ON A PROBLEM OF ORE ON MAXIMAL TREES

S. B. RAO

(Received 18 March 1969)

Communicated by E. Strzelecki

We consider only graphs without loops or multiple edges. Pertinent definitions are given below. For notation and other definitions we generally follow Ore [1].

A connected graph $G = (X, E)$ is said to have the property P if for every maximal tree T of G there exists a vertex a_T of G such that distance between a_T and x is same in T as in G for every x in X. The following problem has been posed by Ore (see [1] page 103, problem 4): Determine the graphs with property P. This paper presents a solution to the above problem in the finite case.

THEOREM 1: *A finite biconnected graph $G = (X, E)$ has the property P if and only if it is a cycle (type I) or a complete bipartite graph $K(V, X-V)$ with $|V| = 2$ and $|X-V| \geqq 2$ (type II).*

PROOF. It is easy to check that the graphs mentioned in the statement of the theorem have the property P.

Conversely, let G be a finite biconnected graph with property P and $d(G)$ its diameter. If $d(G) = 1$, then, G is a triangle. So assume that $d(G) \geqq 2$. We note the following facts.

(1) If T is a maximal tree of G then $d(T) \leqq 2d(G)$ and further if $d(T) = 2d(G)$ then a_T (given by the property P) is the unique centre of T.

(2) Every subgraph of G which is a tree can be extended to a maximal tree of G.

Let x_0, y_0 be vertices of G such that $d_G(x_0, y_0) = d(G)$. Since G is biconnected there is a simple circuit μ containing x_0, y_0 (Theorem 5.4.3 of [1]). Without loss of generality assume that $\mu = [x_0, x_1, x_2, \cdots, x_t, x_0]$. Clearly the length of μ, $L(\mu)$ is greater than or equal to $2d(G)$. We show that $L(\mu) \leqq 2d(G)+1$. Suppose not, then consider the subgraph $\mu[x_0, x_t]$ whose length $\geqq 2d(G)+1$. By (2) and (1) we get a contradiction.

CASE (i). $L(\mu) = 2d(G)+1$. Let $A = \{x_0, x_1, \cdots, x_t\}$. Then $X = A$. For otherwise let y be a vertex of $X-A$ adjacent to some vertex of A, say x_i. Consider the subgraph $\xi = (y, x_i)+\mu[x_i, x_0]+\mu[x_0, x_{i-1}]$ whose diameter $\geqq 2d(G)+1$. By (2) and (1) we get a contradiction.

Now $G = \mu$. Otherwise, let (x_i, x_j) be an edge of G, where j is different from $i-1$ and $i+1$. Consider the subgraph $T = \mu[x_{i+1}, x_j] + (x_j, x_i) + \mu[x_0, x_{i-1}]$. T is a maximal tree of G and $d(T) = 2d(G)$. Since G has the property P, by (2), a_T is the unique centre of T, but here it is not, a contradiction. Hence G is a cycle (Type 1).

CASE (ii). $L(\mu) = 2d(G)$. Let $A = \{x_0, x_1, \cdots, x_t\}$. Define $B_i = \{y : y \in X - A$ and y is adjacent to x_i in $G\}$, for every i, $0 \leqq i \leqq t$ and $B = \bigcup_{i=0}^{t} B_i$. If B is empty $G = \mu$ (as in case (i)). Assume that B is non empty. We show that B is an independent set in G. Let if possible x, y be vertices in B and (x, y) be an edge of G with y in B_{i_0}. Then consider the following subgraph

$$\xi = (x, y) + \mu[x_{i_0}, x_0] + \mu[x_0, x_{i_0 - 1}]$$

of G whose length is $2d(G)+1$; by (2) and (1) this leads to a contradiction. Further, if z is in B_i, (z, x_{i+1}), (z, x_{i-1}) are not edges of G. Since B is an independent set and G is biconnected, z is joined to x_j for some j, $0 \leqq j \leqq t$ and $j \notin \{i-1, i+1\}$. If $d(G) > 2$ consider the subgraph

$$\xi = [x_{j+1}, x_{j+2}, \cdots, x_i, z, x_j, x_{j-1}, \cdots, x_{i+1}].$$

By (2) this can be extended to a maximal tree T of G and $d(T) = 2d(G)$ but a_T is not the unique centre of T – a contradiction. Hence $d(G) = 2$ so $\mu = [x_0, x_1, x_2, x_3, x_0]$. Since B is nonempty at least one of B_i, $0 \leqq i \leqq 3$ is nonempty. Assume that B_0 is non empty. Now if x is in $X - A$ it belongs to B_0 and B_2. Let $V = \{x_0, x_2\}$ then $G = K(V, X-V)$, the complete bipartite graph, with $|X - V| \geqq 2$ (type II). This completes the proof of theorem 1.

THEOREM 2. *A finite connected graph with property P on n vertices is a tree or consists of a subgraph H on n_0 $(3 \leqq n_0 \leqq n)$ vertices of type I or type II to which trees with a total of $n - n_0$ edges are attached at some vertices of H.*

PROOF. Let x be a cut vertex of G. It can be easily shown that at most one leaf with respect to x of G is not a tree. Now theorem 2 follows from theorem 1.

REMARK. Perhaps it is true that $G = K(V, X-V)$, the complete bipartite graph with $|V| = 2$, is the only biconnected graph with property P if X is infinite.

Reference

[1] O. Ore, *Theory of graphs* (American Mathematical Society Colloquium Publication, 38, Providence, Rhode Island, 1962).

Indian Statistical Institute
Calcutta 35
India