# ON RIGHT UNIPOTENT SEMIGROUPS II

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We describe two congruences $\alpha$ and $\gamma$ contained in $\mathcal{L}$ on an arbitrary orthodox semigroup. Let $S$ be a right unipotent semigroup. We show that (i) $\alpha$ is an inverse semigroup congruence and $\gamma$ is the finest fundamental inverse semigroup congruence on $S$, (ii) $S$ is a union of groups if and only if $\gamma = \mathcal{L}$ on $S$ and (iii) $S$ is a band of groups if and only if $\alpha = \mathcal{L}$ on $S$.

Let $A$ be a regular semigroup and let $x \in A$. Throughout the paper $V(x)$ stands for the set of inverses of $x$, and $E(A)$ for the set of idempotents of $A$. $A$ is said to be an orthodox semigroup if $E(A)$ is a subsemigroup of $A$. If $A$ is an orthodox semigroup then $V(b)V(a) \subseteq V(ab)$ for all $a$, $b$ in $A$ [**9**]. For the general terminology and notation, the reader is referred to [**2**].

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**1. The congruences** $\alpha$, $\beta$, $\gamma$, $\delta$ **and** $\mu$. Let $S$ be an orthodox semigroup and let $E = E(S)$. The greatest idempotent-separating congruence $\mu$ on $S$ (Howie [**5**, Theorem VI. 1.17], Meakin [**7**]) is given by the rule:

$$(x, y) \in \mu = \mu(S) \Leftrightarrow x'ex = y'ey \quad \text{and } xex' = yey' \text{ for all } e \in E \text{ and for some } x' \in V(x) \text{ and } y' \in V(y).$$

We define the binary relations $\alpha$ and $\gamma$ on $S$ thus:

$$(x, y) \in \alpha = \alpha(S) \Leftrightarrow x'ex = y'ey \text{ for all } e \in E \text{ and for some } x' \in V(x),\ y' \in V(y);$$

$$(x, y) \in \gamma = \gamma(S) \Leftrightarrow x'exy'ey = x'ex \text{ and } y'eyx'ex = y'ey \text{ for all } e \in E \text{ and for some } x' \in V(x),\ y' \in V(y).$$

The proof of Theorem 1 below is like Meakin's (loc. cit.) derivation of $\mu$ for $S$.

THEOREM 1. *Let $S$ be an orthodox semigroup and let $E = E(S)$. Then*

(1) *$\alpha$ and $\gamma$ are congruences on $S$,*

(2) *$\gamma$ is the greatest congruence contained in $\mathcal{L}$ on $S$.*

*Proof.* Clearly $\alpha$ is reflexive and symmetric. Let $(x, y) \in \alpha$. Then there exist $x' \in V(x)$ and $y' \in V(y)$ such that $x'ex = y'ey$ for all $e \in E$. Let $x'' \in V(x)$ and $y'' \in V(y)$. Taking in turn $e = xx''$ and $e = yy''$, we have $x'x = y'xx''y$ and $y'y = x'yy''x$. Therefore $x'x = x'xy'y = y'y$. If $(y, z) \in \alpha$ then $y^*ey = z'ez$ for all $e \in E$ and some $y^* \in V(y)$, $z' \in V(z)$. So $y^*y = z'z = z'yy'z$. Set $p = y^*yx'$ and $q = z'yy'$. Then $pxp = y^*(yx'xy^*yx') = y^*yx' = p$ and $xpx = xy^*y(x'x) = xy^*y = (xy'y)y^*y = xy'y = x$. Therefore $p \in V(x)$. Similarly $q \in V(z)$. Now, for all $e \in E$, we have $pex = y^*y(x'ex) = y^*y(y'ey) = y^*(yy'e)y = z'(yy'e)z = qez$, giving $(x, z) \in \alpha$. So $\alpha$ is transitive and hence an equivalence relation on $S$.

Let $a \in S$ and $a' \in V(a)$. Then for all $e \in E$ we have $a'(x'ex)a = a'(y'ey)a$ and $x'(a'ea)x = y'(a'ea)y$. Since $a'ea \in E$ [**9**], we get that $\alpha$ is a congruence on $S$.

We now consider $\gamma$, which is clearly reflexive and symmetric. Let $(x, y) \in \gamma$. Then $x'exy'ey = x'ex$ and $y'eyx'ex = y'ey$ for all $e \in E$ and some $x' \in V(x)$, $y' \in V(y)$. If $(y, z) \in \gamma$ then there exist inverses $y^*$ of $y$ and $z'$ of $z$ such that $y^*eyz'ez = y^*ey$ and $z'ezy^*ey = z'ez$. Since $S$ is orthodox, we have $y'ey = y'(eyy^*ey)$. Therefore $x'exz'ez = x'exy'eyz'ez = x'ex(y'eyy^*ey)z'ez = x'ex(y'eyy^*ey) = x'exy'ey = x'ex$. Similarly, $z'ezx'ex = z'ez$. So $(x, z) \in \gamma$, and $\gamma$ is an equivalence relation.

Let $a \in S$ and $a' \in V(a)$. Then since $(x, y) \in \gamma$, for all $e \in E$ we have $x'(a'ea)xy'(a'ea)y = x'(a'ea)x$ and $y'(a'ea)yx'(a'ea)x = y'(a'ea)y$. Therefore $(ax, ay) \in \gamma$. Further $a'x'exaa'y'eya = a'(x'exy'ey)aa'y'eya = a'x'exy'(eya)(a'y'e)(eya) = a'x'exy'(eya) = a'(x'exy'ey)a = a'x'exa$. Similarly, $a'y'eyaa'x'exa = a'y'eya$. Thus $(xa, ya) \in \gamma$, and $\gamma$ is a congruence on $S$.

As for the other part, if $(x, y) \in \gamma$, taking $e = x'x$ we have $x'xy'xx'y = x'x$. So $xy'xx'y = x$ and hence $xy'y = x$. Similarly $yx'x = y$. Therefore $(x, y) \in \mathcal{L}$ and $\gamma \subseteq \mathcal{L}$.

Now let $\eta$ be a congruence on $S$ such that $\eta \subseteq \mathcal{L}$. Let $(x, y) \in \eta$. Then, for all $e \in E$, we have $(ex, ey) \in \mathcal{L}$. So for all inverses $p$ of $ex$ and $q$ of $ey$, we have $pexqey = pex$ and $qeypex = qey$. Taking $p = x^*e$ and $q = y^*e$, where $x^*$ and $y^*$ are inverses of $x$ and $y$ respectively, we get that $(x, y) \in \gamma$ and hence that $\eta \subseteq \gamma$. This completes the proof of the theorem.

Corollary. *If $\mathcal{L}$ is a congruence on an orthodox semigroup, then $\gamma = \mathcal{L}$.*

Remark. Let $S$ be an orthodox semigroup and let $x, y \in S$. Then, since $\gamma \subseteq \mathcal{L}$, we have $(x, y) \in \gamma$ if and only if $x'exy'ey = x'ex$ and $y'eyx'ex = y'ey$ for *all* $e \in E(S)$, $x' \in V(x)$ and $y' \in V(y)$.

Remark. Let $S$ be an orthodox semigroup and let $E = E(S)$. Define the binary relations $\beta$ and $\delta$ on $S$ thus:

$$(x, y) \in \beta \Leftrightarrow xex' = yey' \text{ for all } e \in E \text{ and some } x' \in V(x),\ y' \in V(y);$$

$$(x, y) \in \delta \Leftrightarrow xex'yey' = yey' \text{ and } yey'xex' = xex' \text{ for all } e \in E \text{ and some } x' \in V(x),\ y' \in V(y).$$

Then $\beta$ and $\delta$ are congruences on $S$, and $\delta$ is the greatest congruence contained in $\mathcal{R}$ on $S$. Trivially $\beta \subseteq \delta (\alpha \subseteq \gamma)$, and it follows from the definitions of $\alpha$, $\beta$, $\gamma$, $\delta$ and $\mu$ that $\mu = \alpha \cap \beta = \gamma \cap \delta$.

A semigroup is said to be fundamental if the only congruence on it contained in $\mathcal{H}$ is the trivial congruence. Since $\mu$ is the greatest congruence on $S$ contained in $\mathcal{H}$, it follows that $S$ is fundamental if and only if $\mu$ is the trivial congruence on $S$. We omit the proof of the following result which is similar to the one known for inverse semigroups (Howie [**4**], Munn [**8**]).

Lemma 1. *Let $S$ be an orthodox semigroup. Then $S/\mu$ is a fundamental orthodox semigroup and the band of idempotents of $S/\mu$ is isomorphic to $E(S)$.*

Theorem 2 below shows how certain homomorphic images of $S$ are related.

THEOREM 2. *Let $S$ be an orthodox semigroup. Write $L=S/\mu$, $M=S/\alpha$, $N=S/\gamma$, $\alpha'=\alpha(L)$, $\gamma'=\gamma(L)$ and $\gamma''=\gamma(M)$. Then*

(1) *$L/\alpha'$ is isomorphic to $M$,*

(2) *$L/\gamma'$ and $M/\gamma''$ are isomorphic to $N$.*

*Proof.* By [**2**, Theorem 1.6], the mappings $\theta: L\to M$, $\phi: L\to N$ and $\psi: M\to N$ defined by $(a\mu)\theta=a\alpha$, $(a\mu)\phi=a\gamma$ and $(x\alpha)\psi=x\gamma(a, x\in S)$, are surjective homomorphisms. Let $a, b\in S$. Then, since every idempotent of $M$ is the image of an idempotent of $L$ [**6**], we have $(a,b)\in\alpha$ if and only if $(a\mu, b\mu)\in\alpha'$. So $\theta\circ\theta^{-1}=\alpha'$, proving (1). Since $(a,b)\in\gamma$ if and only if $(a\mu, b\mu)\in\gamma'$, we get the first part of (2). If $x, y\in S$, then $(x,y)\in\gamma$ if and only if $(x\alpha, y\alpha)\in\gamma''$, proving the other part of (2).

REMARK. Write $T=S/\beta$, $\beta'=\beta(L)$ and $\delta'=\delta(T)$. Then proceeding as above we get $L/\beta'$ isomorphic to $T$, and $T/\delta'$ isomorphic to $S/\delta$.

**2. Right unipotent semigroups.** Let $A$ be an orthodox semigroup and let $E=E(A)$. The finest inverse semigroup congruence $\sigma$ on $A$ (Hall [**3**], Yamada [**14**]) is given by the rule: $(x,y)\in\sigma$ if and only if $V(x)=V(y)$.

$A$ is said to be a right (left) unipotent semigroup if every principal right (left) ideal of $A$ has a unique idempotent generator. Let $S$ be a right unipotent semigroup and let $a\in S$. Then (i) $fef=fe$ for any two idempotents $e$, $f$ of $S$, and (ii) $aa'=aa^*$ for any $a'$, $a^*$ in $V(a)$ [**11**]. Further, $\beta$ is the greatest idempotent-separating congruence on $S$ [**12**]. Since $\beta=\delta$ follows from the definition of $S$, we have $\mu=\beta=\delta$ on $S$. That $\mu=\alpha=\beta$ on an inverse semigroup is well known (Howie [**4**]).

THEOREM 3. *Let $S$ be an orthodox semigroup and let $E=E(S)$. Then the following statements are equivalent:*

(A) *$S$ is a right unipotent semigroup;*

(B) *$S/\alpha$ is an inverse semigroup;*

(C) *$S/\gamma$ is an inverse semigroup.*

*Proof.* Assume (A). Let $g, h\in E$. Then, for all $e\in E$, we have $ghehg=ghe=ghegh$ [**11**]. This, since $V(gh)=V(hg)$, implies that $(hg, gh)\in\alpha$. Since every idempotent of $S/\alpha$ is the image of an idempotent of $S$ [**6**], we get (B).

(B) implies (C) since $S/\gamma$ is a homomorphic image of $S/\alpha$ [**2**, p. 17]. That (C) implies (A) follows from [**13**]. For a direct proof, assume (C). Let $g, h\in E$. Then, since $\gamma$ is contained in $\mathcal{L}$, we have $(gh, hg)\in\mathcal{L}$. Therefore $ghg=gh$, proving (A) [**11**].

THEOREM 4. *Let $S$ be a right unipotent semigroup and let $E=E(S)$. Write $P=S/\sigma$ and $\mu'=\mu(P)$. Then*

(1) *$P/\mu'$ is isomorphic to $S/\gamma$,*

(2) *$\gamma$ is the finest fundamental inverse semigroup congruence on $S$.*

*Proof.* By Theorem 3 we have $\sigma\subseteq\gamma$. So, proceeding on the lines of the proof of Theorem 2, we get (1). From this result and Lemma 1 it follows that $S/\gamma$ is a fundamental inverse semigroup. Let $\rho$ be a congruence on $S$ such that $S/\rho$ is a fundamental inverse



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semigroup. Let $x, y \in S$ be such that $(x, y) \in \gamma$. Then, since $S/\rho$ is inverse, we have $(x\rho, y\rho) \in \mu$ on $S/\rho$. This, since $S/\rho$ is fundamental, implies that $(x, y) \in \rho$ and hence that $\gamma \subseteq \rho$.

COROLLARY (*to the proof of* (1)). *The semilattice* $E(S/\sigma)$ *is isomorphic to the semilattice* $E(S/\gamma)$.

**3. Unions of groups.** In this section we consider right unipotent semigroups which are unions of groups.

Let $S$ be a right unipotent semigroup, let $E = E(S)$ and let $x \in S$. Let the symbols $(Px)$ and $(Qx)$ denote statements as follows:
$(Px)$ $exe = ex$ and $ex'e = ex'$ for all $e \in E$ and for some $x' \in V(x)$.
$(Qx)$ $xex' = xx'e$ for all $e \in E$ and $x' \in V(x)$.

Then (i) $(Qx)$ implies $(Px)$ for any $x$ in $S$, (ii) $S$ is a union of groups if and only if $(Px)$ is satisfied for all $x$ in $S$ and (iii) $S/\beta$ is isomorphic to $E$ if and only if $(Qx)$ is satisfied for all $x$ in $S$ [**12**].

Let $S$ be a right unipotent semigroup which is a union of groups. Then Green's relations on $S$ are related thus: $\mathcal{J} = \mathcal{D} = \mathcal{L}$ and $\mathcal{R} = \mathcal{H}$ [**10**].

THEOREM 5. *Let $S$ be a right unipotent semigroup and let $E = E(S)$. Then the following statements are equivalent:*

(A) *$S$ is a union of groups;*
(B) *$S/\sigma$ is a semilattice $Y$ of groups, where $Y$ is isomorphic to $E(S/\sigma)$;*
(C) *$S/\gamma$ is a semilattice;*
(D) *$\gamma = \mathcal{L}$ (equivalently, $\mathcal{L}$ is a congruence on $S$);*
(E) *$S$ is a semilattice $Z$ of left groups, where $Z$ is isomorphic to $S/\gamma$.*

*Proof.* That (A) implies (B) is well known [**2**, pp. 126–129]. Assume (B). Let $x \in S$, $x' \in V(x)$ and $e \in E$. Since the idempotents of $S/\sigma$ are in the centre of $S/\sigma$ [**2**, p. 127], we have $(ex)\sigma = (xe)\sigma$; that is, $V(ex) = V(xe)$. Now both $x'e$ and $ex'$ are inverses of $ex$. This, since $S$ is right unipotent, implies that $exex' = exx'$. Similarly $ex'ex = ex'x$. Therefore

$$(x'ex)(x'xex'x) = x'(exex')x = x'(exx')x = x'ex$$

and

$$(x'xex'x)(x'ex) = x'x(ex'ex) = x'xex'x.$$

So $(x, x'x) \in \gamma$ and hence, by Theorem 3, $S/\gamma$ is a semilattice, proving (C).

Assume (C). We first prove that $(Px)$ is satisfied for all $x \in S$. Let $x \in S$, $x' \in V(x)$ and $e \in E$. Then $(x'e, ex') \in \gamma$. This, since $\gamma \subseteq \mathcal{L}$, implies that $(exx'e)(xex') = exx'e$. Therefore, since $S$ is right unipotent, we have $exex' = exx'$ and hence

$$exe = ex(x'xe) = ex(x'xex'x) = (exex')x = (exx')x = ex.$$

Similarly $ex'e = ex'$. So $(Px)$ is satisfied for all $x \in S$.

Now let $(x, y) \in \mathcal{L}$. If $x' \in V(x)$ and $y' \in V(y)$ then $xy'y = x$ and $yx'x = y$. So, for all $e \in E$, using $(Pxy')$, we have $x'exy'ey = x'(exy'e)y = x'exy'y = x'ex$ and, similarly, $y'eyx'ex = y'ey$. Therefore $(x, y) \in \gamma$ and $\mathcal{L} \subseteq \gamma$. Since $\gamma \subseteq \mathcal{L}$, we get (D).

Assume (D). Let $x \in S$ and $x' \in V(x)$. Then $(x, x'x) \in \mathcal{L}$. Since $\mathcal{L}$ is a congruence, for

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all $e \in E$ we have $(ex, ex'x) \in \mathscr{L}$ and hence $exex'x = ex$. Therefore $ex = ex(x'xex'x) = ex(x'xe) = exe$. Similarly $ex' = ex'e$ and so $(Px)$ is satisfied for all $x \in S$. Thus $S$ is a union of groups. Now (E) follows from [**2**, Theorem 4.6] and [**10**]. Clearly (E) implies (A), proving the theorem.

REMARK. By the corollary to Theorem 4, the semilattices $Y$ and $Z$ that occur in Theorem 5 are isomorphic.

In the proof of Theorem 6 below, for any $x \in S$, $x^{-1}$ denotes the inverse of $x$ in the group $H_x$.

THEOREM 6. *Let $S$ be a right unipotent semigroup and let $E = E(S)$. Then the following statements are equivalent:*

(A) *$S$ is a band of groups*;

(B) *$(Qx)$ is satisfied for all $x$ in $S$*;

(C) $\alpha = \gamma = \mathscr{L}$;

(D) $\beta = \mathscr{H} = \mathscr{R}$ *(equivalently, $\mathscr{R}$ is a congruence on $S$)*;

(E) *$S$ is a band $E$ of groups.*

*Proof.* Assume (A). Then $abS = a^2bS$ for all $a, b$ in $S$ [**1**], [**2**, p. 129]. Let $x \in S$ and $e \in E$. Put $a = x$ and $b = x^{-1}e$. Since $ex^{-1} \in V(xe)$ and $xx^{-1} = x^{-1}x$, from $abS = a^2bS$ we get that $(xx^{-1}e, xex^{-1}) \in \mathscr{R}$. Therefore $xx^{-1}e = xex^{-1}$, giving (B).

Assume (B). Let $x, y \in S$, $x' \in V(x)$ and $y' \in V(y)$. Suppose that $(x, y) \in \mathscr{L}$. Then $xy'y = x$ and $yx'x = y$. Write $a = y'yx'$. Since $a \in V(x)$ and $xy' \in V(yx')$, for all $e \in E$, using $(Qyx')$, we have $aex = y'(yx'exy')y = y'(yx'xy'e)y = y'ey$. Therefore $(x, y) \in \alpha$, proving that $\mathscr{L} \subseteq \alpha$. So we get (C).

Assume (C). We first prove that $(Qx)$ is satisfied for all $x \in S$. Let $x \in S$, $x' \in V(x)$ and $e \in E$. Since $(x', xx') \in \mathscr{L}$, by hypothesis we have $(x', xx') \in \alpha$; that is, $pex' = qexx'$ for some $p \in V(x')$ and $q \in V(xx')$. Premultiplying the equation by $xx'$, we get $xx'pex' = xx'qexx'$. Since $S$ is right unipotent, $x'p = x'x$ and $xx'q = xx'$. Therefore $xex' = xx'exx' = xx'e$, and $(Qx)$ is satisfied for all $x \in S$. This implies that $S$ is a union of groups and $(x, xx^{-1}) \in \beta$ for all $x \in S$ [**12**]. Hence

$$\begin{aligned}(x, y) \in \beta &\Leftrightarrow (xx^{-1}, yy^{-1}) \in \beta\\ &\Leftrightarrow xx^{-1} = yy^{-1}\\ &\Leftrightarrow (x, y) \in \mathscr{R}.\end{aligned}$$

Since $\beta = \mu \subseteq \mathscr{H}$, we get (D).

Assume (D). Let $x \in S$, $x' \in V(x)$ and $e \in E$. Then, since $(x, xx') \in \mathscr{R}$, we have $(xe, xx'e) \in \mathscr{R}$. This implies that $xex' = xx'e$. So $(Qx)$ is satisfied for all $x \in S$ and hence $S$ is a union of groups. Now by [**7**, Theorem 4.3] we have $H_fH_g \subseteq H_{fg}$ for all $f, g \in E$, giving (E). Trivially (E) implies (A). Hence the theorem.

**Note added in proof.** Statement (2) of Theorem 1 is known. See Tôru Saitô, Note on quasi-inverse semigroups, *Semigroup Forum* **6** (1973), 129–132.

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## REFERENCES

**1.** A. H. Clifford, Bands of semigroups, *Proc. Amer. Math. Soc.* **5** (1954), 499–504.

**2.** A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Math. Surveys of the Amer. Math. Soc. 7 (Providence R.I., 1961 (Vol. 1) and 1967 (Vol. 2)).

**3.** T. E. Hall, On regular semigroups whose idempotents form a subsemigroup, *Bull. Austral. Math. Soc.* (1969), 195–208.

**4.** J. M. Howie, The maximum idempotent-separating congruence on an inverse semigroup, *Proc. Edinburgh Math. Soc.* (2) **14** (1964), 71–79.

**5.** J. M. Howie, *Introduction to semigroup theory* (Academic Press, 1976).

**6.** G. Lallement, Congruences et équivalences de Green sur un demi-groupe régulier, *C.R. Acad. Sci. Paris Sér. A.* **262** (1966), 613–616.

**7.** J. C. Meakin, Congruences on orthodox semigroups, *J. Austral. Math. Soc.* **12** (1971), 323–341.

**8.** W. D. Munn, Fundamental inverse semigroups, *Quart. J. Math. Oxford* (2) **21** (1970), 157–170.

**9.** N. R. Reilly and H. E. Scheiblich, Congruences on regular semigroups, *Pacific J. Math.* **23** (1967), 349–360.

**10.** P. S. Venkatesan, On decomposition of semigroups with zero, *Math. Z.* **92** (1966), 164–174.

**11.** P. S. Venkatesan, Right (left) inverse semigroups, *J. Algebra* **31** (1974), 209–217.

**12.** P. S. Venkatesan, On right unipotent semigroups, *Pacific J. Math.* **63** (1976), 555–561.

**13.** R. J. Warne, *L*-unipotent semigroups, *Nigerian J. Sci.* (2) **5** (1972), 245–248.

**14.** M. Yamada, On a regular semigroup in which the idempotents form a band. *Pacific J. Math.* **33** (1970), 261–272.

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