytic.

Due to the analyticity of $f(\lambda)$ on σ_p, there are only a finite number of solutions of $f(\lambda) = 0$. It has been observed numerically that we actually have a unique such solution which can be computed with as good an approximation as we wish. Varying p in γ we get a continuous curve connecting the points $[0\!:\!-2\!:\!1]$ and a point $\Delta_2 \in \Delta$ whose coordinates are given numerically as in [5]. At this point the phase portrait is W_{18}. Numerical computations show that the curve $\mathcal{G}_3$ has the qualitative shape indicated in Figures 2 and 3.

We divide the curve Δ into two arcs: Δ_1, which is the open arc of Δ having endpoints $[0\!:\!-1\!:\!0]$ and the point Δ_2, and the open arc Δ_3 having endpoints Δ_2 and $[0\!:\!-5\!:\!2]$.

Now we consider a sufficiently small open segment $l = $ constant, having endpoints on both sides of the curve Δ_3. Thus at the endpoint inside R_9 the phase portrait is W_{14} and at the endpoint inside R_8 and very near Δ_3 the phase portrait is W_{17}. Then moving from W_{17} to W_{14} on this vertical segment we obtain that an infinite saddle and an infinite node of W_{17} collide on Δ_3 giving a saddle-node, and by continuity we obtain the phase portrait W_{16} on Δ_3. If we continue the motion the saddle-node at infinity disappears and we get the phase portrait W_{14}.

We note that the phase portraits W_{14}, W_{16} and W_{17} have a limit cycle, due to the instability of the strong focus and the behavior of the separatrix winding around the strong focus and whose ω-limit set cannot thus be this strong focus. The uniqueness of this limit cycle follows from previous arguments given at the beginning of this section.

Let $\mathcal{G} = \Delta_1 \cup \Delta_2 \cup \mathcal{G}_3$. $\mathcal{G}$ is the bifurcation curve which separates in D the phase portraits having limit cycles from those without limit cycles. Then $\mathcal{G} = \Delta_1 \cup \Delta_2 \cup \mathcal{G}_3$. We also denote $\mathcal{G}_1 = \Delta_1$ and $\mathcal{G}_2 = \Delta_2$.

From the above arguments, modulo uniqueness of the curve $\mathcal{G}_3$ which is observed numerically, in 7.2 and 7.3 we have thus proved the following result.

Theorem 12   *The bifurcation set of the class* QW3 *is formed by an algebraic set on which we have bifurcations of saddle-node singularities and by an analytic set ($\mathcal{G}_3$) of saddle-to-saddle connections. The bifurcation diagram of the class* QW3*, viewed in the quarter of the disc D is as pictured in Figure 2 with its grey area enlarged in Figure 3, the uniqueness and shape of the connection curve $\mathcal{G}_3$ being observed numerically.*

8   Summing Up the Global Geometrical Properties of Quadratic Systems with Third Order Foci: the Classification Theorems

In a bifurcation diagram we may have topologically equivalent phase portraits belonging to distinct regions of the parameter space. In identifying two of the phase

portraits, counted as distinct in [5], we were helped by the integer-valued invariants which we constructed. It is thus necessary to have a classification in terms of simple and, if possible integer-valued invariants. We give below such a classification.

A crude grouping of the systems into classes according to their properties, was given in terms of the values of $\mathbb{N}_{\mathbb{C}}$ in Corollary 7 of the preceding section. We shall now split the three classes obtained into finer ones by using the remaining three invariants: $\mathbb{N}_{\mathbb{R},f}$, $\deg(DI_f)$, $\mathbb{N}_{\mathbb{R},\infty}$, the four invariants being ordered according to their weight in the classification problem. Knowledge of the first invariants sometimes determines the remaining ones and even implies, up to topological equivalence, a unique phase portrait as indicated in the next theorem. Using these four invariants we obtain a partition of the parameter space $\mathbb{RP}^2$.

Theorem 13   *Consider the family* QW3 *of all quadratic systems with a weak focus of third order. The values of the affine invariant* $\mathcal{J}(S) = \big(\mathbb{N}_{\mathbb{C}}, \mathbb{N}_{\mathbb{R},f}, \deg(DI_f), \mathbb{N}_{\mathbb{R},\infty}\big)(S)$ *given in the following diagram yield a partition of the family* QW3^2 *as follows:*

$$
\mathbb{N}_{\mathbb{C}}(S) = \begin{cases}
7 \text{ and } \mathbb{N}_{\mathbb{R},f}(S) = \begin{cases}
2 \text{ and } \deg\big(DI_f(S)\big) = \begin{cases}
0(W_1), \\
2 \text{ and } \mathbb{N}_{\mathbb{R},\infty}(S) = \begin{cases} 1(W_{14}), \\ 3, \end{cases}
\end{cases} \\[1em]
4 \text{ and } \deg\big(DI_f(S)\big) = \begin{cases}
-2(W_8), \\
0(W_5), \\
2(W_9),
\end{cases}
\end{cases} \\[3em]
6 \text{ and } \mathbb{N}_{\mathbb{R},f}(S) = \begin{cases}
1(W_2), \\
2, \\
3 \text{ and } \deg\big(DI_f(S)\big) = \begin{cases}
-1(W_3), \\
0(W_7), \\
1(W_6), \\
2(W_{10}),
\end{cases}
\end{cases} \\[3em]
5(W_4).
\end{cases}
$$

Furthermore, for each value of $\mathcal{J}(S)$ *in this diagram with the exception of two cases:* (i) $\mathcal{J}(S) = (7, 2, 2, 3)$ *which occurs in* R_8, *and* (ii) $\mathbb{N}_{\mathbb{C}}(S) = 6$ *and* $\mathbb{N}_{\mathbb{R},f}(S) = 2$ *which occurs on* Δ, *there corresponds a single phase portrait; i.e., if S and S' are such that* $\mathcal{J}(S) = \mathcal{J}(S')$ *and this common value does not satisfy* (i) *or* (ii) *above, S and S' are topologically equivalent. This unique phase portrait has neither limit cycles nor graphics (for the definition of a graphic see* [15]), *with the exception of the phase portrait* W_{14} *corresponding to* $\mathcal{J}(S) = (7, 2, 2, 1)$ *which has a unique limit cycle. The phase portrait* W_{14} *is the only one in the family* QW3 *having (two) complex singularities at infinity in addition to a real one.*

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[2]We point out that in the cases other than (i) and (ii), whenever just the first component (or the first two components) of $\mathcal{J}(S)$ suffice to yield a single phase portrait, we do not write the values of the remaining invariants.

The above result follows from a careful analysis of the bifurcation diagram in Figures 2 and 3 of Section 9. ∎

We now consider the cases (i) and (ii) left out in Theorem 13, *i.e.*, the systems S whose normal form (6) corresponds to $\lambda \in R_8 \cup \Delta$. We define below some new concepts.

Definition   We consider the Poincaré compactification on the sphere. Let $\bar{H} = H \cup \mathbb{S}^1$ where H is the upper hemisphere and $\mathbb{S}^1$ is its equator. Let

$$D_{\text{Sep},\infty} = \sum_{W \in \mathbb{S}^1} s(W)W, \quad D_{\text{Sep},f} = \sum_{W \in H} s(W)W, \quad D_{\text{Sep}} = \sum_{W \in H} s(W)W$$

where $s(W)$ is the number of global nonequilibrium separatrices contained in H (or in $\mathbb{R}^2$) which start or end at W. Let $m_\infty = \max_{W \in \mathbb{S}^1}\{s(W)\}$ where $s(W)$ is defined as above.

There is a great difference between the divisor $D_{\text{Sep},\infty}$ on $\mathbb{S}^1$, the zero-cycles $D_{\text{Sep},f}$ on $\mathbb{R}^2$, D_{Sep} on the semi-algebraic variety $\bar{H}$, and the previously defined zero-cycles or divisors. Indeed, $D_{\text{Sep},\infty}, D_{\text{Sep},f}, D_{\text{Sep}}$ depend on nontrivial global solutions which are separatrices of the phase portrait. Clearly m_∞ is a global integer-valued affine invariant.[3]

Remark 14   With the exception of two phase portraits: W_3 and W_7, the type of the zero-cycle D_{Sep} distinguishes the 18 phase portraits of quadratic systems having a weak focus of third order. These two phase portraits are well distinguished by the degree $\deg(DI_f)$ of the zero-cycle DI_f. More precisely $\deg(DI_f) = 0$ for W_3 and $\deg(DI_f) = -1$ for W_7.

It is not difficult to prove that the only separatrices of polynomial vector fields on the Poincaré sphere are singular points, limit cycles and the boundary orbits of the hyperbolic sectors of singularities; for more details see [26].

We denote by p_i' the infinite singular point diametrically opposed to the infinite singular point p_i for the Poincaré compactification of a planar polynomial vector field.

We divide the open set R_8 as follows: R_8^u is the open subset of R_8 whose boundary is formed by the curves $\delta_2 \cup n_8 \cup \mathcal{L}_2$; R_8^d is the open subset of R_8 whose boundary is formed by the curves $\mathcal{L}_2 \cup \mathcal{G}$; and R_8^{dd} is the open subset defined by $R_8 \setminus \left(R_8^u \cup \text{cl}(R_8^d) \right)$, where as usual $\text{cl}(A)$ denotes the closure of the set A.

We now define the full multi-integer-valued invariant which classifies topologically all systems in QW3: $\mathfrak{I} = \left(\mathbb{N}_{\mathbb{C}}, \mathbb{N}_{\mathbb{R},f}, \deg(DI_f), \mathbb{N}_{\mathbb{R},\infty}, \deg(D_{\text{sep}}), m_\infty \right)$. Due to Theorem 13, we only need to check the cases covered by (i) and (ii), left out in Theorem 13. For W_i with $i = 11, 12, 13$, m_∞ takes the values 1, 2, 3 respectively. For W_{16} and W_{17} we have $D_{\text{sep}} = 7$ but m_∞ is 2 respectively 1. For W_{15} and W_{18}, D_{sep} is 6 but m_∞ is 2, respectively 1. In short, we have obtained the following:

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[3]For a good definition of a separatrix of a 2-dimensional flow see, for instance, Newman [31].

Theorem 15 *Let S and S′ be quadratic systems having a weak focus of third order. Then $\mathfrak{I}(S) = \mathfrak{I}(S′)$ if and only if S and S′ are topologically equivalent.*

We now sum up the global geometrical characteristics we have obtained for the class QW3.

Theorem 16 *The class* QW3 *is partitioned in the following three subclasses:*

(I) *Systems without a limit cycle and without graphics. We have a total of twelve such phase portraits: W_i with $i = 1, \ldots, 12$. The systems with W_i, $i = 1, \ldots, 10$ are classified by $\mathfrak{J}(S)$ (see Theorem 13). $\mathfrak{J}(S) = (7, 2, 2, 3)$ for W_{11} and for W_{12} and these phase portraits are distinguished by m_∞. $m_\infty = 1$ for W_{11} and $m_\infty = 2$ for W_{12}.*

(II) *Systems with a limit cycle. These have no graphic and the limit cycle is unique. These yield three phase portraits which are topologically classified by $\mathfrak{J}(S)$. More precisely we have: (II.1) $\mathfrak{J}(S) = (7, 2, 2, 1)$ with phase portrait W_{14}; (II.2) $\mathfrak{J}(S) = (7, 2, 2, 3)$ with phase portrait W_{17}; (II.3) $\mathfrak{J}(S) = (6, 2, 2, 2)$ with phase portrait W_{16} occurring as a bifurcation from (II.2) to (II.1), when two of the three points at infinity collide. (In Figure 3 the region where we have limit cycles is delimited by the curves $\mathcal{G}$ and $a = 0$: $\lambda \in R_9 \cup \Delta_3 \cup R_8^{dd}$).*

(III) *Systems with a graphic. These have no limit cycle and the graphic is unique, surrounding a strong focus. We have exactly three phase portraits in this class. These are: W_{18} with $\mathfrak{J}(S) = (7, 2, 2, 3)$ and W_{13}, W_{15}, both with $\mathfrak{J}(S) = (6, 2, 2, 2)$. W_{13} is distinguished from W_{15} by m_∞. For W_{13}, $m_\infty = 3$; for W_{15}, $m_\infty = 2$.*

We note that if $\mathfrak{J}(S) = (7, 2, 2, 3)$, S could be of any of the types (I), (II), (III).

9 Perturbations of Systems in the Class QW3

In this section we study the limit cycles which can bifurcate from a system in QW3 when we perturb it inside the class of all quadratic system with the topology of the coefficients.

Proposition 17 *The following statement holds.*

(a) *At most one limit cycle could arise near the graphic, from a quadratic system corresponding to a point on $\mathcal{G}_1$, in any perturbation of the system inside the class of all quadratic systems. Furthermore, if this limit cycle exists, it is hyperbolic.*

In addition there is numerical evidence for the following affirmation:

(b) *From the graphic corresponding to a quadratic system having its parameters on the curve $\mathcal{G}_2 \cup \mathcal{G}_3$, at most one limit cycle could arise near the graphic, in any perturbation of the system inside the class of all quadratic systems. When this limit cycle exists it is hyperbolic.*

Proof We know explicitly the points of the algebraic curve $\mathcal{G}_1$. The graphics of the systems associated to these parameters (see W_{13}) are elementary (in the sense of [15]),

having two singular points, a saddle and a saddle-node, which satisfy the assumptions of Theorem 1 of [15]. Therefore, the cyclicity of such a graphic is one, *i.e.*, by a small perturbation, at most one limit cycle could bifurcate near the graphic. If this limit cycle exists it is hyperbolic (see again [15]). This proves statement (a).

The fact that $\mathcal{G}_3$ is analytic has been proved. But we could only determine its position numerically. The graphic of a system W_{18} associated to a point of the analytic curve $\mathcal{G}_3$, (see W_{18}) is an elementary graphic having two saddles. Since the position of $\mathcal{G}_3$ is only numerically determined, the assumptions in Theorem 1 of [15] are only numerically verified for points on $\mathcal{G}_3$. The graphic of the system associated to the point $\mathcal{G}_2$, (see W_{15}) is elementary having two singular points, a saddle and a saddle-node. The assumptions in Theorem 1 of [15] were numerically verified for the point $\mathcal{G}_2$. So, we have (b). ∎

Theorem 18 *The following statements hold.*

(a) *Inside the class of all quadratic systems there exists a neighborhood U_1 of the systems in* QW3 *which have neither a limit cycle nor a graphic, i.e., whose phase portrait is W_i for some $i = 1,\dots,12$ such that any quadratic system in U_1 has at most three limit cycles.*

(b) *Inside the class of all quadratic systems there exists a neighborhood U_2 of the family of systems in* QW3 *with phase portrait W_{13}, such that any quadratic system in U_2 has at most four limit cycles.*

In addition there is numerical evidence for the following two affirmations:

(c) *Inside the class of all quadratic systems, there exists a neighborhood U_3 of the systems in* QW3 *with phase portraits W_i for $i = 15, 18$ such that any quadratic system in U_3 has at most four limit cycles.*

(d) *Inside the class of all quadratic systems, there exists a neighborhood U_4 of the systems in* QW3 *with phase portraits W_i for $i = 14, 16, 17$, such that any quadratic system in U_4 has at most four limit cycles.*

Proof Consider a system corresponding to W_i for some $i = 1,\dots,10$. This system has a unique focus, the weak focus of third order and it has neither graphics, nor limit cycles. For systems in a sufficiently small neighborhood of the system, their only limit cycles are those produced by the weak focus and in view of Bautin's theorem [6], there could be at most three.

Consider now a system W_i for some $i = 11, 12$. This system has two foci, the weak focus of third order and a strong focus, and it has neither graphics nor limit cycles. For a sufficiently small perturbation within the quadratic family, the only limit cycles which could be obtained are those produced by the weak focus. So, statement (a) is proved.

A system with portrait W_{13} has two foci, the weak focus of third order and a strong focus which is surrounded by a graphic. This system has no limit cycles. By Proposition 17(a), if the perturbation of W_{13} is sufficiently small within the quadratic family, the only limit cycles which could be obtained are those produced by the weak focus

plus the unique one produced by the graphic. So, the perturbed system can have at most four limit cycles. Hence, statement (b) is proved.

The point (c) is obtained by a similar argument used to show (b) but now since the part (b) of Proposition 17(b) is numerical, we only have numerical evidence for the statement (c).

A system with portrait W_i for $i = 14, 16, 17$ has two foci, (the weak focus of third order and a strong focus which is surrounded by a unique limit cycle) and it has no graphic. We have numerical evidence that this limit cycle is hyperbolic. Therefore, if the perturbation of W_i is sufficiently small within the quadratic family, the only limit cycles which could be obtained are those produced by the weak focus, and the one which comes from the hyperbolic limit cycle. So, the perturbed system can have at most four limit cycles. Hence, the statement (d) is obtained. ∎

We remark that the statement (d) would be an analytical result if we knew that the limit cycle of W_i for $i = 14, 16, 17$ is hyperbolic. In fact we only know, due to Proposition 17(a), that the limit cycle of a system with W_{14} is hyperbolic in a neighborhood of the algebraic curve $\mathcal{G}_1$.

Definition As in [15] we call *polycycle* a graphic in the sense of [15] with a Poincaré return map on a semitransversal section.

The next result follows easily from Theorem 18.

Corollary 19 *Inside the class of all quadratic systems there exists a neighborhood of the class of quadratic systems with a weak focus of third order and without any polycycle or limit cycle such that any quadratic system in this neighborhood has at most* 4 *limit cycles.*

10 Concluding Comments

(I) We chose the class QW3 for two reasons: first, because it seems to be next in line to study after the family of systems with a center, systems studied in [50], [39] and [33]. Second, this class plays a significant role in Hilbert's 16-th problem. Indeed, all the quadratic systems which have been proven to have most limit cycles (at least four) are obtained by perturbing systems in QW3. The first known examples of quadratic systems for which it was possible to show that they have at least four limit cycles are the example of Shi Songling [44] and that of Chen and Wang [9]. Both these examples are produced by perturbing systems S in the region of QW3 determined by $\mathcal{J}(S) = (7, 2, 2, 1)$, *i.e.*, in R_9 of Figures 2 and 3. In the early 1980s it became clear from the work [3] that there are two other distinct subsets of QW3 from which two new types of phase portraits with at least four limit cycles could be obtained by perturbations of systems in the class QW3. In this work we see these sets clearly on Figures 2 and 3. These are the systems in R_8^{dd} and on Δ_3. The region where we have limit cycles is bounded by the curve $\mathcal{G}$ and by points on $a = 0$ which correspond to symmetric systems with two centers.

It is interesting to note that part of the boundary of the region in the parameter space where we have limit cycles for QW3 is formed by systems with center and that it

is basically by perturbing systems with a center that the limit cycles are obtained. This shows the close connection existing between the problem of the center and Hilbert's 16-th problem. Among the systems with center located on $a = 0$ in Figures 2 and 3, the more degenerate ones lie, as expected, in the boundary of several regions R_i. Naturally we expect that the study of their perturbations in the quadratic class will be more complex. It is thus not surprising that the program outlined in [15] and pursued in numerous articles has stopped short (so far) of proving the finite cyclicity of graphics present in these more degenerate systems, such as for example the system corresponding to the point $\lambda = [0:1:0]$ in Figure 1.

(II) Zhang's theorem (see point (vii) in Section 7.1) in [52, 53] was very helpful for proving more results here than in the previous studies [3, 2, 5], done before this result was available. We also proved here the analyticity of the saddle-to-saddle connection which was not proven in the above-mentioned works and gave the bifurcation diagram in just one picture representing the projective space, the adequate parameter space for this classification problem. The previous bifurcation diagrams contained certain inaccuracies. Two phase portraits denoted by W_6 and W_9, and counted as distinct in [3], [5] are in fact topologically identical. The topological equivalence of these phase portraits was only observed after we saw that these portraits were put into the same class by the first components of the multi-integer-valued invariant $\mathcal{I}(S)$. Two other phase portraits in the list in [5] turn out to be topologically equivalent: W_{13} and W_{14}. They are in fact distinct C^{∞} phase portraits (we have a node and a focus in one and two foci in the other) but topologically equivalent.

In [2] Andronova studied the class QW3 and constructed a bifurcation diagram for this class. This was done in $\mathbb{R}^3$ by taking sections of the space obtained by making l (denoted by k in [2]) constant for $l = 0$ or $l \neq 0$. However her arguments on pages 122 and 124 of [2] where she deals with the more delicate cases, encountered by us in the part of the half disc which we denoted by $\Delta \cup R_8$, are incomplete. She says[4]:

> *If the separatrices coalesce, then there must exist still another bifurcation curve, at each point of which, a separatrix of a saddle (or of a saddle-node) goes from it to a saddle. If the coalescence of separatrices does not occur, then on the curve $\Delta = 0$ for $|a| > -l/(12\sqrt{5})$ there could exist only the decomposition $36, \ldots$ It is exactly in this last case that we get the bifurcation diagram in Figure 22.*

This paragraph contains an "if ... if" situation and she chooses one of the two cases for the bifurcation diagram. We see numerically that actually it is the other situation which occurs. Due to this, her description of the phase portraits on the bifurcation curve where we only have two points at infinity, (our curve Δ in Figure 1), is correct in only one part of this curve, as she says that the phase portraits on $\Delta = 0$ with the exception of only one point, are all topologically equivalent. Actually on the above mentioned curve we have three phase portraits. The phase portrait we denoted here by W_{13} is missing from Andronova's list. Although she notes the presence of a bifurcation curve of connections in the area which in our Figure 3 is "inside" $\mathcal{L}_2$ but not "inside" Δ, she does not observe that this curve may actually coincide with Δ at

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[4]This is an almost word for word translation of Andronova's russian text by the authors.

some of the points. We see that this happens and so on the curve $\mathcal{G}$ we have three phase portraits instead of just two.

(III) This work shows the importance of numerical work in such studies. It is clear from her pictures that Andronova did not rely on numerical computations for drawing the connection bifurcation and it is in part due to this that her work contains the inaccuracy indicated above. The numerical computations performed in [5] which could be made as accurate as we wish were very helpful is seeing more clearly what actually occurs.

(IV) The invariants we used in this work are very simple and have a clear geometric meaning. In contrast to them, the algebraic invariants and comitants used in the work of K. S. Sibirsky and his school (*cf.* [47]) are much more complicated. Those invariants and comitants are defined using tensors with their multi-index notation and rather artificial-looking polynomial expressions defined in terms of them. Classifications are done using such polynomial expressions, see for example [32]. This method is technically very complex, and the geometrical meaning is mostly missing. The power of the method resides in its computational aspect: no matter how the system may be presented, independent of particular chart and normal form, one can compute the corresponding algebraic invariants and comitants and the computation can be programmed on a computer.

Given the merits of both kinds of invariants it is thus natural to try to merge the geometrical ones with the algebraic ones which are computationally very powerful. Such a merger would yield easily accessible computations of the geometrical invariants of systems for whatever normal form and would lend geometrical meaning to the algebraic invariants.

This was done for the problem of classifying the quadratic differential systems according to the topology of their phase curves in the neighborhood of the infinity [43]. This problem was first studied in [32] using only algebraic invariants. In [43], both geometric (integer-valued ones like here) and algebraic invariants were used and the algebraic invariants were chosen to fit better with the geometric ones. A complete dictionary translating geometric into algebraic invariants is given. An analogous work by Artés, Llibre and Vulpe for the classification of the quadratic differential systems according to the topology of their phase curves in the neighborhood of their finite singularities is almost completed. These works enable us to compute most integer-valued invariants defined in this article for systems given in whatever normal form.

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