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An Algebraic Approach to Weakly Symmetric Finsler Spaces

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Abstract. In this paper, we introduce a new algebraic notion, weakly symmetric Lie algebras, to give an algebraic description of an interesting class of homogeneous Riemann–Finsler spaces, weakly symmetric Finsler spaces. Using this new definition, we are able to give a classification of weakly symmetric Finsler spaces with dimensions 2 and 3. Finally, we show that all the non-Riemannian reversible weakly symmetric Finsler spaces we find are non-Berwaldian and with vanishing S-curvature. This means that reversible non-Berwaldian Finsler spaces with vanishing S-curvature may exist at large. Hence the generalized volume comparison theorems due to Z. Shen are valid for a rather large class of Finsler spaces.

Introduction

The notion of a weakly symmetric Finsler space is the Finslerian analogue of that of a Riemannian weakly symmetric space. A Riemannian weakly symmetric space is, according to Selberg [16], a (connected) Riemannian manifold (M, Q) with the property that there exists a subgroup G of the full group $I(M, Q)$ of isometries such that G acts transitively on M and there exists an isometry f of (M, Q), with $f^2 \in G$ and $fGf^{-1} = G$, such that for any two points $p, q \in M$, there exists $g \in G$ satisfying $g(p) = f(q)$ and $g(q) = f(p)$. This condition is equivalent to the simple geometric characterization that for any pair of points (p, q) there exists an isometry f such that $f(p) = q$ and $f(q) = p$, in short, f interchanges p and q (see [4]). Originally, the study of Riemannian weakly symmetric spaces was mainly focused on harmonic analysis. Much recent work has been done on the geometry of such spaces (see [3,4,15,24]). This research shows that weakly symmetric Riemannian manifolds possess rather interesting geometrical properties. In particular, a classification of Riemannian weakly symmetric spaces with semisimple (resp. reductive) isometric group was obtained (see [22,23]).

However, up to now we have not had an algebraic notion related to this interesting class of homogeneous Riemannian manifolds. The classification of certain special classes, including those with semisimple (resp. reductive) isometric groups, are all obtained through a careful technical analysis of results about other kinds of homogeneous manifolds. For example, Yakimove established the classification of Riemannian weakly symmetric spaces with reductive isometric group (see [23]). It is based on the classification of spherical homogeneous spaces, which is obtained through the efforts of several people (see [14]). Berndt and Vanhecke obtained a classification of

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Riemannian weakly symmetric spaces with dimension less than six (see [4]). This is based on the observation that a Riemannian weakly symmetric space is necessarily a Riemannian g.o. space, (*i.e.,* Riemannian manifolds any of whose geodesics is the orbit of a one-parameter subgroup of the full group of isometries), and the converse is also true, provided that the dimension is less than six. Then the classification is reduced to that of the Riemannian g.o. spaces (see [13]). This kind of approach cannot help us to understand the intrinsic features of weakly symmetric spaces, and most importantly, it is very difficult to get a complete classification of such spaces using similar methods. Therefore, an algebraic approach to this problem is very important.

In this paper, we will study this interesting problem. Contrary to the current settings, which consider only the Riemannian metrics, we study this problem for the more general class of Finsler spaces. The corresponding algebraic object we obtain, which is a pair of (real) Lie algebras together with some appropriate conditions, will be called a weakly symmetric Lie algebra. It turns out that this notion is very useful for studying weakly symmetric Finsler spaces. In particular, every weakly symmetric Finsler space gives rise to a weakly symmetric Lie algebra and conversely, given any weakly symmetric Lie algebra, one can construct a (not unique) weakly symmetric Finsler space. With this intrinsic algebraic interpretation we are able to give a classification of weakly symmetric Finsler spaces of dimensions two and three. As an application of these results, we find many new examples of reversible non-Berwaldian spaces with vanishing S-curvature. The problem whether there exists such a space was posed by Z. Shen (see [17]), and we have pointed out that such spaces do exist by constructing a series of explicit examples (see [9]).

The arrangement of this paper is as follows. In Section 1 we recall some basic definitions and results on Finsler metrics, connections, S-curvature and weakly symmetric Finsler spaces. In Section 2 we introduce the weakly symmetric Lie algebras and prove two main theorems which reduce the study of weakly symmetric Finsler spaces to that of weakly symmetric Lie algebras. In Section 3 we present some examples of weakly symmetric Lie algebras, which will be useful in the classification of lower dimensional weakly symmetric spaces. In Section 4 we obtain a classification of weakly symmetric Finsler spaces of dimensions two and three, which contains a complicated computation using the structure theory of real lie algebras. Finally, in Section 5 we prove that the non-Riemannian and non-symmetric Finsler metrics constructed in Section 4 are all non-Berwaldian and with vanishing S-curvature.

1 Preliminaries

In this section, we recall some definitions and fundamental results needed in this paper.

1.1 Finsler Spaces

Definition 1.1 Let V be an n-dimensional real vector space. A Minkowski norm on V is a functional F on V which is smooth on $V - \{0\}$ and satisfies the following conditions:

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(i) $F(u) \geq 0, \forall u \in V$;
(ii) $F(\lambda u) = \lambda F(u), \forall \lambda > 0$;
(iii) For any basis $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ of V, write $F(y) = F(y^1, y^2, \dots, y^n)$ for $y = y^j \varepsilon_j$. Then the Hessian matrix

$$(g_{ij}) := \left(\left[\frac{1}{2} F^2 \right]_{y^i y^j} \right)$$

is positive-definite at any point of $V - \{0\}$.

It can be shown (see [2]) that for a Minkowski norm F, we have $F(u) > 0, \forall u \neq 0$. Furthermore, $F(u_1 + u_2) \leq F(u_1) + F(u_2)$, where the equality holds if and only if $u_2 = \alpha u_1$ or $u_1 = \alpha u_2$ for some $\alpha \geq 0$.

For any Minkowski norm F on real vector space V we define

$$C_{ijk} = \frac{1}{4}[F^2]_{y^i y^j y^k}.$$

Then for any $y \neq 0$, we can define two tensors on V, namely,

$$g_y(u, v) = g_{ij}(y) u^i v^j \quad \text{and} \quad C_y(u, v, w) = C_{ijk}(y) u^i v^j w^k.$$

They are called the fundamental form and the Cartan torsion, respectively.

Definition 1.2 Let M be a (connected) smooth manifold. A Finsler metric on M is a function $F \colon TM \to [0, \infty)$ such that

(i) F is C^∞ on the slit tangent bundle $TM - \{0\}$;
(ii) the restriction of F to any $T_x M$, $x \in M$ is a Minkowski norm.

Let (M, F) be a Finsler space and $(x^1, x^2, \dots, x^n)$ a local coordinate system on an open subset U of M. Then $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ form a basis for the tangent space at any point in U. For $y \in T_x(M), x \in U$, write $y = y^j \frac{\partial}{\partial x^j}$. Then $(x^1, x^2, \dots, x^n, y^1, y^2, \dots, y^n)$ is a (standard) coordinate system on TU. Using the coefficients g_{ij} and C_{ijk}, we define

$$C^i{}_{jk} = g^{is} C_{sjk},$$

where (g^{ij}) is the inverse matrix of (g_{ij}). The formal Christofell symbols of the second kind are

$$\gamma^i{}_{jk} = g^{is} \frac{1}{2} \left(\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j} \right).$$

They are functions on $TU - \{0\}$. We can also define some other quantities on $TU - \{0\}$ by $N^i{}_j(x, y) := \gamma^i{}_{jk} y^k - C^i{}_{jk} \gamma^k{}_{rs} y^r y^s$, where $y = y^i \frac{\partial}{\partial x^i} \in T_x(M) - \{0\}$.

Now the slit tangent bundle $TM - \{0\}$ is a fibre bundle over the manifold M with the natural projection π. Since TM is a vector bundle over M, we have a pull-back bundle $\pi^* TM$ over $TM - \{0\}$. The pull-back bundle $\pi^* TM$ admits a unique linear connection, called the Chern connection, which is torsion free and almost

g-compatible (see [1]). The coefficients of the connection in the standard coordinate system are

$$\Gamma^l{}_{jk} = \gamma^l{}_{jk} - g^{li}\left(A_{ijs}\frac{N^s{}_k}{F} - A_{jks}\frac{N^s{}_i}{F} + A_{kis}\frac{N^s{}_j}{F}\right).$$

Let V be an n-dimensional real vector space and F a Minkowski norm on V. For a basis $\{b_i\}$ of V, let

$$\sigma_F = \frac{\mathrm{Vol}(B^n)}{\mathrm{Vol}\{(y^i) \in \mathbb{R}^n \mid F(y^i b_i) < 1\}},$$

where Vol means the volume of a subset in the standard Euclidean space $\mathbb{R}^n$ and B^n is the open ball of radius 1. This quantity is generally dependent on the choice of the basis $\{b_i\}$. But it is easily seen that

$$\tau(y) = \ln \frac{\sqrt{\det(g_{ij}(y))}}{\sigma_F}, \quad y \in V - \{0\}$$

is independent of the choice of the basis. So $\tau = \tau(y)$ is called the distortion of (V, F). Now let (M, F) be a Finsler space. Let $\tau(x, y)$ be the distortion of the Minkowski norm F_x on $T_x(M)$. For $y \in T_x(M) - \{0\}$, let $\sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$. Then the quantity

$$S(x, y) = \frac{d}{dt}\left[\tau(\sigma(t), \dot{\sigma}(t))\right]\Big|_{t=0}$$

is called the S-curvature of the Finsler space (M, F). It is a function on the slit tangent bundle $TM - \{0\}$. The notion of S-curvature was introduced by Z. Shen (see [17], see also [5]). It is a useful tool for studying the volume comparison theorems in Finsler geometry. It is proved in [19] that for any Berwald space the S-curvature vanishes. On the other hand, there exists Randers' metric which is non-Berwaldian and whose S-curvature vanishes. Z. Shen raised the open problem whether there exists reversible non-Berwaldian Finsler spaces with vanishing S-curvature. In our previous paper, we gave an affirmative answer to this problem by constructing some examples of such spaces. In this paper, we will find many new examples of such spaces. This fact implies that reversible non-Berwaldian S-isotropic Finsler spaces may exist at large.

1.2 Weakly Symmetric Finsler Spaces

The notion of a weakly symmetric space is a natural generalization of Selberg's definition of a Riemannian weakly symmetric space. Let (M, F) be a Finsler space and $I(M, F)$ be the full group of isometries. Then (M, F) is called weakly symmetric if for any two points p, q in M there exists an isometry $\sigma \in I(M, F)$ such that $\sigma(p) = q$ and $\sigma(q) = p$. In this section, we will recall some geometric criteria for weakly symmetric spaces which will be used in this paper (see [9]).

Proposition 1.3 *A Finsler space (M, F) is a weakly symmetric space if and only if for every maximal geodesic γ in M and any point $m \in \gamma$ there exists an isometry $\sigma \in I(M, F)$ satisfying $\sigma(\gamma) \subset \gamma$, $\sigma(m) = m$, $\sigma|_\gamma \neq \mathrm{id}$, $\sigma^2|_\gamma = \mathrm{id}$, where* id *denotes the identity transformation. In other words, σ is a non-trivial involution along γ fixing m.*

Proposition 1.4 *A Finsler space (M, F) is weakly symmetric if and only if for any $m \in M$ and $X \in T_m(M)$ there exists an isometry σ of (M, F) such that $\sigma(m) = m$ and $d\sigma(X) = -X$.*

The following result is very useful for studying invariant weakly symmetric Finsler metrics on homogeneous manifolds.

Proposition 1.5 *Let G be a Lie group and H a closed subgroup of G. Suppose that the coset space G/H is reducible,* i.e., *there exists a subspace $\mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of the Lie group G such that $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ (direct sum), and $\mathrm{Ad}(h)(\mathfrak{p}) \subset \mathfrak{p}, \forall h \in H$. If for any $X \in \mathfrak{p}$, there exists $h \in H$ such that $\mathrm{Ad}(h)(X) = -X$, then any G-invariant Finsler metric on G/H is weakly symmetric.*

Proof Since G/H is homogeneous, we only need to prove that the condition of Proposition 1.4 is satisfied at the origin $o = eH$. Identifying $Y \in \mathfrak{p}$ with the tangent vector $\frac{d}{dt} \exp(tY) \cdot o|_{t=0}$, we get a linear isomorphism between $\mathfrak{p}$ and the tangent space $T_o(G/H)$. Under this isomorphism the action of H on $T_o(G/H)$ $(h \mapsto dh)$ corresponds to the adjoint action of H on $\mathfrak{p}$. From this the proposition follows. ■

2 Weakly Symmetric Lie Algebras

In this section, we will introduce the definition of a weakly symmetric Lie algebra and use this notion to give an algebraic description of simply connected weakly symmetric Finsler spaces. As usual, we only consider connected manifolds.

Definition 2.1 Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$, and suppose that there exists a subspace $\mathfrak{p}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ (direct sum) and $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$. Then $(\mathfrak{g}, \mathfrak{h})$ is called a weakly symmetric Lie algebra if there exists a finite set of automorphisms of $\mathfrak{g}$, $\{\sigma_0, \sigma_1, \sigma_2, \dots, \sigma_s\}$ $(\sigma_0 = \mathrm{id})$, satisfying the following conditions:

(W1) any σ_i, $0 \leq i \leq s$ keeps the subspaces $\mathfrak{h}$ and $\mathfrak{p}$, i.e., $\sigma_i(\mathfrak{h}) = \mathfrak{h}$, $\sigma_i(\mathfrak{p}) = \mathfrak{p}$;
(W2) for any pair i, j, $0 \leq i, j \leq s$, there exists a k, $0 \leq k \leq s$ and a vector $X_{ij} \in \mathfrak{h}$ such that $\sigma_i \sigma_j = e^{\mathrm{ad}\, X_{ij}} \sigma_k$;
(W3) for any $Y \in \mathfrak{p}$ there exists $X_Y \in \mathfrak{h}$ and an index m_Y such that $e^{\mathrm{ad}\, X_Y} \cdot \sigma_{m_Y}(Y) = -Y$.

We usually say that $(\mathfrak{g}, \mathfrak{h})$ is weakly symmetric with respect to $\{\sigma_0, \sigma_1, \dots, \sigma_s\}$. Moreover, a weakly symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h})$ is called Riemannian if in addition $\mathfrak{h}$ is a compactly embedded subalgebra of $\mathfrak{g}$.

We now give some remarks concerning the above definition.

Remarks (1) The definition of a Riemannian weakly symmetric Lie algebra is a natural generalization of an orthogonal symmetric Lie algebra (see [10, p. 213]).

In fact, if $(\mathfrak{g}, \sigma)$ is an orthogonal symmetric Lie algebra, then the fixed points of the involution σ, denoted by $\mathfrak{h}$, is a compactly embedded subalgebra of $\mathfrak{g}$. Let $\mathfrak{p}$ be the eigenspace of σ with the eigenvalue -1. Then $\mathfrak{p}$ satisfies the condition in the above definition. In fact, we only need to define $\sigma_0 = \mathrm{id}$, $\sigma_1 = \sigma$. Then it is easy to verify that (W1), (W2), and (W3) are satisfied. In the next section, we will construct a series of examples of Riemannian weakly symmetric Lie algebras that are not orthogonal symmetric.

(2) A subalgebra $\mathfrak{h}$ of a real Lie algebra $\mathfrak{g}$ is called compactly embedded if the analytic subgroup H^* of the adjoint group $\mathrm{Int}\,\mathfrak{g}$ of $\mathfrak{g}$ that corresponds to the subalgebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$ of $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$ is a compact Lie group (see [10, p. 130]).

(3) In the definition, we do not require the set of automorphisms $\{\sigma_0, \sigma_1, \dots, \sigma_s\}$ to be unique. However, if $(\mathfrak{g}, \mathfrak{h})$ is a weakly symmetric Lie algebra, then we can choose a reduced set of automorphisms $(\tau_0(=\mathrm{id}), \tau_1, \tau_2, \dots, \tau_l)$ that satisfy (W1), (W2), (W3) and the following additional condition:

(W4) For any $j \neq k$, $\tau_j(\tau_k)^{-1}$ cannot be written in the form e^{adX}, $X \in \mathfrak{h}$.

It is easily seen that each set of automorphisms in the definition can be reduced to a reduced set. In the following, we will usually select a reduced set of automorphisms to study weakly symmetric Lie algebras.

Now we can give the algebraic description of weakly symmetric Finsler spaces.

Theorem 2.2 *Let (M, F) be a weakly symmetric Finsler space. Then there exists a Lie group G and a closed subgroup H of G such that $M = G/H$ and F is G-invariant. Further, the Lie algebra pair $(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{g} = \mathrm{Lie}\,G$, $\mathfrak{h} = \mathrm{Lie}\,H$, is a Riemannian weakly symmetric Lie algebra.*

Proof Fix $x \in M$. By Proposition 1.5, for any $v \in T_x(M)$, there exists an isometry τ of (M, F) such that $\tau(x) = x$ and $d\tau(v) = -v$. Let G be the full group of isometries of (M, F) and H the isotropy subgroup of G at x. Then by [6], G is a Lie transformation group of M and H is a compact subgroup of G. Since a weakly symmetric Finsler spaces is homogeneous, G acts transitively on M. Hence M is diffeomorphic to the coset space G/H and F is G-invariant. Now we prove that the pair $(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{g} = \mathrm{Lie}\,G$, $\mathfrak{h} = \mathrm{Lie}\,H$, is a weakly symmetric Lie algebra. Since H is compact, G/H is a reductive homogeneous manifold. Hence there exists a subspace $\mathfrak{p}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ (direct sum) and $\mathrm{Ad}(h)(\mathfrak{p}) \subset \mathfrak{p}\, \forall h \in H$. In particular, $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$. Now we identify the space $\mathfrak{p}$ with the tangent space $T_x(M)$ by the mapping $Y \mapsto d/dt(\exp tY) \cdot x|_{t=0}$, $Y \in \mathfrak{p}$. Then isotropic action of H on $T_x(M)$ corresponds to the adjoint action of H on $\mathfrak{p}$. Let e be the unit element and H_e be the identity component of H. Then H_e is a normal subgroup of H and the quotient group H/H_e is finite, because H, as a compact Lie group, has at most finitely many number of components. Now let $\{e, h_1, \dots, h_s\}$ be a set of H such that $\{eH, h_1H, \dots, h_sH\}$ are all the (distinct) elements of the quotient group. Let σ_0 be the identity transformation of $\mathfrak{g}$ and $\sigma_j = \mathrm{Ad}(h_j)$, $j = 1, 2, \dots, s$. Then the set $\{\sigma_0, \sigma_1, \dots, \sigma_s\}$ satisfies the conditions (W1), (W2), (W3). In fact, (W1) is obviously satisfied. For any pair i, j, suppose that in the quotient group H/H_e we have $h_iH_e \cdot h_jH_e = h_kH_e$, then there exists $m_1, m_2, m_3 \in H_e$, such that $h_im_1h_jm_2 = h_km_3$, *i.e.*, $h_ih_j = h_k(m_3m_2^{-1}(h_jm_1^{-1}h_j^{-1}))$. Since $h_jm_1^{-1}h_j^{-1} \in$

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H_e, we have $m = m_3 m_2^{-1}(h_j m_1^{-1} h_j^{-1}) \in H_e$. Since H_e is a connected compact Lie group, the exponential mapping is surjective. Hence there exists $X_{ij} \in \mathfrak{h}$ such that $\exp(X_{ij}) = m$. Then $\mathrm{Ad}(m) = e^{\mathrm{ad}\, X_{ij}}$. Therefore,

$$\sigma_i \sigma_j = \sigma_k e^{\mathrm{ad}\, X_{ij}} = \sigma_k e^{\mathrm{ad}\, X_{ij}} \sigma_k^{-1} \cdot \sigma_k = e^{\mathrm{ad}(\sigma_k(X_{ij}))} \sigma_k,$$

i.e., (W2) is satisfied. Now we prove (W3). By Proposition 1.4, for any $Y \in \mathfrak{p}$ we can select $h \in H$ such that $\mathrm{Ad}(h)(Y) = -Y$. Suppose h lies in the component $h_i H_e$. Then there exists $h_0 \in H_e$ such that $h = h_i h_0 = h_i h_0 h_i^{-1} h_i$. Since $h_i h_0 h_i^{-1} \in H_e$, we can write $h = \exp(X_y) h_i$ for some $X_y \in \mathfrak{h}$. From this we easily see that (W3) is satisfied. This completes the proof of the theorem. ■

Next we will show that any Riemannian weakly symmetric Lie algebra can give rise to a weakly symmetric Finsler space, although in general the spaces constructed from a weakly symmetric Lie algebra are not unique.

Theorem 2.3 *Let $(\mathfrak{g}, \mathfrak{h})$ be a Riemannian weakly symmetric Lie algebra. Suppose that G is a connected simply connected Lie group with Lie algebra $\mathfrak{g}$ and H is the (unique) connected Lie subgroup of G with Lie algebra $\mathfrak{h}$. If H is closed in G (this is the case if $C(\mathfrak{g}) = 0$), then there exists a G-invariant Riemannian metric Q on the coset space G/H such that $(G/H, Q)$ is a Riemannian weakly symmetric space. Furthermore, if there exists a non-trivial subspace of $\mathfrak{g}/\mathfrak{h}$ which is invariant under the actions of $\mathfrak{h}$ and a set of reduced automorphisms of the weakly symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h})$ (see Remark (3) after Definition 2.1), then there exists a non-Riemannian G-invariant Finsler metric F on G/H such that $(G/H, F)$ is a weakly symmetric Finsler space.*

Proof Since $\mathfrak{h}$ is a compactly embedded subalgebra of $\mathfrak{g}$, the Lie algebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$ is compact. If we identify the tangent space $T_o(G/H)$ with $\mathfrak{p}$, where $\mathfrak{p}$ is as in Definition 2.1 and $o = eH$ is the origin of the coset space G/H, then $\mathrm{Ad}(H)$ is a compact group of linear transformations of $\mathfrak{p}$. Hence there exists an $\mathrm{Ad}(H)$-invariant inner product $\langle\ ,\ \rangle_1$ on $\mathfrak{p}$. Fix a reduced set of automorphisms $\{\tau_0, \tau_1, \dots, \tau_l\}$ ($\tau_0 = \mathrm{id}$) as in Remark (3), and define an inner product $\langle\ ,\ \rangle$ on $\mathfrak{p}$ as follows:

$$\langle X, Y \rangle = \sum_{j=0}^{l} \langle \tau_j X, \tau_j Y \rangle_1, \quad X, Y \in \mathfrak{p}.$$

By condition (W2) we see that $\langle\ ,\ \rangle$ is invariant under the action of each τ_i, $i = 1, 2, \dots, l$. We assert that $\langle\ ,\ \rangle$ is also $\mathrm{Ad}(H)$-invariant. In fact, for any $h \in H$ and any index j, we have $\mathrm{Ad}\, h \cdot \tau_j = \tau_j \cdot (\tau_j^{-1} \cdot \mathrm{Ad}\, h \cdot \tau_j)$. Since H is connected, it is generated by the elements $\exp(X), X \in \mathfrak{h}$. Hence h can be written as $\exp X_1 \exp X_2 \cdots \exp X_k$, where $X_i \in \mathfrak{h}, i = 1, 2, \dots, k$. Then we have

$$\tau_j^{-1} \cdot \mathrm{Ad}\, h \cdot \tau_j = e^{\mathrm{ad}\, \tau_j X_1} e^{\mathrm{ad}\, \tau_j X_2} \cdots e^{\mathrm{ad}\, \tau_j X_k}.$$

From this our assertion follows. Now using $\langle\ ,\ \rangle$ we can construct a G-invariant Riemannian metric Q on the coset space G/H (here the condition that H is closed is required, otherwise it may happen that there is no differentiable structure on G/H) whose restriction to $T_o(G/H) = \mathfrak{p}$ is equal to $\langle\, ,\rangle$ (see [12]). We assert that the homogeneous Riemannian manifold constructed above is weakly symmetric. Note that G is connected and simply connected, each automorphism τ_j of $\mathfrak{g}$ can be lifted to an automorphism of G (see [10]), denoted by $\tilde{\tau}_j$, $j = 0, 1, 2, \ldots$. Since $\tau_j(\mathfrak{h}) = \mathfrak{h}$, we easily see that $\tilde{\tau}_j(H) \subset H$. Hence $\tilde{\tau}_j$ induces a diffeomorphism of G/H, which we denote by $\hat{\tau}_j$, by sending gH to $\tilde{\tau}_j(g)H$. The diffeomorphism $\hat{\tau}_j$ keeps the origin $o = eH$ invariant, and its differential at o is just the restriction of τ_j to $\mathfrak{p}$. From this we see that $\hat{\tau}_j$ keeps the Riemannian metric Q invariant, or in other words, $\hat{\tau}_j$ lies in the isotropic subgroup (at o) of the full group of isometries of $(G/H, Q)$. By (W3) for any $Y \in \mathfrak{p} = T_o(G/H)$, we can choose $X_Y \in \mathfrak{h}$ and index i_Y such that $e^{\operatorname{ad} X_Y}\tau_{i_Y}(Y) = -Y$. This means that the isometry $\tau_{\exp(X_Y)} \cdot \hat{\tau}_{i_Y}$ of the Riemannian manifold $(G/H, Q)$, where $\tau_h(gH) = hgH$, $h \in H$, reverse the tangent vector Y. Since $(G/H, Q)$ is homogeneous for any $m \in G/H$ and $Y \in T_m(G/H)$, there exists an isometry f such that $f(m) = m$ and $df|_m(Y) = -Y$. Thus by Proposition 1.4 $(G/H, Q)$ is weakly symmetric.

Now we prove the second assertion. The conditions imply that there exists a non-trivial subspace $\mathfrak{p}'$ of $\mathfrak{p}$ which is invariant under $\operatorname{ad}_{\mathfrak{g}}(\mathfrak{h})$ and τ_j, $j = 0, 1, 2, \ldots$. By the above argument, we see that there exists an inner product $(\ ,\ )_0$ on $\mathfrak{p}$ such that $\operatorname{Ad}(h)$ and τ_j are all orthogonal with respect to it. Then the orthogonal complement of $\mathfrak{p}'$ with respect to $(\ ,\ )_0$, $(\mathfrak{p}')^{\perp}$, is also invariant under the action of $\operatorname{Ad}(h)$ and τ_j. Hence $\mathfrak{p}$ has a decomposition

$$\mathfrak{p} = \mathfrak{p}_0 + \mathfrak{p}_1 + \cdots + \mathfrak{p}_m \quad \text{(direct sum)}, \tag{2.1}$$

where $\mathfrak{p}_0$ is the set of fixed points of $\operatorname{Ad}(H)$ and $\mathfrak{p}_j$, $j = 1, 2, \ldots, m$ are irreducible invariant subspaces of $\operatorname{Ad}(H)$ and τ_j, $j = 0, 1, 2, \ldots, n$. Without loss of generality, we can assume that $m \geq 1$. Now we define a functional F_1 on $\mathfrak{p}$ by

$$F_1(X) = \sqrt{\sum_{j=0}^{m}(X_j, X_j)_0 + \sqrt{\sum_{j=0}^{m}(X_j, X_j)_0^2}},$$

where $X = X_0 + X_1 + \cdots + X_m$ is the decomposition of X corresponding to (2.1). It is easy to check that F_1 is a Minkowski norm on $\mathfrak{p}$ which is obviously non-Euclidean. Using F_1 we can define a G-invariant Finsler metric on G/H (see [7]) and as above we can prove that it is a weakly symmetric Finsler space.

Finally, if $C(\mathfrak{g}) = 0$, then H is closed in G; see [10, pp. 213–214]. ■

3 Examples of Weakly Symmetric Lie Algebras

In this section, we present some examples of weakly symmetric Lie algebras. Some of these will be used in the classification of three-dimensional weakly symmetric Finsler spaces in the next section.

Example 1 Let $\mathfrak{h}$ be the one-dimensional real Lie algebra,

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{su}(2) \quad \text{(direct sum)}.$$

Define the Lie brackets as follows. The Lie brackets between the elements of $\mathfrak{su}(2)$ are defined as usual. Let X be a non-zero element in $\mathfrak{h}$ and

$$\varepsilon_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon_2 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad \varepsilon_3 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}.$$

Then we define

$$[X, \varepsilon_1] = 0, \quad [X, \varepsilon_2] = -\varepsilon_3, \quad [X, \varepsilon_3] = \varepsilon_2.$$

These brackets can be extended linearly to a skew symmetric binary operation on $\mathfrak{g}$. It is easy to check that the Jacobi identities hold for this operation. Hence $\mathfrak{g}$ is a Lie algebra. Obviously $[\mathfrak{h}, \mathfrak{su}(2)] \subset \mathfrak{su}(2)$. Now we define an endomorphism τ on $\mathfrak{g}$ by

$$\tau(X) = -X, \quad \tau(\varepsilon_1) = -\varepsilon_1, \quad \tau(\varepsilon_2) = -\varepsilon_2, \quad \tau(\varepsilon_3) = \varepsilon_3.$$

Then it is easy to check that τ keeps the Lie brackets invariant, hence is a Lie algebra automorphism, and $\tau^2 = \mathrm{id}$. Now we prove that $(\mathfrak{g}, \mathfrak{h})$ is a weakly symmetric Lie algebra with respect to $S = \{\mathrm{id}, \tau\}$. Since (W1) and (W2) are obviously satisfied, we only need to check (W3). Note that the action of $e^{\mathrm{ad}(tX)}$ on $\mathfrak{p} = \mathfrak{su}(2)$ keeps the subspaces $V_1 = \mathrm{span}(\varepsilon_1)$ and $V_2 = \mathrm{span}(\varepsilon_2, \varepsilon_3)$ invariant and on V_1 it is equal to the identity transformation. On V_2 it has the matrix

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

with respect to the basis $\varepsilon_2, \varepsilon_3$. Thus $e^{\mathrm{ad}(tX)}$ is a rotation of t angle if we define an inner product on V_2 such that ε_2, ε_3 form an orthonormal basis. Now given an element $\varepsilon = a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_3$ in $\mathfrak{p}$, we have $\tau(\epsilon) = -a\epsilon_1 - b\epsilon_2 + c\epsilon_3$. Since $-b\varepsilon_2 + c\varepsilon_3$ is an element in V_2 with the same length as $-b\varepsilon_2 - c\varepsilon_3$, there exists appropriate $t_0 \in \mathbb{R}$ such that

$$e^{\mathrm{ad}(t_0X)}(-b\varepsilon_2 + c\varepsilon_3) = -b\varepsilon_2 - c\varepsilon_3.$$

Then

$$e^{(t_0X)}\tau(\varepsilon) = -a\varepsilon_1 - b\varepsilon_2 - c\varepsilon_3 = -\varepsilon.$$

Therefore $(\mathfrak{g}, \mathfrak{h})$ is a weakly symmetric Lie algebra. Note that the action of $\mathrm{ad}\, X$ on $\mathfrak{p}$ has skew symmetric matrix with respect to the basis $\varepsilon_1, \varepsilon_2, \varepsilon_3$. Thus $\mathfrak{h}$ is a compactly embedded subalgebra of $\mathfrak{h}$. Hence $(\mathfrak{g}, \mathfrak{h})$ is a Riemannian weakly symmetric Lie algebra.

Example 2 This example is similar to Example 1. Here we let

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{sl}(2, \mathbb{R}) \quad \text{(direct sum)}.$$

The Lie brackets are defined similarly. In $\mathfrak{sl}(2, \mathbb{R})$ we use the usual Lie operations. Let

$$\varepsilon_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \varepsilon_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then define

$$[X, \varepsilon_1] = 0, \quad [X, \varepsilon_2] = -\varepsilon_3, \quad [X, \varepsilon_3] = \varepsilon_2.$$

Define an endomorphism τ on $\mathfrak{g}$ by

$$\tau(X) = -X, \quad \tau(\varepsilon_1) = -\varepsilon_1, \quad \tau(\varepsilon_2) = -\varepsilon_2, \quad \tau(\varepsilon_3) = \varepsilon_3.$$

Then it can be checked directly that τ is an automorphism of $\mathfrak{g}$. Similarly as in Example 1, we can prove that $(\mathfrak{g}, \mathfrak{h})$ is a Riemannian weakly symmetric Lie algebra with respect to $\{\mathrm{id}, \tau\}$.

Example 3 In this example we consider Heisenberg Lie algebras. Let $\mathfrak{n}$ be a $(2n+1)$-dimensional real Lie algebra with a basis $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, z$ and the brackets

$$[x_i, y_j] = \delta_{ij} z, \quad [x_i, x_j] = [y_i, y_j] = 0, [x_i, z] = [y_i, z] = 0, \quad i, j = 1, 2, \ldots, n.$$

Then $\mathfrak{n}$ is a 2-step nilpotent Lie algebra. Let $\mathfrak{g} = \mathfrak{u}(n) + \mathfrak{n}$ (direct sum of subspaces) and define the brackets as follows. The brackets among the elements in $\mathfrak{u}(n)$ are the usual operations. For $A \in \mathfrak{u}(n)$ we define $[A, z] = 0$ and for the element

$$w = \sum_{i=1}^{n} (a_i x_i + b_i y_i), \quad a_i, b_i \in \mathbb{R}$$

we denote

$$z_i = a_i + \sqrt{-1} b_i, i = 1, 2, \ldots, n. \tag{3.1}$$

Let $(z_1', z_2', \ldots, z_n') = (z_1, z_2, \ldots, z_n)A$ and write $z_i' = a_i' + \sqrt{-1} b_i', a_i', b_i' \in \mathbb{R}$. Then we define

$$[A, w] = w' = \sum_{i=1}^{n} (a_i' x_i + b_i' y_i).$$

It is easy to check that the Jacobi identities hold among these brackets. Therefore these brackets together with brackets of $\mathfrak{n}$ define a Lie algebra structure on $\mathfrak{g}$ and we have $[\mathfrak{u}(n), \mathfrak{n}] \subset \mathfrak{n}$. Now we define an endomorphism τ of $\mathfrak{g}$ by

$$\tau(A) = \bar{A}, \quad \tau(x_i) = -x_i, \quad \tau(y_i) = y_i, \quad \tau(z) = -z,$$

where $A \in \mathfrak{u}(n)$ and $\bar{A}$ is the complex conjugate matrix of A. It is easy to check that τ is an (real) automorphism of the real Lie algebra $\mathfrak{g}$ and $\tau^2 = \mathrm{id}$. Now we prove that $(\mathfrak{g}, \mathfrak{u}(n))$ is a weakly symmetric Lie algebra with respect to $\{\mathrm{id}, \tau\}$. Since (W1) and (W2) are obviously satisfied, we only need to check (W3). Note that the action of $\mathfrak{u}(n)$ in the subspace $V_2 = \mathrm{span}(x_1, y_1, \dots, x_n, y_n)$ is just the usual operation of $\mathfrak{u}(n)$ on the complex linear space $\mathbb{C}^n$ if we write the corresponding coordinates as complex numbers as in (3.1). Hence the action of $e^{\mathrm{ad}(tA)}, A \in \mathfrak{u}(n)$ on V_2 is the operation $w \mapsto w \times \exp(tA)$ (matrices multiplication), *i.e.,* it is just the usual action of the unitary Lie group $U(n)$ on $\mathbb{C}^n$. Now for any $w + cz$, $w \in V_2$, $c \in \mathbb{R}$, we have

$$\tau(w + cz) = \tau(w) - cz,$$

where $\tau(w)$ is in V_2 with the same length as w when we identify V_2 with $\mathbb{C}^n$ (with the usual Hermitian metric). Since the group $U(n)$ acts as the identity transformation on span (z) and acts transitively on the unit sphere in $\mathbb{C}^n$, there exists A_0 in $\mathfrak{u}(n)$ such that

$$e^{\mathrm{ad}(A_0)}(\tau(w)) = -w, \quad e^{\mathrm{ad}(A_0)}(z) = z.$$

Thus $e^{\mathrm{ad}(A_0)} \cdot \tau(w + cz) = -(w + cz)$. This proves our assertion.

For $n = 1$, $\dim \mathfrak{n} = 3$. The Lie algebra $\mathfrak{u}(1)$ is one-dimensional. Let x, y, z be the basis of $\mathfrak{n}$ such that $[x, y] = z$, $[x, z] = [y, z] = 0$. Then we can choose an element X of $\mathfrak{u}(1)$ such that $[X, z] = 0$, $[X, x] = -y$, $[X, y] = x$. The automorphism τ satisfies $\tau(X) = -X$, $\tau(x) = -x$, $\tau(y) = y$. This weakly symmetric Lie algebra will appear in our classification of three-dimensional weakly symmetric Finsler spaces in the next section.

4 Classification of Weakly Symmetric Finsler Spaces of Low Dimensions

As an application of the notion of a weakly symmetric Lie algebra, we get a classification of weakly symmetric Finsler spaces of dimension ≤ 3.

We begin with some general observations. Suppose that (M, F) is a weakly symmetric Finsler space. Let $\tilde{G} = I(M, F)$ be the full group of isometries and $\tilde{H}$ be the isotropic subgroup at certain point $x \in M$. Then we can write M as $\tilde{G}/\tilde{H}$ and F can be viewed as a $\tilde{G}$-invariant metric on $\tilde{G}/\tilde{H}$.

Lemma 4.1 *The isotropic representation ρ of $\tilde{H}$ on $T_x(M)$ is faithful,* i.e., *the mapping $\rho\colon h \mapsto dh|_x$, $h \in \tilde{H}$ is one-to-one.*

The proof is similar to the Riemannian case. Just note that an isometry sends a geodesic to a geodesic and that M is homogeneous, hence complete. See [10].

Since M is connected, the identity component G of $\tilde{G}$ acts transitively on M. The identity component H of $\tilde{H}$ is obviously equal to the isotropic subgroup of G at x. Hence $M = G/H$ and F can also be viewed as a G-invariant Finsler metric on G/H.

The two-dimensional case is settled by the following.

Theorem 4.2 *Let (M, F) be a two-dimensional connected simply connected weakly symmetric Finsler space. Then one of the following holds:*

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(i) (M, F) *is a reversible Minkowski space;*

(ii) F *is Riemannian and* (M, F) *is isometric to a globally symmetric Riemannian space of rank* 1.

In each case, (M, F) *must be a globally symmetric Finsler space.*

Proof Let G, H, ρ be as above. Then by Lemma 4.1 we have $\dim H = \dim \rho(H)$. Since $\rho(H)$ is a compact group of linear transformations of $T_x(M)$, $\dim \rho(H) \leq 1$. Thus we have only two cases.

Case 1: $\rho(H)$ is 0-dimensional. Then $H = \{e\}$. Hence M is itself equal to the Lie group G. Then by Theorem 2.2 the Lie algebra $\mathfrak{g}$ admits finitely many of automorphisms $\tau_0(= \mathrm{id}), \tau_1, \tau_2, \dots, \tau_s$ such that for any $Y \in \mathfrak{g}$ there exists an index j_Y such that $\tau_{j_Y}(Y) = -Y$. Let $V_j = \{Y \in \mathfrak{g} \mid \tau_j(Y) = -Y\}$. Then V_j are subspaces of $\mathfrak{g}$ and we have $\mathfrak{g} = \bigcup V_j$. Therefore there must be a j_0 such that $V_{j_0} = \mathfrak{g}$. Thus for any $X, Y \in \mathfrak{g}$, we have

$$-[X, Y] = \tau_{j_0}([X, Y]) = [\tau_{j_0}(X), \tau_{j_0}(Y)] = [-X, -Y] = [X, Y].$$

This means that $\mathfrak{g}$ is abelian. Hence G is a two-dimensional connected simply connected commutative Lie group, *i.e.,* $G = \mathbb{R}^2$ (here $\mathbb{R}^2$ is viewed as an additive group.) Since F is invariant under G, we see that F is indeed defined by a Minkowski norm in the canonical way. Hence in this case (M, F) is a reversible Minkowski space.

Case 2: $\rho(H)$ is one-dimensional. Since $\rho(H)$ is connected and compact, $\rho(H)$ is isomorphic to S^1. Hence $\rho(H)$ acts transitively on the indicatrix

$$\mathfrak{I}_x = \{X \in T_x(M) | \, F(X) = 1\}$$

of F at $T_x(M)$. On the other hand, by the compactness of $\rho(H)$ there exists a $\rho(H)$-invariant inner product on $T_x(M)$. So $\rho(H)$ also acts transitively on the unit circle of $T_x(M)$ with respect to this inner product. Thus $F|_{T_x(M)}$ is a Euclidean norm. Since (M, F) is homogeneous, F is Riemannian. Further, if p_1, p_2 are two points in M such that $d(x, p_1) = d(x, p_2)$, then we can select two unit vectors X_1, X_2 in $T_x(M)$ such that

$$p_1 = \mathrm{Exp}(tX_1), \quad p_2 = \mathrm{Exp}(tX_2),$$

where $t = d(x, p_1)$. Since $\rho(H)$ acts transitively on the unit circle of $T_x(M)$, we can choose $h \in H$ such that $\rho(h)(X_1) = X_2$. Hence $h(p_1) = p_2$. This argument and the fact that (M, F) is homogeneous imply that (M, F) is actually a two-point homogeneous Riemannian manifold, *i.e.,* for any two pairs of points (q_1, q_2) and (q_1', q_2') in M satisfying $d(q_1, q_2) = d(q_1', q_2')$ there exists an isometry f such that $f(q_1) = q_1'$, $f(q_2) = q_2'$. By the classification results of H. C. Wang [21] and J. Tits [20] on two-point homogeneous Riemannian manifolds, if (M, F) is not the Euclidean space, then it must be a globally symmetric Riemannian manifold of rank 1. This proves the theorem. ■

Next we consider the three-dimensional case. The situation here is much more complicated. Let (M, F) be a three-dimensional connected simply connected weakly symmetric Finsler space and let $\tilde{G}$, $\tilde{H}$, G, H, ρ be as above. Then $\rho(H)$ is a connected compact Lie subgroup of $\mathrm{GL}(T_x(M))$. Hence it is a connected compact Lie subgroup of $\mathrm{SL}(T_x(M))$. By the conjugacy of maximal compact subgroup of semisimple Lie groups [10], there exists an element $g \in \mathrm{SL}(T_x(M))$ such that $g\rho(H)g^{-1} \subset \mathrm{SO}(T_x(M)))$, where we have fixed an inner product in $T_x(M)$ and $\mathrm{SO}(T_x(M))$ is defined as usual. Without loss of generality, we can assume that $\rho(H) \subset \mathrm{SO}(T_x(M))$. Hence there are only two cases,

Case 1: $\rho(H) = \mathrm{SO}(T_x(M))$. In this case, H acts transitively on the indicatrix of F at $T_x(M)$. Similarly as in the two-dimensional case, we can prove that F is Riemannian and (M, F) is isometric to a two-point homogeneous Riemannian manifold.

Case 2: $\rho(H) \neq \mathrm{SO}(T_x(M))$. According to a result of Montgomery and Samelson (see [21]), O_N has no proper subgroup of dimension greater than $\frac{1}{2}(N-1)(N-2)$, where $N > 2$. In particular, $\mathrm{SO}(T_x(M))$ has no proper subgroup of dimension greater than $\frac{1}{2} \times 2 \times 1 = 1$. Therefore we have either $\dim \rho(H) = 0$ or $\dim \rho(H) = 1$. Next we tackle these two cases.

The case for $\dim \rho(H) = 0$ is easy. In fact, we can proceed in exactly the same way as in Theorem 4.2 to prove that in this case (M, F) is just a reversible Minkowski space.

So we are left with the case for $\dim \rho(H) = 1$. In this case, we need some complicated reasoning and computation. Since ρ is a faithful representation, $\dim H = 1$. So in the weakly symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h})$ we have $\dim \mathfrak{h} = 1$. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ be the corresponding decomposition. As pointed out in the proof of Theorem 2.3, there exists an inner product in $\mathfrak{p}$ which is invariant under the actions of $\rho(H)$ and τ_j, $j = 0, 1, 2, \dots, s$, where $T = \{\tau_0, \tau_1, \tau_2, \dots, \tau_s\}$ is a reduced set of automorphisms in Definition 2.1. Then for any $Z \in \mathfrak{h}$, $\operatorname{ad} Z$ is skew-symmetric with respect to this inner product. Fix $X \neq 0$, $X \in \mathfrak{h}$. Then $\operatorname{ad} X|_{\mathfrak{p}}$ must have 0 as an eigenvalue and the corresponding eigenspace V_0 is either one-dimensional or three-dimensional. If $\dim V_0 = 3$, then $\mathfrak{g}$ is the direct sum of the ideals $\mathfrak{h}$ and $\mathfrak{p}$. Since $e^{\operatorname{ad} tX}$ acts as identity transformation on $\mathfrak{p}$ for any $t \in \mathbb{R}$, there must be a τ_i, $1 \leq i \leq s$ such that $\tau_i(Y) = -Y$ for any $Y \in \mathfrak{p}$. This means that for any $Y_1, Y_2 \in \mathfrak{p}$, we have

$$-[Y_1, Y_2] = \tau_i([Y_1, Y_2]) = [\tau_i(Y_1), \tau_i(Y_2)] = [-Y_1, -Y_2] = [Y_1, Y_2].$$

Hence $\mathfrak{p}$ is an abelian ideal of $\mathfrak{g}$. If $\dim V_0 = 1$, then it is easily seen that there exists a basis $\varepsilon_1, \varepsilon_2, \varepsilon_3$ of $\mathfrak{p}$ such that $\operatorname{ad} X$ has the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

with respect to this basis.

Lemma 4.3 *The reduced set T can be selected to consist of only two automorphisms τ_0 $(= \mathrm{id})$, τ_1. Moreover, $\tau_1(\varepsilon_1) = -\varepsilon_1$.*

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Proof First, there exists an index j_0 and $t \in \mathbb{R}$ such that

$$(*) \qquad e^{t\,\mathrm{ad}\,X}\tau_{j_0}(\varepsilon_1) = -\varepsilon_1.$$

Then we assert that $\tau_{j_0}(\varepsilon_1) = -\varepsilon_1$. In fact, if $\tau_{j_0}(\varepsilon_1) = a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_3$, where $\varepsilon = b\varepsilon_2 + c\varepsilon_3$ is a non-zero vector in $V = \mathrm{span}(\varepsilon_2, \varepsilon_3)$, then $\varepsilon' = e^{t\,\mathrm{ad}\,X}(\varepsilon)$ is also a non-zero vector in V. But $-\varepsilon_1 = e^{t\,\mathrm{ad}\,X}\tau_{j_0}(\varepsilon_1) = a\varepsilon_1 + \varepsilon'$. This contradicts the assumption that $\varepsilon_1, \varepsilon_2, \varepsilon_3$ is a base of $\mathfrak{p}$. Hence $\tau_{j_0}(\varepsilon_1) = a\varepsilon_1$. Substituting this into $(*)$ yields $a = -1$. This proves the lemma. ■

Now we will proceed to give a classification of the weakly symmetric Lie algebras corresponding to three-dimensional weakly symmetric Finsler spaces in the case of $\dim H = 1$ and $\dim V_0 = 1$.

The Lie algebra $\mathfrak{g}$ has a basis $X, \varepsilon_1, \varepsilon_2, \varepsilon_3$ where X spans $\mathfrak{h}$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3$ span $\mathfrak{p}$. Moreover, we know the following brackets:

$$[X, \varepsilon_1] = 0, \quad [X, \varepsilon_2] = -\varepsilon_3, \quad [X, \varepsilon_3] = \varepsilon_2.$$

Also, the automorphism τ_1 keeps the subspaces $\mathfrak{h}$ and $\mathfrak{p}$. So $\tau_1(X) = dX$, $d \in \mathbb{R}$. Since τ_1 keeps the inner product invariant and $\tau_1(\varepsilon_1) = -\varepsilon_1$, τ_1 must keep the space $V = \mathrm{span}(\varepsilon_2, \varepsilon_3)$ invariant and $\tau_1|_V$ is an orthogonal linear transformation with respect to the inner product. Hence $\tau_1|_V$ has the matrix

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -\cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in \mathbb{R}$$

with respect to the base $\varepsilon_2, \varepsilon_3$.

To determine the structure of the Lie algebra $\mathfrak{g}$, we only need to determine the Lie brackets $[\varepsilon_i, \varepsilon_j]$, where $i = 1, 2, 3$. Taking into account the skew-symmetry of the Lie bracket, we are to determine $[\varepsilon_1, \varepsilon_2]$, $[\varepsilon_1, \varepsilon_3]$, $[\varepsilon_1, \varepsilon_3]$.

By the Jacobi identity, we have

$$[X, [\varepsilon_1, \varepsilon_2]] = [[X, \varepsilon_1], \varepsilon_2] + [\varepsilon_1, [X, \varepsilon_2]].$$

Thus

$$(4.1) \qquad [X, [\varepsilon_1, \varepsilon_2]] = -[\varepsilon_1, \varepsilon_3].$$

Similarly, we have

$$(4.2) \qquad [X, [\varepsilon_1, \varepsilon_3]] = [\varepsilon_1, \varepsilon_2].$$

Since $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{p}$, we have $[\varepsilon_1, \varepsilon_2] \in \mathfrak{p}$ and $[\varepsilon_1, \varepsilon_3] \in \mathfrak{p}$. Suppose

$$[\varepsilon_1, \varepsilon_2] = a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_3.$$

Then by (4.1) we have $[\varepsilon_1, \varepsilon_3] = -c\varepsilon_2 + b\varepsilon_3$. Substituting this into (4.2) yields

$$[\varepsilon_1, \varepsilon_2] = b\varepsilon_2 + c\varepsilon_3.$$

Therefore $a = 0$. Now from

$$[X, [\varepsilon_2, \varepsilon_3]] = [[X, \varepsilon_2], \varepsilon_3] + [\varepsilon_2, [X, \varepsilon_3]] = [-\varepsilon_3, \varepsilon_3] + [\varepsilon_2, \varepsilon_2] = 0,$$

we easily see that $[\varepsilon_2, \varepsilon_3] \in \operatorname{span}(X, \varepsilon_1)$. Therefore we can write

$$[\varepsilon_2, \varepsilon_3] = a_1 X + a_2 \varepsilon, \quad a_1, a_2 \in \mathbb{R}.$$

First we consider the case that $\tau_1|_V$ has the matrix

$$\begin{pmatrix} -\cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in \mathbb{R}$$

with respect to the basis $\varepsilon_2, \varepsilon_3$. Since τ_1 is an automorphism, we have

$$-\tau_1(\varepsilon_3) = \tau_1([X, \varepsilon_2]) = [\tau_1(X), \tau_1(\varepsilon_2)] = d[X, \tau_1(\varepsilon_2)].$$

Thus $(\sin\theta)\varepsilon_2 - (\cos\theta)\varepsilon_3 = -d(\sin\theta)\varepsilon_2 + d(\cos\theta)\varepsilon_3$. This implies that

$$d\cos\theta = -\cos\theta, \quad d\sin\theta = -\sin\theta.$$

Thus $d = -1$.

Next applying τ_1 to both sides of $[\varepsilon_1, \varepsilon_2] = b\varepsilon_2 + c\varepsilon_3$, we get

$$(b\cos\theta - c\sin\theta)\varepsilon_2 + (c\cos\theta + b\sin\theta)\varepsilon_3 = (-b\cos\theta - c\sin\theta)\varepsilon_2 + (c\cos\theta - b\sin\theta)\varepsilon_3.$$

Therefore we have $b\cos\theta = 0$, and $b\sin\theta = 0$. Thus $b = 0$.

What we have obtained about the Lie brackets of the Lie algebra $\mathfrak{g}$ can be summarized as

$$[X, \varepsilon_1] = 0, \quad [X, \varepsilon_2] = -\varepsilon_3, \quad [X, \varepsilon_3] = \varepsilon_2,$$
$$[\varepsilon_1, \varepsilon_2] = c\varepsilon_3, \quad [\varepsilon_1, \varepsilon_3] = -c\varepsilon_2, \quad [\varepsilon_2, \varepsilon_3] = a_1 X + a_2 \varepsilon_1,$$

where c, a_1, a_2 are real numbers. If $a_1 \neq 0$, we set

$$\varepsilon_1' = X + \frac{a_2}{a_1}\varepsilon_1 \quad \text{and} \quad \mathfrak{p}' = \operatorname{span}(\varepsilon_1', \varepsilon_2, \varepsilon_3).$$

Then we have $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}'$, $[\mathfrak{h}, \mathfrak{p}'] \subset \mathfrak{p}'$. Also the above brackets are still valid and the action of τ_1 on $\mathfrak{p}'$ is the same as on $\mathfrak{p}$ with respect to the basis $\varepsilon_1', \varepsilon_2, \varepsilon_3$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3$. Thus we can assume that $a_1 = 0$. Now we have the following five cases.

Case 1: $a_2 = 0$. In this case, we set $\varepsilon_1'' = cX + \varepsilon_1$, $\mathfrak{p}'' = \operatorname{span}(\varepsilon_1'', \varepsilon_2, \varepsilon_3)$. Then $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}''$ and it is easy to check that $\mathfrak{p}''$ is an abelian ideal of $\mathfrak{g}$ and the action of X on $\mathfrak{p}''$ is

$$[X, \varepsilon_1''] = 0, \quad [X, \varepsilon_2] = -\varepsilon_3, \quad [X, \varepsilon_3] = \varepsilon_2.$$

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Case 2: $a_2 \neq 0$ and $c = 0$. Without loss of generality, we can assume that $a_2 = 1$ (otherwise, we use $a_2\varepsilon_1$ to substitute ε_1). Then $\mathfrak{p}$ is a three-dimensional Heisenberg Lie algebra ($[\varepsilon_1, \varepsilon_2] = [\varepsilon_1, \varepsilon_3] = 0$, $[\varepsilon_2, \varepsilon_3] = \varepsilon_1$), and the action of X is

$$[X, \varepsilon_1] = 0, \quad [X, \varepsilon_2] = -\varepsilon_3, \quad [X, \varepsilon_3] = \varepsilon_2.$$

Case 3: $a_2 c > 0$. Set $\varepsilon_1' = \frac{2}{c}\varepsilon_1$,$\varepsilon_2' = \frac{2}{\sqrt{a_2 c}}\varepsilon_2$, $\frac{2}{\sqrt{a_2 c}}\varepsilon_3$. We get

$$[\varepsilon_1', \varepsilon_2'] = 2\varepsilon_3', \quad [\varepsilon_1', \varepsilon_3'] = -2\varepsilon_2', \quad [\varepsilon_2', \varepsilon_3'] = 2\varepsilon_1'.$$

Thus $\mathfrak{p}$ is an ideal of $\mathfrak{g}$ which is isomorphic to the (real) compact simple Lie algebra $\mathfrak{su}(2)$ of all skew-Hermitian traceless complex matrices. For simplicity, we just suppose $\mathfrak{p} = \mathfrak{s}u(2)$. Let

$$H_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$$

be the standard basis of $\mathfrak{s}u(2)$. Then

$$[H_1, X_1] = 2Y_1, \quad [H_1, Y_1] = -2X_1, \quad [X_1, Y_1] = 2H_1.$$

And the action of X on $\mathfrak{p}$ is

$$[X, H_1] = 0, \quad [X, X_1] = -Y_1, \quad [X, Y_1] = X_1.$$

Case 4: $a_2 c < 0$. Similarly as in Case 3, we can prove that $\mathfrak{p}$ is an ideal of $\mathfrak{g}$ which is isomorphic with the real simple Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ of traceless 2×2 real matrices. Let

$$H_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then H_2, X_2, Y_2 form a basis of $\mathfrak{p}$ and

$$[H_2, X_2] = -2Y_2, \quad [H_2, Y_2] = 2X_2, \quad [X_2, Y_2] = 2H_2.$$

The action of X on $\mathfrak{p}$ is

$$[X, H_2] = 0, \quad [X, X_2] = -Y_2, \quad [X, Y_2] = X_2.$$

It remains to consider the case in which the restriction to $V = \mathrm{span}(\varepsilon_2, \varepsilon_3)$ of the automorphism τ_1 has the matrix

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \theta \in \mathbb{R}$$

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with respect to the basis $\varepsilon_2, \varepsilon_3$. In this case, applying τ_1 to both sides of

$$[\varepsilon_1, \varepsilon_2] = b\varepsilon_2 + \varepsilon_3$$

we get

$$b\cos\theta + c\sin\theta = 0, \quad -b\sin\theta + c\cos\theta = 0.$$

Therefore we have $b = c = 0$. Thus $[\varepsilon_1, \varepsilon_2] = [\varepsilon_1, \varepsilon_3] = 0$. If $[\varepsilon_2, \varepsilon_3] = 0$, then $\mathfrak{g}$ is abelian. Otherwise, we can assume that $[\varepsilon_2, \varepsilon_3] = \varepsilon_1$ (see Case 2 above), so $\mathfrak{p}$ is a Heisenberg Lie algebra. From this we see that no new structure will appear in this case.

Combining the above analysis with the examples in Section 3, we have the following.

Theorem 4.4 *Let* $(\mathfrak{g}, \mathfrak{h})$ *be a Riemannian weakly symmetric Lie algebra with* $\dim \mathfrak{h} = 1$ *and* $\dim \mathfrak{g} = 4$. *Then* $(\mathfrak{g}, \mathfrak{h})$ *must be one of the following:*

(i) $\mathfrak{g}$ *is an abelian Lie algebra. In this case the reduced set of automorphisms can be chosen to be* $\{\mathrm{id}, -\mathrm{id}\}$.

(ii) $\mathfrak{g}$ *has a decomposition* $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ *(direct sum), where* $\mathfrak{p}$ *is an abelian ideal of* $\mathfrak{g}$ *and we can select a base* X *of* $\mathfrak{h}$ *and* $\varepsilon_1, \varepsilon_2, \varepsilon_3$ *of* $\mathfrak{p}$ *such that*

$$[X, \varepsilon_1] = 0, \quad [X, \varepsilon_2] = -\varepsilon_3, \quad [X, \varepsilon_3] = \varepsilon_2.$$

In this case, the reduced set of automorphisms can be chosen to be $\{\mathrm{id}, \tau_1\}$, *where* τ_1 *is defined by*

$$\tau_1(X) = -X, \quad \tau_1(\varepsilon_1) = -\varepsilon_1, \quad \tau_1(\varepsilon_2) = -\varepsilon_2, \quad \tau_1(\varepsilon_3) = \varepsilon_3.$$

(iii) $\mathfrak{g}$ *has a decomposition* $\mathfrak{g} = \mathfrak{h} + \mathfrak{su}(2)$ *(direct sum), where* $\mathfrak{su}(2)$ *is an ideal of* $\mathfrak{g}$. *In this case, let*

$$\varepsilon_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon_2 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad \varepsilon_3 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$$

be the standard basis of $\mathfrak{su}(2)$. *Then the action of a nonzero vector* X *of* $\mathfrak{h}$ *on* $\mathfrak{su}(2)$ *is*

$$[X, \varepsilon_1] = 0, \quad [X, \varepsilon_2] = -\varepsilon_3, \quad [X, \varepsilon_3] = \varepsilon_2.$$

The reduced set of automorphisms can be chosen the same as case (ii).

(iv) $\mathfrak{g}$ *has a decomposition* $\mathfrak{g} = \mathfrak{h} + \mathfrak{sl}(2, \mathbb{R})$ *(direct sum), where* $\mathfrak{sl}(2, \mathbb{R})$ *is an ideal of* $\mathfrak{g}$. *In this case, let*

$$\varepsilon_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \varepsilon_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

be the standard basis of $\mathfrak{sl}(2, \mathbb{R})$. *Then the action of a nonzero vector* X *of* $\mathfrak{h}$ *on* $\mathfrak{sl}(2, \mathbb{R})$ *is the same as in the case* (iii). *The reduced set of automorphisms can be chosen the same as case* (ii).

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(v) *$\mathfrak{g}$ has a decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{n}$ (direct sum), where $\mathfrak{n}$ is an ideal of $\mathfrak{g}$ and can be identified with the three-dimensional Heisenberg Lie algebra consisting of the real matrices*

$$\begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, \quad a, b, c \in \mathbb{R}.$$

Let

$$\varepsilon_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \varepsilon_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \varepsilon_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

be the standard basis of $\mathfrak{n}$. Then the action of a nonzero vector X of $\mathfrak{h}$ on $\mathfrak{n}$ is the same as in the case (iii). *The reduced set of automorphisms can be chosen the same as case* (ii).

Using Theorem 4.4, we can give the classification of three-dimensional weakly symmetric Finsler spaces.

Theorem 4.5 *Let (M, F) be a three-dimensional connected simply connected weakly symmetric Finsler space. Then one of the following holds:*

(i) *(M, F) is a (reversible) globally symmetric Finsler space.*

(ii) *M is the compact simple Lie group $\mathrm{SU}(2)$ and F is a left invariant Finsler metric on M. The restriction of F at the unit element of M is a reversible Minkowski norm on the Lie algebra $\mathfrak{su}(2)$ satisfying*

$$(**) \quad \begin{cases} F(a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_3) = F\big(a\varepsilon_1 + (b\cos\theta + c\sin\theta)\varepsilon_2 + (c\cos\theta - b\sin\theta)\varepsilon_3\big) \\ F(y, y) \neq d\sqrt{-B(y, y)}, \quad \forall d > 0 \end{cases}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are the basis as in Theorem 4.4(iii) *and a, b, c, θ are arbitrary real numbers. In this case, there are infinitely many Riemannian metrics as well as infinitely many non-Riemannian Finsler metrics. All these metrics are non-symmetric.*

(iii) *M is the universal covering group of $\mathrm{SL}(2, \mathbb{R})$ and F is a left invariant Finsler metric on the Lie group M. The restriction of F at the unit element e of M is a reversible Minkowski norm on the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ satisfying*

$$F(a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_3) = F(a\varepsilon_1 + (b\cos\theta + c\sin\theta)\varepsilon_2 + (c\cos\theta - b\sin\theta)\varepsilon_3),$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are the basis as in Theorem 4.4(iv) *and a, b, c, θ are arbitrary real numbers. In this case, there are infinitely many Riemannian metrics as well as infinitely many non-Riemannian Finsler metrics. Further, all these metrics are non-symmetric.*

(iv) *M is the three-dimensional Heisenberg Lie group*

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R}.$$

F is a left invariant Finsler metric on M. The restriction of F at the unit element e of M is a reversible Minkowski norm on the three-dimensional Heisenberg Lie algebra satisfying

$$F(a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_3) = F(a\varepsilon_1 + (b\cos\theta + c\sin\theta)\varepsilon_2 + (c\cos\theta - b\sin\theta)\varepsilon_3),$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are the basis as in Theorem 4.4(v) *and a, b, c, θ are arbitrary real numbers. In this case, in the sense of isometric diffeomorphism, there exists a unique (up to a positive scalar) Riemannian metric but there are infinitely many non-Riemannian Finsler metrics. All these metrics are non-symmetric.*

Proof Let (M, F) be a three-dimensional connected and simply connected weakly symmetric Finsler space. As before let $\tilde{G}$ be the full group of isometries and $\tilde{H}$ the isotropic subgroup at a fixed point in M. Let H, G be the identity component of $\tilde{H}$, $\tilde{G}$, respectively. Then, as we have pointed out, $\dim H = 0, 1$, or 3. If $\dim H = 3$, then (M, F) is a two-point homogeneous Riemannian manifold. If $\dim H = 0$, then M itself is a commutative Lie group. Hence (M, F) is just a reversible Minkowski space. If $\dim H = 1$, then by Theorem 2.2, we see that $(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{g} = \mathrm{Lie}\, G$, $\mathfrak{h} = \mathrm{Lie}\, H$, is a Riemannian weakly symmetric Lie algebra. By Theorem 4.4, there are only five kinds of structures for $(\mathfrak{g}, \mathfrak{h})$. In the case (i), (M, F) is obviously a reversible Minkowski space. Therefore we only need to consider the cases (ii)–(v).

We first consider the case (ii). Let P be the connected Lie subgroup of G with Lie algebra $\mathfrak{p}$. Then P is a commutative normal subgroup of G and G is the semiproduct of H and P. Hence the coset space $M = G/H$ is diffeomorphic with the Lie group P. This means that P is a connected simply connected commutative Lie group, *i.e.,* $P = \mathbb{R}^3$ (as an additive group). Thus (M, F) must be the Euclidean space $\mathbb{R}^3$ endowed with a reversible Minkowski norm which is invariant under the actions of $\tilde{H}$ and τ_1. Hence it is a globally symmetric Finsler space.

Next we consider the case (iii). Similarly as in the case (ii), we see that (M, F) must be a left invariant Finsler metric on the Lie group SU(2) whose restriction on the tangent space at the unit element ($= \mathfrak{su}(2)$) is invariant under the actions of $\tilde{H}$ and τ_1. In particular, it is invariant under the action of H. Hence F satisfies $(**)$. On the other hand, by Proposition 1.5, any Minkowski norm on $\mathfrak{s}u(2)$ satisfying $(**)$ defines a weakly symmetric left invariant Finsler metric on SU(2). However, if $F(y) = d\sqrt{-B(y, y)}$ for some positive number d, then this metric is globally symmetric and the isotropic subgroup at e contains Ad(SU(2)), which is three-dimensional. Therefore the corresponding weakly symmetric Lie algebra cannot be the type (iii). Hence $F(y) \neq d\sqrt{-B(y, y)}$ for any positive number d. It is easily seen that among these metrics there are infinitely many Riemannian ones as well as non-Riemannian ones. Finally, if one of such metrics is (globally) symmetric, then the Lie group pair $(\tilde{G}, \tilde{H})$ is a Riemannian symmetric pair. (For the Riemannian case, this can be found in [10]. For the general case, see [8]). Thus the Lie algebra pair $(\mathfrak{g}, \mathfrak{h})$ is an orthogonal symmetric Lie algebra [10]. But this is impossible, because in an orthogonal Lie algebra we have $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}$ and in our case we have $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p}$. Thus all the metrics in this case are non-symmetric.

The above arguments are all valid in the case (iv) except the proof that such metrics are non-symmetric. Now we proceed as follows. If one of such metrics on the universal covering of $SL(2,\mathbb{R})$ is symmetric. Then as in the case (iii), the isotropic subgroup at the unit element must be three-dimensional, otherwise it will contradicts the fact that $(\mathfrak{g},\mathfrak{h})$ cannot be an orthogonal symmetric Lie algebra. If the isotropic group is three-dimensional, then the space (M,F) is a two-point homogeneous Riemannian manifold. Hence it is a simply connected non-compact symmetric space of rank 1, *i.e.,* it is the three-dimensional hyperbolic space. In particular, it is a homogeneous Riemannian space of negative constant curvature -1. Now we can deduce a contraction as follows. Note that the Lie group $SL(2,\mathbb{R})$, endowed with the left invariant Riemannian metric Q whose restriction at e satisfying $F(Y) = \sqrt{Q(Y,Y)}$, $Y = \mathfrak{s}l(2,\mathbb{R})$, is locally isometric to (M,F). Hence $(SL(2,\mathbb{R}),Q)$ is of constant curvature -1. This is impossible, because a classical result of S. Kobayashi asserts that a homogeneous Riemannian manifold of strictly negative curvature is necessarily simply connected [11]. This completes the proof for the case (iv).

Finally, for the case (v), the assertion that the weakly symmetric Riemannian metric is unique up to positive factor follows from the fact that on the three-dimensional Heisenberg Lie group any two left-invariant Riemannian metric are isometric (up to a positive factor). See [4], where it was also proved that this metric is not symmetric. But then any of the non-Riemannian ones cannot be symmetric. Otherwise a contraction will arise in both the cases for the dimension of isotropic subgroup is 2 (the Lie algebra pair $(\mathfrak{g},\mathfrak{h})$ cannot be an an orthogonal symmetric Lie algebra) or is 3 (Riemannian metric). ■

5 An Open Problem of Z. Shen

Z. Shen posed the following open problem. Is there any reversible non-Berwaldian Finsler space with vanishing S-curvature? We explain here the merits of this problem. The notion of S-curvature of a Finsler space was introduced by Z. Shen [17]. It is a quantity to measure the rate of change of the volume form of a Finsler space along the geodesics. Shen showed that the Bishop–Gromov volume comparison theorem is true for a Finsler space with vanishing S-curvature. It is therefore important to find examples of Finsler spaces with vanishing S-curvature. He proved that every Berwald space has vanishing S-curvature. More recently Shen pointed out that there exists Randers spaces (which are automatically non-reversible) that are non-Berwaldian but with vanishing S-curvature. Hence it is a natural problem to ask whether there exists any reversible non-Berwaldian Finsler space with vanishing S-curvature.

In our previous paper, we showed that the non-Riemannian reversible Finsler metrics constructed in Theorem 4.5(ii) satisfy the conditions. In this section, we will show that this is also the case for the non-Riemannian Finsler metrics constructed in Theorem 4.5(iii) or (iv). Thus reversible non-Berwaldian Finsler spaces with vanishing S-curvature may exist at large.

First of all, any maximal (constant speed) geodesic in a weakly symmetric Finsler space must be the orbit of a one-parameter subgroup of the full group of isometries. This fact can be proved in exactly the same way as in the Riemannian case (see [3,9]). This implies that the S-curvature of any Finsler metrics constructed in Theorem 4.5 vanishes [9]. Thus we only need to prove that the non-Riemannian metrics in (ii), (iii) or (iv) must be non-Berwaldian.

In [9], we presented a proof for the case (ii). Here we can give a simpler and more direct proof which is also valid for other cases. Suppose that F is a non-Riemannian metric constructed in Theorem 4.5(ii), (iii), or (iv). If F is Berwaldian, then the linear connection ∇ of F is also the Levi–Civita connection of a Riemannian metric g on M (see [19]). Then by the main theorem of [19], the connection ∇ must be either holonomy reducible or the connection of a globally Riemannian symmetric metric of rank ≥ 2. In Theorem 4.5, we have pointed out that such metrics are not symmetric. Hence ∇ must be holonomy reducible. According to the generalized de Rham decomposition theorem for Berwald spaces [19], (M, F) can be written as the product of irreducible Berwald spaces. In particular, (M, F) can be written as the product of a (reversible) one-dimensional Berwald space (M_1, F_1) and a reversible two-dimensional Berwald space (M_2, F_2). It is then obvious that (M_1, F_1) and (M_2, F_2) are totally geodesic submanifolds of (M, F), hence they are both weakly symmetric. The one-dimensional reversible Berwald space (M_1, F_1) is obviously globally symmetric. Further, by the classification Theorem 4.2, we conclude that (M_2, F_2) is also a globally symmetric (reversible) Berwald space. Therefore, (M, F) is a globally symmetric Berwald space. This is a contradiction with the conclusions in Theorem 4.5.

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