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A DEFECT RELATION FOR MEROMORPHIC MAPS ON GENERALIZED p-PARABOLIC MANIFOLDS INTERSECTING HYPERSURFACES IN COMPLEX PROJECTIVE ALGEBRAIC VARIETIES

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Abstract We establish a defect relation for algebraically non-degenerate meromorphic maps over generalized p-parabolic manifolds that intersect hypersurfaces in smooth projective algebraic varieties, extending certain results of H. Cartan, L. Ahlfors, W. Stoll, M. Ru, P. M. Wong and Philip P. W. Wong and others.

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1. Introduction

In 1929, Nevanlinna established the first and second main theorems for meromorphic functions on the complex plane, and derived a defect relation with 2 as a precise upper bound. In 1933, Cartan [**4**, **8**] showed that a linearly non-degenerate holomorphic curve $f\colon \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ that intersects q $(> n)$ hyperplanes, say, $H_j \in \mathbb{P}^n(\mathbb{C})$ for $j = 1, 2, \ldots, q$, located in general position satisfies the defect relation

$$\sum_{j=1}^{q} \delta(H_j, f) \leqslant n + 1. \tag{1.1}$$

Later, Ahlfors [**1**], using an innovative geometric method, generalized this result to linearly non-degenerate meromorphic maps on $\mathbb{C}^m$. This theory was greatly extended by Carlson, Griffiths and King [**3**, **7**] around 1972–1973, when they studied the value distribution of algebraically non-degenerate meromorphic maps $f\colon M \to N$ from affine algebraic varieties to smooth projective algebraic varieties with the additional assumption that $\dim M \geqslant \dim N$, which turns out to be crucial and is extremely difficult to remove. Subsequently, Stoll [**16**, **17**] generalized all those results to algebraically non-degenerate meromorphic maps defined on parabolic manifolds.

Recently, Ru [**14**] removed this dimensional restriction and obtained the following result.

Theorem 1.1. *Let $f\colon \mathbb{C} \to V \subseteq \mathbb{P}^N(\mathbb{C})$ be an algebraically non-degenerate holomorphic curve to a non-singular projective algebraic variety V such that $\dim V = n \geqslant 1$, and let $D_1, D_2, \ldots, D_q \in \mathbb{P}^N(\mathbb{C})$ be q $(> n)$ hypersurfaces located in general position. Then, it follows that*

$$\sum_{j=1}^{q} \delta(D_j, f) \leqslant n + 1, \tag{1.2}$$

where the term 'in general position' here denotes the condition

$$\bigcap_{k=0}^{n} \operatorname{supp}(D_{j_k}) \cap V = \emptyset$$

for any distinct subset $\{j_0, j_1, \ldots, j_n\}$ of $\{1, 2, \ldots, q\}$.

Actually, this result answers a question by Stoll [**17**] on an extension for the complex projective space $\mathbb{P}^n(\mathbb{C})$ concerning hypersurfaces, and extends a former one [**13**] that solved a conjecture by Shiffman [**15**] which weakened the desired sharper defect relation concerning hypersurfaces in smooth complex projective algebraic varieties conjectured by Griffiths [**3**, **7**]. (See also [**5**].)

In this paper, we apply the notion 'generalized p-parabolic manifolds', introduced by Wong and Wong [**21**], to extend Theorem 1.1 for algebraically non-degenerate meromorphic maps defined on such manifolds.

The parabolic manifold (see [**16**, **17**]) has the affine algebraic variety as a prototype, and the concept of parabolicity is based on the very existence of a non-negative plurisubharmonic exhaustion τ defined on a Kähler manifold (M, ω) such that $\phi := \log \tau$ satisfies the following complex Monge–Ampère equation

$$(dd^c\phi)^m \equiv 0 \tag{1.3}$$

on $M \setminus \{\tau = 0\}$, where $m := \dim M$, yet $(dd^c\phi)^{m-1} \not\equiv 0$. If we can weaken this condition, we should get something new. The concept of p-parabolicity depends on the existence of a non-negative plurisubharmonic exhaustion τ such that, for $1 \leqslant p \leqslant m$, $\phi := \log \tau$ satisfies the generalized complex Monge–Ampère equation

$$(dd^c\phi)^p \wedge \omega^{m-p} \equiv 0 \tag{1.4}$$

on $M \setminus \{\tau = 0\}$. Note that m-parabolicity is just the classical notion of parabolicity. One thing of interest is that (see [**21**, Theorem 2.10]) for a parabolic Stein manifold M of dimension m with a strictly plurisubharmonic parabolic exhaustion τ_M, the holomorphic vector bundle E of rank $r \geqslant 2$ over M, its dual vector bundle E^* and the corresponding projectivizations $\mathbb{P}(E)$ and $\mathbb{P}(E^*)$ over M are not parabolic but do satisfy identities analogous to the one above: for example, for $\mathbb{P}(E)$ we have

$$(dd^c\phi)^{m-1} \wedge \omega^{r-1} \not\equiv 0 \quad \text{and} \quad (dd^c\phi)^m \wedge \omega^{r-1} \equiv 0, \tag{1.5}$$

where ϕ is the pull-back of $\phi_M := \log \tau_M$ on M, and ω is some Kähler metric on $\mathbb{P}(E)$ (see [**21**, Lemma 2.9]). See [**2**] and the references therein for more details on this matter. With this new concept, Wong and Wong [**21**] obtained certain first and second main theorems, and showed that a linearly non-degenerate meromorphic map $f\colon M \to \mathbb{P}^n(\mathbb{C})$ defined on M, an algebraic vector bundle over an affine algebraic variety or its projectivization (of rank no less than 2 if necessary for p-parabolicity) that intersects q $(> n)$ hyperplanes, say, $H_j \in \mathbb{P}^n(\mathbb{C})$ for $j = 1, 2, \ldots, q$, in general position satisfies the defect relation

$$\sum_{j=1}^{q} \delta(H_j, f) \leqslant n + 1. \tag{1.6}$$

Our main result is the following.

Theorem 1.2. *Let $f\colon M \to V \subseteq \mathbb{P}^N(\mathbb{C})$ be an algebraically non-degenerate meromorphic map from M, either an affine algebraic variety or an algebraic vector bundle over an affine algebraic variety or its projectivization, to a smooth projective algebraic variety V with $\dim V = n \geqslant 1$, and let $D_1, D_2, \ldots, D_q \in \mathbb{P}^N(\mathbb{C})$ be q $(> n)$ hypersurfaces in general position. Then, we have*

$$\sum_{j=1}^{q} \delta(D_j, f) \leqslant n + 1, \tag{1.7}$$

where the term 'in general position' means the same as it did in Theorem 1.1.

Remark 1.3. Note that when M represents the projectivizations of an algebraic vector bundle E over an affine algebraic variety or its dual bundle E^*, we shall assume $\operatorname{rank}(E) \geqslant 2$ to guarantee the existence of some Kähler metric on M.

2. Generalized manifolds

Following [**21**], we give the definition of generalized p-parabolic manifolds below. In addition, §§2 and 3 basically follow [**20**] and [**21**].

Definition 2.1. A Kähler (complex) manifold (M, ω) of dimension m is said to be a generalized p-parabolic manifold for $1 \leqslant p \leqslant m$ if there exists a plurisubharmonic function ϕ such that

(i) $\{\phi = -\infty\}$ is a closed subset of M with strictly lower dimension,

(ii) ϕ is smooth on the open dense set $M \setminus \{\phi = -\infty\}$, with $dd^c\phi \geqslant 0$, such that

$$(dd^c\phi)^{p-1} \wedge \omega^{m-p} \not\equiv 0 \quad \text{and} \quad (dd^c\phi)^p \wedge \omega^{m-p} \equiv 0. \tag{2.1}$$

Accordingly, we shall define

$$\tau := \mathrm{e}^{\phi} \quad \text{and} \quad \sigma := d^c\phi \wedge (dd^c\phi)^{p-1} \wedge \omega^{m-p}, \tag{2.2}$$

where τ is non-negative and is called a p-parabolic exhaustion on M, and we have

$$(dd^c\tau)^p \wedge \omega^{m-p} \not\equiv 0, \qquad d\sigma = (dd^c\phi)^p \wedge \omega^{m-p} \equiv 0, \tag{2.3}$$

$$(dd^c\tau)^j = \tau^j\{(dd^c\phi)^j + jd\phi \wedge d^c\phi \wedge (dd^c\phi)^{j-1}\} \quad \text{for } j = 1, 2, \ldots, p, \tag{2.4}$$

where we naturally set $\Omega := (dd^c\tau)^p \wedge \omega^{m-p}$ to be the volume form on M.

Correspondingly, for any positive real number $r > 0$, we write

$$\begin{aligned} M[r] &:= \{x \in M \mid \tau(x) \leqslant r^2\}, \\ M(r) &:= \{x \in M \mid \tau(x) < r^2\}, \\ M\langle r\rangle &:= M[r] \setminus M(r) = \{x \in M \mid \tau(x) = r^2\}. \end{aligned}$$

From Stokes's formula and the second equality in (2.3), for any $r > 0$ it follows immediately that

$$\int_{M\langle r\rangle} \sigma = \kappa, \tag{2.5}$$

where κ is a constant dependent only upon the structure of M.

The Green–Jensen formula on generalized p-parabolic manifolds is as follows (see [**21**, Theorem 1.3]). Let M be a generalized p-parabolic manifold of dimension m with $1 \leqslant p \leqslant m$, and let Υ be a plurisubharmonic or plurisuperharmonic function on M. Then, for $r > s > 0$, we have

$$\int_s^r \frac{\mathrm{d}t}{t^{2p-1}} \int_{M[t]} dd^c[\Upsilon] \wedge (dd^c\tau)^{p-1} \wedge \omega^{m-p} = \frac{1}{2}\int_{M\langle r\rangle} \Upsilon\sigma - \frac{1}{2}\int_{M\langle s\rangle} \Upsilon\sigma, \tag{2.6}$$

where the operation $dd^c[\Upsilon]$ is taken in the sense of currents (distributions).

We will now list some general assumptions that we will work with in this paper.

First, we introduce the notion of associated maps. Let $f\colon M \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate meromorphic map defined on a generalized p-parabolic manifold M of dimension m, and let $\tilde{f}\colon M \to \mathbb{C}^{n+1}$ be a reduced representation of it. Then, for some global meromorphic $(m-1, 0)$-form B on M, we define the first B-derivative $\tilde{f}'_B$ of $\tilde{f}$, say, on the local holomorphic coordinate chart (z, U_z), by

$$\mathrm{d}\tilde{f} \wedge B = \tilde{f}'_B \,\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \cdots \wedge \mathrm{d}z_m,$$

and define inductively the kth B-derivative $\tilde{f}_B^{(k)}$ of $\tilde{f}$ by

$$\mathrm{d}\tilde{f}_B^{(k-1)} \wedge B = \tilde{f}_B^{(k)} \,\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \cdots \wedge \mathrm{d}z_m$$

for $k = 1, 2, \ldots, n$. They are independent of the choice of the local holomorphic coordinate chart, and thus are globally well defined. As a consequence, for $k = 1, 2, \ldots, n$, the kth preassociated map $\tilde{f}_k$ of f is defined by

$$\tilde{f}_k := \tilde{f} \wedge \tilde{f}'_B \wedge \cdots \wedge \tilde{f}_B^{(k)}\colon M \to \wedge^{k+1}\mathbb{C}^{n+1},$$

and, correspondingly, the generated kth associated map f_k of f is defined by

$$f_k := [\tilde{f}_k]\colon M \to \mathbb{P}(\wedge^{k+1}\mathbb{C}^{n+1}) = \mathbb{P}^{n_k}(\mathbb{C}) \quad \text{for } n_k = \binom{n+1}{k+1} - 1. \tag{2.7}$$

We now give some general conditions under which we shall proceed.

($\mathcal{A}_1$) (M,τ,ω) denotes a generalized p-parabolic manifold which possesses a globally defined meromorphic $(m-1,0)$-form B such that, for any linearly non-degenerate meromorphic map $f\colon M \to \mathbb{P}^n(\mathbb{C})$, the kth associated map f_k is well defined for $k = 0,1,\dots,n$, where we set $f_0 := f$ and where f_n is constant.

($\mathcal{A}_2$) There exists a Hermitian holomorphic line bundle $(\mathfrak{L},\hbar)$ that admits a holomorphic section μ such that, for some increasing function $Y(\tau)$, we have

$$m i_{m-1}|\mu|_{\hbar}^2 B \wedge \bar{B} \leqslant Y(\tau)(dd^c\tau)^{p-1} \wedge \omega^{m-p}, \tag{2.8}$$

where we write the index

$$i_{m-1} := \left(\frac{\mathrm{i}}{2\pi}\right)^{m-1}(m-1)!(-1)^{(m-1)(m-2)/2}.$$

Remark 2.2. Note here that the existence of B is guaranteed only if we choose it to be meromorphic rather than holomorphic. The reason for the second assumption is that, as we define the associated maps via a meromorphic form B, we need to compensate its singularities by some holomorphic section of some holomorphic line bundle that vanishes exactly at those points. Furthermore, for an affine algebraic variety (which is parabolic), or an algebraic vector bundle over an affine algebraic variety or its projectivization (of rank $r \geqslant 2$, if needed, which are p-parabolic), B can be given rationally, $(\mathfrak{L},\hbar)$ can be chosen to be algebraic, A_p (see (3.7)) is algebraic, and $Y(\tau) \leqslant (1+\tau)^{n_M}$ for some positive integer n_M (see [**21**, Remark 6.7 and Corollary 7.2]). In particular, under the hypothesis of Theorem 1.2, conditions ($\mathcal{A}_1$) and ($\mathcal{A}_2$) are automatically satisfied.

3. Value distribution theory

In this section we give some fundamental notation, facts and results in value distribution theory on generalized p-parabolic manifolds following [**20**, **21**].

Let $f\colon M \to \mathbb{P}^n(\mathbb{C})$ be an algebraically non-degenerate meromorphic map defined on a generalized p-parabolic manifold such that $\dim M = m$ and $1 \leqslant p \leqslant m$, let ω_{FS} be the Fubini–Study metric on $\mathbb{P}^n(\mathbb{C})$ and let $D \in \mathbb{P}^n(\mathbb{C})$ be a hypersurface of degree d. The characteristic function of f, the proximity function and the counting function of f with respect to D are defined, respectively, for a fixed $s > 0$ and any $r > s$, as

$$T_f(r,s) := \int_s^r \frac{\mathrm{d}t}{t^{2p-1}} \int_{M[t]} f^*\omega_{\mathrm{FS}} \wedge (dd^c\tau)^{p-1} \wedge \omega^{m-p}, \tag{3.1}$$

$$m_f(D;r) := \frac{1}{2}\int_{M\langle r\rangle} \log\frac{1}{\|f;D\|^2}\sigma, \tag{3.2}$$

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and

$$N_f(D;r,s) := \int_s^r \frac{\mathrm{d}t}{t^{2p-1}} \int_{M[t]} \theta_f^D \wedge (dd^c\tau)^{p-1} \wedge \omega^{m-p}, \tag{3.3}$$

where we write the globally well-defined norm

$$\|f;D\| := \frac{|\langle f,D\rangle|}{\|f\|^d\|D\|}, \tag{3.4}$$

which falls into the closed region $\mathbb{R}[0,1]$, such that, on the local holomorphic coordinate chart (z,U_z),

$$\|f\||_{U_z} := \sqrt{\sum_{s=0}^{n} |\hat{f}_s|^2}$$

for a reduced representation $\tilde{f}\colon = (\hat{f}_0, \hat{f}_1, \dots, \hat{f}_n)$ of f on (z,U_z), and

$$\|D\| := \sqrt{\sum |\hat{d}_t|^2}$$

for the coefficients $\hat{d}_t$ of the homogenous form $\tilde{D}$ defining D, and set

$$\theta_f^D|_{U_z} := \operatorname{div} \tilde{D}(\tilde{f})|_{U_z} = \operatorname{div}\langle f,D\rangle|_{U_z} = dd^c[\log|\langle f,D\rangle||^2_{U_z}]$$

in the sense of currents by the classical Poincaré–Lelong formula.

From the Green–Jensen formula, we immediately derive the following.

Theorem 3.1 (first main theorem). *Let $f\colon M \to \mathbb{P}^n(\mathbb{C})$ be a non-constant meromorphic map defined on a generalized p-parabolic manifold M, and let $D \in \mathbb{P}^n(\mathbb{C})$ be a hypersurface of degree d such that $f(M) \not\subseteq D$. Then, for $r > s > 0$, we have*

$$dT_f(r,s) = N_f(D;r,s) + m_f(D;r) - m_f(D;s). \tag{3.5}$$

Proof. The proof is just standard. Actually, on the local holomorphic coordinate chart (z,U_z), by (3.4), it follows that

$$dd^c[\log\|f;D\||^2_{U_z}] = dd^c[\log|\langle f,D\rangle||^2_{U_z}] - ddd^c[\log\|f\||^2_{U_z}],$$

which yields the conclusion by proper integration and application of the Green–Jensen formula, since $dd^c[\log\|f\||^2_{U_z}] = f^*\omega_{\mathrm{FS}}|_{U_z}$ by definition. □

Then, the defect of f with respect to the hypersurface D is defined as

$$\delta(D,f) := \liminf_{r\to+\infty} \frac{m_f(D;r)}{dT_f(r,s)} = 1 - \limsup_{r\to+\infty} \frac{N_f(D;r,s)}{dT_f(r,s)}. \tag{3.6}$$

Now, for each $0 \leqslant k \leqslant n-1$, we turn to a linearly non-degenerate meromorphic map on a generalized p-parabolic manifold M of dimension m with $1 \leqslant p \leqslant m$ satisfying the general conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, and define an important auxiliary function

$$\varPsi_k := \frac{mi_{m-1} f_k^*\omega_{\mathrm{FS}}^k \wedge B \wedge \bar{B}}{(dd^c\tau)^p \wedge \omega^{m-p}} = \frac{\|\tilde{f}_{k-1}\|^2\|\tilde{f}_{k+1}\|^2}{\|\tilde{f}_k\|^4}\frac{1}{A_p}, \tag{3.7}$$

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where ω_{FS}^k is the Fubini–Study metric on $\mathbb{P}(\wedge^{k+1}\mathbb{C}^{n+1})$, and A_p, $1 \leqslant p \leqslant m$, is the pth symmetric polynomial of the matrix $(\tau_{a\bar{b}})$ with respect to the Kähler metric ω. Actually, A_1 is the trace of $(\tau_{a\bar{b}})$, while A_m is the determinant $\det(\tau_{a\bar{b}})$ (>0).

The corresponding analytic Plücker formula now shows the following.

Theorem 3.2 (Wong and Wong [21, Theorem 5.2]). *Let $f\colon M \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate meromorphic map on a generalized p-parabolic manifold M of dimension m with $1 \leqslant p \leqslant m$ satisfying the general conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$. Then, for $r > s > 0$ and $0 \leqslant k \leqslant n-1$, we have*

$$N_k(r,s) + T_{f_{k-1}}(r,s) + T_{f_{k+1}}(r,s) - 2T_{f_k}(r,s) - \mathrm{Ric}_p(r,s) = S_k(r,s), \tag{3.8}$$

where

$$S_k(r,s) := \frac{1}{2}\int_{M\langle r\rangle} \log \Psi_k \sigma - \frac{1}{2}\int_{M\langle s\rangle} \log \Psi_k \sigma, \tag{3.9}$$

$T_{f_k}(r,s)$ denotes the characteristic function of f_k, the kth associated map of f, such that $T_{f_{-1}}(r,s) = T_{f_n}(r,s) = 0$ and $T_{f_0}(r,s) = T_f(r,s)$, $N_k(r,s)$ denotes the counting function of the kth globally defined stationary divisor

$$\varsigma_k := d_{k+1} + d_{k-1} - 2d_k \tag{3.10}$$

such that $d_{-1} + d_0 = 0$, where d_k is a global divisor defined, on the local holomorphic coordinate chart (z, U_z), by some meromorphic function g_k such that $d_k|_{U_z} = \operatorname{div} g_k$ with $\tilde{f}_k|_{U_z} = g_k \xi_k$ for a reduced representation ξ_k of f_k on (z, U_z), and $\mathrm{Ric}_p(r,s)$ denotes the counting function of $\operatorname{div} A_p$ for the holomorphic function A_p, that is,

$$\mathrm{Ric}_p(r,s) := \int_s^r \frac{\mathrm{d}t}{t^{2p-1}} \int_{M[t]} \theta_{A_p}^0 \wedge (dd^c\tau)^{p-1} \wedge \omega^{m-p}. \tag{3.11}$$

Then, summing up (3.8) with the telescoping trick yields the following.

Corollary 3.3 (Wong and Wong [21, Lemma 5.4]). *Under the assumptions of Theorem 3.2 and for $r > s > 0$, we have*

$$N_{\mathrm{Ram}\,f}(r,s) - (n+1)T_f(r,s) - \tfrac{1}{2}n(n+1)\,\mathrm{Ric}_p(r,s) = \sum_{k=0}^{n-1}(n-k)S_k(r,s), \tag{3.12}$$

where $N_{\mathrm{Ram}\,f}(r,s)$ is the counting function of the ramification divisor $\operatorname{div} \tilde{f}_n$.

Next, we prove a version of the calculus lemma that suffices for us.

Proposition 3.4 (Wong and Stoll [20, Corollary 2.4]). *Let h be a non-negative measurable function defined on a generalized p-parabolic manifold M of dimension m with $1 \leqslant p \leqslant m$ such that $h(dd^c\tau)^p \wedge \omega^{m-p}$ is locally integrable, and let $s > 0$ be such that*

$$T(r) := \int_s^r \frac{\mathrm{d}t}{t^{2p-1}} \int_{M[t]} h(dd^c\tau)^p \wedge \omega^{m-p} \tag{3.13}$$

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is well defined for all $r \geqslant s$. Then, $h\sigma$ and $(\log h)\sigma$ are integrable over $M\langle r\rangle$ for almost all $r \geqslant s$ such that, for the constant κ given in (2.5), we have

$$\int_{M\langle r\rangle} (\log h)\sigma \mathrel{.\leqslant.} \kappa \log^{+} \frac{1}{\kappa} \int_{M\langle r\rangle} h\sigma \mathrel{.\leqslant.} \kappa(1+\varepsilon)(\log^{+} T(r) + \log^{+}\log^{+} r), \tag{3.14}$$

where $\varepsilon > 0$ is arbitrary, and the notation '$.\leqslant.$' indicates that the estimate holds outside a set of positive real numbers in $\mathbb{R}^{+}$ with finite Lebesgue measure.

Proof. Following Wong and Stoll (see [**20**, Corollary 2.4]), we can see that

$$r^{2p-1}\frac{\mathrm{d}}{\mathrm{d}r}T(r) = \int_{M[r]} h(dd^{c}\tau)^{p} \wedge \omega^{m-p}, \tag{3.15}$$

so that, by (2.4) for the case $j = p$, which yields

$$(dd^{c}\tau)^{p} \wedge \omega^{m-p} = p\tau^{p}\,\mathrm{d}\phi \wedge d^{c}\phi \wedge (dd^{c}\phi)^{p-1} \wedge \omega^{m-p} = p\tau^{p-1}\,\mathrm{d}\tau \wedge \sigma, \tag{3.16}$$

and Fubini's theorem, we obtain

$$\int_{M[r]} h(dd^{c}\tau)^{p} \wedge \omega^{m-p} = 2p\int_{0}^{r} t^{2p-1}\,\mathrm{d}t \int_{M\langle t\rangle} h\sigma, \tag{3.17}$$

which implies that $h\sigma$ is integrable over $M\langle r\rangle$ for almost all $r > 0$, and hence, by the concavity of the logarithmic function, so is $(\log h)\sigma$. In addition, $r^{2p-1}\mathrm{d}T(r)/\mathrm{d}r$ is absolutely continuous by (3.15) and (3.17), so that it is differentiable. Hence, we can follow [**20**, Corollary 2.3] and its equality (2.8) to get the desired result. □

From Theorem 3.2 and Proposition 3.4, we can estimate $T_{f_k}(r,s)$ in terms of $T_f(r,s)$. Actually, we can get a more precise estimate analogous to its counterpart as proved in [**20**, Corollary 3.9]. However, for our purpose, the following one is adequate.

Proposition 3.5 (Stoll [17, Proposition 10.9]). *Under the assumptions of Theorem 3.2, and for $r > s > 0$ and $0 \leqslant k \leqslant n-1$, it follows that*

$$T_{f_k}(r,s) \mathrel{.\leqslant.} 3^{k}T_f(r,s) + \tfrac{1}{2}(3^{k}-1)(m_0(\mathfrak{L};r,s) + Q_0(r,s)), \tag{3.18}$$

where, for μ and $Y(\tau)$ stated in (2.8), we write

$$m_0(\mathfrak{L};r,s) := \frac{1}{2}\int_{M\langle r\rangle} \log\frac{1}{|\mu|_{\hbar}^{2}}\sigma - \frac{1}{2}\int_{M\langle s\rangle} \log\frac{1}{|\mu|_{\hbar}^{2}}\sigma \tag{3.19}$$

and $Q_0(r,s) := \mathrm{Ric}_p(r,s) + \kappa\log^{+} Y(r^2) + \kappa\log^{+}\log^{+} r$.

Proof. In order to explore the method in [**17**, Proposition 10.9], we need to check that $N_k(r,s) \geqslant 0$ in (3.8), as here we get the associated maps f_k through a meromorphic $(m-1,0)$-form B. For this reason, we need the following equality:

$$\Psi_k = \frac{\|\tilde{f}_{k-1}\|^{2}\|\tilde{f}_{k+1}\|^{2}}{\|\tilde{f}_k\|^{4}}\frac{1}{A_p} = \frac{\|\tilde{f}_k \wedge \tilde{f}_k'\|^{2}}{\|\tilde{f}_k\|^{4}}\frac{1}{A_p} \quad (0 \leqslant k \leqslant n-1). \tag{3.20}$$

Here the first equality is given by [**21**, (5.4)]. The second one follows from [**18**, Lemma 6.1]. For completeness, we outline the proof. Actually,

$$\tilde{f}_k = \tilde{f} \wedge \tilde{f}'_B \wedge \cdots \wedge \tilde{f}_B^{(k-1)} \wedge \tilde{f}_B^{(k)} \quad \text{and} \quad \tilde{f}'_k = \tilde{f} \wedge \tilde{f}'_B \wedge \cdots \wedge \tilde{f}_B^{(k-1)} \wedge \tilde{f}_B^{(k+1)}.$$

Also, by the Gram–Schmidt method, there is an orthonormal basis $e_0, e_1, \dots, e_n$ of $\mathbb{C}^{n+1}$ such that

$$\tilde{f}_B^{(u)} = \sum_{l=0}^{u} a_{u,l} e_l \quad \text{for } u = 0, 1, \dots, k+1,$$

which implies that

$$\begin{aligned}
&\tilde{f}_v = a_{0,0} a_{1,1} \cdots a_{v,v} e_0 \wedge e_1 \wedge \cdots \wedge e_v \quad \text{for } v = k-1,\ k \text{ or } k+1,\\
&\tilde{f}'_k = a_{0,0} a_{1,1} \cdots a_{k-1,k-1}(a_{k+1,k} e_0 \wedge e_1 \wedge \cdots \wedge e_k + a_{k+1,k+1} e_0 \wedge \cdots \wedge e_{k-1} \wedge e_{k+1}),
\end{aligned}$$

and

$$\tilde{f}_k \wedge \tilde{f}'_k = (-1)^{k(k-1)/2} a_{0,0}^2 a_{1,1}^2 \cdots a_{k-1,k-1}^2 a_{k,k} a_{k+1,k+1} e_0 \wedge e_1 \wedge \cdots \wedge e_k \wedge e_{k+1},$$

so that

$$\begin{aligned}
\|\tilde{f}_k \wedge \tilde{f}'_k\| &= |a_{0,0}^2 a_{1,1}^2 \cdots a_{k-1,k-1}^2 a_{k,k} a_{k+1,k+1}|\\
&= |a_{0,0} a_{1,1} \cdots a_{k-1,k-1}||a_{0,0} a_{1,1} \cdots a_{k-1,k-1} a_{k,k} a_{k+1,k+1}|\\
&= \|\tilde{f}_{k-1}\| \|\tilde{f}_{k+1}\|.
\end{aligned}$$

Now, on the local holomorphic coordinate chart (z, U_z), we write $\tilde{f}_k|_{U_z} = g_k \xi_k$ for a reduced representation ξ_k of f_k on (z, U_z), that is,

$$\dim\{\xi_k^{-1}\{0\} \cup \xi_k^{-1}\{\infty\}\} \leqslant m-2.$$

It then follows immediately that $\xi_k \wedge \xi'_k = g\xi$ for a reduced representation ξ of $f_k \wedge f'_k$ on (z, U_z) such that $g^{-1}\{\infty\} \subseteq \xi_k^{-1}\{\infty\}$ so that g is a holomorphic function by the Remmert Extension Theorem. Hence, the proof of [**17**, Lemma 7.2], along with our equality (3.20), applies to show that the kth stationary divisor defined by (3.10) satisfies

$$\varsigma_k = \operatorname{div} g, \tag{3.21}$$

so that $N_k(r, s) \geqslant 0$, where by abuse of notation $\xi_k^{-1}\{0\}$ and $\xi_k^{-1}\{\infty\}$ denote the sets of common zeros and poles of the ξ_k coordinate functions, respectively.

On the other hand, we need to estimate the term $S_k(r, s)$ as in [**17**, Proposition 10.8]. As a matter of fact, by (2.8) and (3.7), it is straightforward to show that

$$\begin{aligned}
\int_s^r \frac{\mathrm{d}t}{t^{2p-1}} \int_{M[t]} \Psi_k |\mu|_\hbar^2 (dd^c\tau)^p \wedge \omega^{m-p} &= \int_s^r \frac{\mathrm{d}t}{t^{2p-1}} \int_{M[t]} m i_{m-1} |\mu|_\hbar^2 f_k^* \omega_{\mathrm{FS}}^k \wedge B \wedge \bar{B}\\
&\leqslant \int_s^r \frac{\mathrm{d}t}{t^{2p-1}} \int_{M[t]} Y(\tau) f_k^* \omega_{\mathrm{FS}}^k \wedge (dd^c\tau)^{p-1} \wedge \omega^{m-p}\\
&\leqslant Y(r^2) T_{f_k}(r, s),
\end{aligned} \tag{3.22}$$

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which, together with (3.9), (3.14) for $\varepsilon > 0$ and (3.19), gives the following estimate:

$$\begin{aligned}
S_k(r,s) &= m_0(\mathfrak{L};r,s) + \frac{1}{2}\int_{M\langle r\rangle} \log(\Psi_k|\mu|^2_{\hbar})\sigma - \frac{1}{2}\int_{M\langle s\rangle} \log(\Psi_k|\mu|^2_{\hbar})\sigma \\
&\mathrel{.{\leqslant}.} m_0(\mathfrak{L};r,s) + \tfrac{1}{4}\kappa(2+\varepsilon)(\log^+(Y(r^2)T_{f_k}(r,s)) + \log^+\log^+ r) \\
&\qquad - \frac{1}{2}\int_{M\langle s\rangle} \log(\Psi_k|\mu|^2_{\hbar})\sigma \\
&\mathrel{.{\leqslant}.} m_0(\mathfrak{L};r,s) + \tfrac{1}{2}\kappa(1+\varepsilon)(\log^+ Y(r^2) + \log^+ T_{f_k}(r,s) + \log^+\log^+ r). \qquad (3.23)
\end{aligned}$$

Hence, for $\varepsilon = 1$, noting that $N_k(r,s) \geqslant 0$, by (3.8) we can derive

$$\begin{aligned}
T_{f_{k+1}}(r,s) &\mathrel{.{\leqslant}.} 2T_{f_k}(r,s) + \mathrm{Ric}_p(r,s) + S_k(r,s) \\
&\mathrel{.{\leqslant}.} 3T_{f_k}(r,s) + m_0(\mathfrak{L};r,s) + \mathrm{Ric}_p(r,s) + \kappa\log^+ Y(r^2) + \kappa\log^+\log^+ r, \qquad (3.24)
\end{aligned}$$

which implies the desired estimate (3.18) by a routine induction. □

As stated earlier, application of the proof in [**20**, Corollary 3.9] gives a more precise estimate of $T_{f_k}(r,s)$ than (3.18). However, we only need it for the situation for $\log^+ T_{f_k}(r,s)$, otherwise (3.18) suffices. On the other hand, from (3.10), (3.21) and the telescoping trick, the ramification divisor $\operatorname{div}\tilde{f}_n$ is non-negative, so that $N_{\mathrm{Ram}f}(r,s) \geqslant 0$.

Finally, the corresponding Ahlfors–Stoll estimate gives the following result.

Theorem 3.6 (Wong and Wong [21, Theorem 6.8]). *Let $f\colon M \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate meromorphic map defined on a generalized p-parabolic manifold M of dimension m with $1 \leqslant p \leqslant m$ satisfying the general conditions ($\mathcal{A}_1$) and ($\mathcal{A}_2$), and let $H \in \mathbb{P}^n(\mathbb{C})$ be a hyperplane. Then, for any real number $0 < \lambda < 1$, and for $r > s > 0$ and $0 \leqslant k \leqslant n-1$, we have*

$$\int_s^r \frac{\mathrm{d}t}{t^{2p-1}} \int_{M[t]} |\mu|^2_{\hbar} \frac{\|f_{k+1};H\|^2}{\|f_k;H\|^{2-2\lambda}} \Psi_k (dd^c\tau)^p \wedge \omega^{m-p} \mathrel{.{\leqslant}.} Y(r^2)\left(\frac{\lambda+2}{\lambda}T_{f_k}(r,s) + \frac{2\log 2}{\lambda}\right), \qquad (3.25)$$

where (see (2.7)), on the local holomorphic coordinate chart (z, U_z), we define

$$f_k(z) \llcorner H|_{U_z} := \langle \tilde{f}_k(z), \tilde{H}\rangle|_{U_z} \colon \wedge^{k+1}\mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \wedge^k\mathbb{C}^{n+1} \qquad (3.26)$$

basically by operation between dual bases, and write the global norm

$$\|f_k;H\| := \frac{|f_k \llcorner H|}{\|f_k\|\|H\|} \in \mathbb{R}[0,1], \qquad (3.27)$$

*which is given by [**20**, (1.17)] or [**21**, (6.3)].*

An immediate consequence of Theorem 3.6 is the following result.

Corollary 3.7. *Under the assumptions of Theorem 3.6, for any real numbers $\varepsilon > 0$ and $0 < \lambda < 1$, and for $r \geqslant s > 0$ and $0 \leqslant k \leqslant n-1$, it follows that*

$$\log^+ \int_{M\langle r\rangle} |\mu|_{\hbar}^2 \frac{\|f_{k+1}; H\|^2}{\|f_k; H\|^{2-2\lambda}} \Psi_k \sigma \mathrel{.{\leqslant}.} (1+\varepsilon)(\log^+ T_f(r,s) + Q_1(r,s)), \tag{3.28}$$

where

$$Q_1(r,s) := \log^+ m_0(\mathfrak{L}; r,s) + \log^+ Y(r^2) + \log^+ \mathrm{Ric}_p(r,s) + \log^+ \log^+ r.$$

Proof. From (3.25), and Propositions 3.4 and 3.5, we derive that

$$\begin{aligned}\log^+ \int_{M\langle r\rangle} |\mu|_{\hbar}^2 \frac{\|f_{k+1}; H\|^2}{\|f_k; H\|^{2-2\lambda}} \Psi_k \sigma &\mathrel{.{\leqslant}.} \frac{2+\varepsilon}{2}(\log^+(Y(r^2) T_{f_k}(r,s)) + \log^+ \log^+ r) \\ &\mathrel{.{\leqslant}.} (1+\varepsilon)(\log^+ T_f(r,s) + Q_1(r,s)).\end{aligned}$$

□

4. A second main theorem

In this section, using the notation, facts and results in §§2 and 3, we give a second main theorem, slightly improving on that of Wong and Wong [**21**, Theorem 7.1], using the main idea from [**12**] (see also [**10**, **19**]).

Theorem 4.1 (second main theorem). *Let $f\colon M \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate meromorphic map defined on a generalized p-parabolic manifold M satisfying the general conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, and let $H_j \in \mathbb{P}^n(\mathbb{C})$ be q arbitrary hyperplanes for $j = 1, 2, \dots, q$. Then, for $r > s > 0$, we have*

$$\begin{aligned}&\int_{M\langle r\rangle} \max_K \log \prod_{j\in K} \frac{1}{\|f; H_j\|}\sigma \\ &\quad \mathrel{.{\leqslant}.} (n+1)T_f(r,s) - N_{\mathrm{Ram}\,f}(r,s) \\ &\qquad + \tfrac{1}{2}n(n+1)m_0(\mathfrak{L}; r,s) + \tfrac{1}{2}n(n+1)\,\mathrm{Ric}_p(r,s) + \tfrac{1}{2}\kappa n(n+1)\log^+ T_f(r,s) \\ &\qquad + \tfrac{1}{2}\kappa n(n+1)(\log^+ m_0(\mathfrak{L}; r,s) + \log^+ Y(r^2) + \log^+ \mathrm{Ric}_p(r,s) + \log^+ r),\end{aligned} \tag{4.1}$$

where the maximum is taken over all subsets K of $\{1, 2, \dots, q\}$ such that the generating linear forms of the hyperplanes in each set are linearly independent.

Proof. Let $K \subseteq \{1, 2, \dots, q\}$ be such that the corresponding linear forms are linearly independent. Without loss of generality, we set $q > n$ and $\#K = n+1$. Now, let $\mathfrak{T}$ be the set of all injective maps $\iota\colon \{0, 1, \dots, n\} \to \{1, 2, \dots, q\}$ such that the linear forms $\tilde{H}_{\iota(0)}, \tilde{H}_{\iota(1)}, \dots, \tilde{H}_{\iota(n)}$ are linearly independent. Furthermore, we write

$$\Gamma := \max_{1\leqslant j\leqslant q} \left\{ \sum_{k=0}^{n-1} m_{f_k}(H_j; s) \right\},$$

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a constant, and

$$\Lambda(r) := \min_{0\leqslant k\leqslant n-1}\left\{\frac{1}{1+T_{f_k}(r,s)}\right\} \ (<1),$$

where $T_{f_k}(r,s)$ is the characteristic function of the kth associated map of f given by (2.7), and $m_{f_k}(H;s)$ is defined by (3.2) only switching the norm (3.4) to (3.27).

Then, for the case where the hyperplanes involved are located in general position, the Ru–Stoll product-to-sum estimate (see [**12**, Lemma 3.2])

$$\prod_{j=1}^{q}\frac{1}{\|f;H_j\|}\leqslant c_q\sum_{\iota\in\mathfrak{T}}\prod_{v=0}^{n}\frac{1}{\|f;H_{\iota(v)}\|} \tag{4.2}$$

will immediately yield Wong and Wong's second main theorem (see [**21**, Theorem 7.1]) in a slightly more precise form, since the estimate

$$\sum_{k=0}^{n-1}(n-k)S_k(r,s) \ .\!\leqslant\!. \ O(\log^+(rT_f(r,s)))$$

which Wong and Wong applied for their proof does not necessarily hold in general, where the constant c_q depends only on the q hyperplanes H_j. Also, from the Green–Jensen formula, we see that, by definition, the term $m_0(\mathfrak{L};r,s)$ is equal to

$$m_0(\mathfrak{L};r,s)=T(\mathfrak{L};r,s)-N(\theta_\mu^0;r,s), \tag{4.3}$$

where $T(\mathfrak{L};r,s)$ is defined via the pull-back of the first Chern form on $(\mathfrak{L},\hbar)$.

Below, we follow [**10**, Theorem 2.1] to outline the proof.

For any $\iota\in\mathfrak{T}$ and fixed $z\notin\mathfrak{I}_f$, the indeterminacy of f, we can apply the Wong–Stoll product-to-sum estimate (see [**20**, Lemma 1.12]), that is, for some $d_k>1$,

$$\prod_{v=0}^{n}\frac{\|f_{k+1}(z);H_{\iota(v)}\|^2}{\|f_k(z);H_{\iota(v)}\|^{2-2\Lambda(r)}}\leqslant d_k\left(\sum_{v=0}^{n}\frac{\|f_{k+1}(z);H_{\iota(v)}\|^2}{\|f_k(z);H_{\iota(v)}\|^{2-2\Lambda(r)}}\right)^{n-k}, \tag{4.4}$$

in order to derive that (noting that f_n is a constant), for some constant $d>1$,

$$\begin{aligned}\prod_{v=0}^{n}&\frac{1}{\|f(z);H_{\iota(v)}\|^2}\\ &=\prod_{k=0}^{n-1}\prod_{v=0}^{n}\left(\frac{\|f_{k+1}(z);H_{\iota(v)}\|^2}{\|f_k(z);H_{\iota(v)}\|^{2-2\Lambda(r)}}\frac{1}{\|f_k(z);H_{\iota(v)}\|^{2\Lambda(r)}}\right)\\ &\leqslant d\prod_{k=0}^{n-1}\left(\sum_{v=0}^{n}\frac{\|f_{k+1}(z);H_{\iota(v)}\|^2}{\|f_k(z);H_{\iota(v)}\|^{2-2\Lambda(r)}}\right)^{n-k}\prod_{k=0}^{n-1}\prod_{v=0}^{n}\frac{1}{\|f_k(z);H_{\iota(v)}\|^{2\Lambda(r)}}.\end{aligned} \tag{4.5}$$

As a consequence, we have

$$\begin{aligned}\int_{M\langle r\rangle}\max_{K}\log\prod_{j\in K}\frac{1}{\|f;H_j\|^2}\sigma &= \int_{M\langle r\rangle}\max_{\iota\in\mathfrak{T}}\log\prod_{v=0}^{n}\frac{1}{\|f;H_{\iota(v)}\|^2}\\ &\leqslant \sum_{k=0}^{n-1}\int_{M\langle r\rangle}\max_{\iota\in\mathfrak{T}}\log\left(\sum_{v=0}^{n}\frac{\|f_{k+1};H_{\iota(v)}\|^2}{\|f_k;H_{\iota(v)}\|^{2-2\Lambda(r)}}\right)^{n-k}\sigma\\ &\quad+\sum_{k=0}^{n-1}\sum_{v=0}^{n}\int_{M\langle r\rangle}\max_{\iota\in\mathfrak{T}}\log\frac{1}{\|f_k;H_{\iota(v)}\|^{2\Lambda(r)}}\sigma+O(1).\end{aligned}\tag{4.6}$$

For the first term, by definitions of $S_k(r,s)$ and $m_0(\mathfrak{L};r,s)$, we see that

$$\begin{aligned}&\sum_{k=0}^{n-1}\int_{M\langle r\rangle}\max_{\iota\in\mathfrak{T}}\log\left(\sum_{v=0}^{n}\frac{\|f_{k+1};H_{\iota(v)}\|^2}{\|f_k;H_{\iota(v)}\|^{2-2\Lambda(r)}}\right)^{n-k}\sigma\\ &\qquad=\sum_{k=0}^{n-1}\int_{M\langle r\rangle}\max_{\iota\in\mathfrak{T}}\log\left(\sum_{v=0}^{n}|\mu|_{\hbar}^{2}\frac{\|f_{k+1};H_{\iota(v)}\|^2}{\|f_k;H_{\iota(v)}\|^{2-2\Lambda(r)}}\Psi_k\right)^{n-k}\sigma\\ &\qquad\quad-2\sum_{k=0}^{n-1}(n-k)(S_k(r,s)-m_0(\mathfrak{L};r,s))+O(1),\end{aligned}\tag{4.7}$$

which, together with (3.12), (3.14) (for $\varepsilon=1$) and (3.28), implies that

$$\begin{aligned}&\frac{1}{2}\sum_{k=0}^{n-1}\int_{M\langle r\rangle}\max_{\iota\in\mathfrak{T}}\log\left(\sum_{v=0}^{n}\frac{\|f_{k+1};H_{\iota(v)}\|^2}{\|f_k;H_{\iota(v)}\|^{2-2\Lambda(r)}}\right)^{n-k}\sigma\\ &\quad .\leqslant.\ \kappa\sum_{k=0}^{n-1}\tfrac{1}{2}(n-k)\max_{\iota\in\mathfrak{T}}\log^{+}\frac{1}{\kappa}\int_{M\langle r\rangle}\sum_{v=0}^{n}|\mu|_{\hbar}^{2}\frac{\|f_{k+1};H_{\iota(v)}\|^2}{\|f_k;H_{\iota(v)}\|^{2-2\Lambda(r)}}\Psi_k\sigma\\ &\qquad+\sum_{k=0}^{n-1}(n-k)m_0(\mathfrak{L};r,s)-\sum_{k=0}^{n-1}(n-k)S_k(r,s)+O(1)\\ &\quad .\leqslant.\ \kappa\sum_{k=0}^{n-1}\tfrac{1}{2}(n-k)\max_{1\leqslant j\leqslant q}\log^{+}\int_{M\langle r\rangle}|\mu|_{\hbar}^{2}\frac{\|f_{k+1};H_j\|^2}{\|f_k;H_j\|^{2-2\Lambda(r)}}\Psi_k\sigma\\ &\qquad+\kappa\tfrac{1}{4}n(n+1)\log q+\tfrac{1}{2}n(n+1)m_0(\mathfrak{L};r,s)\\ &\qquad+(n+1)T_f(r,s)+\tfrac{1}{2}n(n+1)\operatorname{Ric}_p(r,s)-N_{\mathrm{Ram}f}(r,s)+O(1)\\ &\quad .\leqslant.\ \tfrac{1}{2}\kappa n(n+1)(\log^{+}T_f(r,s)+\log^{+}m_0(\mathfrak{L};r,s)+\log^{+}Y(r^2)\\ &\qquad+\log^{+}\operatorname{Ric}_p(r,s)+\log^{+}\log^{+}r)+\tfrac{1}{2}n(n+1)m_0(\mathfrak{L};r,s)\\ &\qquad+(n+1)T_f(r,s)+\tfrac{1}{2}n(n+1)\operatorname{Ric}_p(r,s)-N_{\mathrm{Ram}f}(r,s)+O(1).\end{aligned}\tag{4.8}$$

For the second term, we need the following estimate:

$$T_{f_k}(r,s)\geqslant N_{f_k}(H;r,s)+m_{f_k}(H;r)-m_{f_k}(H;s),\tag{4.9}$$

which can be proved in exactly the same way as [**17**, (8.21)], via proper integration, as the proofs on pp. 151–153 of [**17**] only involve reduced representations of the associated maps. Thus, by definitions of Γ and $\Lambda(r)$, it follows from (4.9) that

$$\begin{aligned}\frac{1}{2}\sum_{k=0}^{n-1}\sum_{v=0}^{n}\int_{M\langle r\rangle}\max_{\iota\in\mathfrak{T}}\log\frac{1}{\|f_k;H_{\iota(v)}\|^{2\Lambda(r)}}\sigma \\ &\leqslant \sum_{\iota\in\mathfrak{T}}\sum_{k=0}^{n-1}\sum_{v=0}^{n}\Lambda(r)\int_{M\langle r\rangle}\log\frac{1}{\|f_k;H_{\iota(v)}\|}\sigma \\ &\leqslant q!\sum_{k=0}^{n-1}\sum_{j=0}^{q}\Lambda(r)m_{f_k}(H_j;r) \\ &\leqslant q!\sum_{k=0}^{n-1}\sum_{j=0}^{q}\Lambda(r)(T_{f_k}(r,s)+m_{f_k}(H_j;s))+O(1)=O(1), \qquad (4.10)\end{aligned}$$

which together with (4.8) yields the desired result. □

5. Chow weight and Hilbert weight

In this section, we state some necessary facts in algebraic geometry, and refer the reader to [**14**] and the references therein for more details.

Let $Y\subseteq\mathbb{P}^N(\mathbb{C})$ be a smooth complex projective algebraic variety of dimension n and degree Δ (for the case $Y=\mathbb{P}^N(\mathbb{C})$, it follows that $n=N$ and $\Delta=1$). Then, up to a constant scalar, we can associate it with a unique polynomial

$$P_Y(\boldsymbol{u}_0,\boldsymbol{u}_1,\dots,\boldsymbol{u}_n)=P_Y(u_{00},u_{01},\dots,u_{0N};\dots;u_{n0},u_{n1},\dots,u_{nN}) \qquad (5.1)$$

in $n+1$ blocks of $N+1$ variables $\boldsymbol{u}_i=(u_{i0},u_{i1},\dots,u_{iN})$ for $i=0,1,\dots,n$, which is called the (Cayley–Bertini-van der Waerden–)Chow form of Y such that

1. P_Y is irreducible in the ring $\mathbb{C}[u_{00},u_{01},\dots,u_{0N},u_{10},\dots,u_{n-1N},u_{n0},u_{n1},\dots,u_{nN}]$ of complex polynomials with $(n+1)(N+1)$ variables, and is homogenous of degree Δ in each block $\boldsymbol{u}_i=(u_{i0},u_{i1},\dots,u_{iN})$ for $i=0,1,\dots,n$; and

2. $P_Y(\boldsymbol{u}_0,\boldsymbol{u}_1,\dots,\boldsymbol{u}_n)=0$ if and only if $Y\cap H_{\boldsymbol{u}_0}\cap H_{\boldsymbol{u}_1}\cap\cdots\cap H_{\boldsymbol{u}_n}\neq\emptyset$, where, for $i=0,1,\dots,n$, $H_{\boldsymbol{u}_i}$ is the hyperplane generated by $\boldsymbol{u}_i$ as the zero set of

$$\tilde{H}_{\boldsymbol{u}_i}(\boldsymbol{w}):=\boldsymbol{u}_i\boldsymbol{w}=u_{i0}w_0+u_{i1}w_1+\cdots+u_{iN}w_N$$

for the coordinate $\boldsymbol{w}=(w_0,w_1,\dots,w_N)$ of $\mathbb{C}^{N+1}$.

Now, for a tuple of non-negative real numbers $\boldsymbol{c}=(c_0,c_1,\dots,c_N)\in\mathbb{R}^{N+1}_{\geqslant 0}$ and an auxiliary variable t, we consider the decomposition of Chow form as

$$\begin{aligned}P_Y(t^{c_0}u_{00},t^{c_1}u_{01},\dots,t^{c_N}u_{0N};\dots;t^{c_0}u_{n0},t^{c_1}u_{n1},\dots,t^{c_N}u_{nN}) \\ =t^{e_0}G_0(\boldsymbol{u}_0,\boldsymbol{u}_1,\dots,\boldsymbol{u}_n)+t^{e_1}G_1(\boldsymbol{u}_0,\boldsymbol{u}_1,\dots,\boldsymbol{u}_n)+\cdots \\ +t^{e_r}G_r(\boldsymbol{u}_0,\boldsymbol{u}_1,\dots,\boldsymbol{u}_n), \qquad (5.2)\end{aligned}$$

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where $G_0, G_1, \ldots, G_r \in \mathbb{C}[u_{00}, u_{01}, \ldots, u_{0N}, u_{10}, \ldots, u_{n-1N}, u_{n0}, u_{n1}, \ldots, u_{nN}]$ and $e_0 > e_1 > \cdots > e_r \geqslant 0$. Then, the Chow weight of Y with respect to $\boldsymbol{c}$ is defined by

$$e_Y(\boldsymbol{c}) := e_0. \tag{5.3}$$

On the other hand, for each distinct subset $J = \{j_0, j_1, \ldots, j_n\}$ of $\{0, 1, \ldots, N\}$ with the order $j_0 < j_1 < \cdots < j_n$, we define the bracket

$$[J] = [J](\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n) := \det(u_{ij_k})_{i,k=0,1,\ldots,n}, \tag{5.4}$$

where $\boldsymbol{u}_i = (u_{i0}, u_{i1}, \ldots, u_{iN})$ for $i = 0, 1, \ldots, n$. Then, for the integer

$$\alpha := \binom{N+1}{n+1},$$

we set $J_1, J_2, \ldots, J_\alpha$ to be all the subsets of $\{0, 1, \ldots, N\}$ with cardinality $n+1$. It is known that the Chow form P_Y can be rewritten as a homogenous polynomial of degree Δ in the variables $\{[J_1], [J_2], \ldots, [J_\alpha]\}$. Further, for any $J \in \{J_1, J_2, \ldots, J_\alpha\}$ and any $\boldsymbol{c} = (c_0, c_1, \ldots, c_N) \in \mathbb{R}^{N+1}_{\geqslant 0}$, we have

$$\begin{aligned} &[J](t^{c_0} u_{00}, t^{c_1} u_{01}, \ldots, t^{c_N} u_{0N}; \ldots; t^{c_0} u_{n0}, t^{c_1} u_{n1}, \ldots, t^{c_N} u_{nN}) \\ &\qquad = t^{\sum_{j \in J} c_j} [J](u_{00}, u_{01}, \ldots, u_{0N}; \ldots; u_{n0}, u_{n1}, \ldots, u_{nN}). \end{aligned} \tag{5.5}$$

Note that for the case $Y = \mathbb{P}^N(\mathbb{C})$, it is trivially a scalar-multiplication property of the determinant of the matrix $(u_{ij})_{i,j=0,1,\ldots,N}$ (see [**9**, Theorem IV, p. 41]).

Next, we denote by $\mathcal{I}_Y$ the prime ideal in $\mathbb{C}[w_0, w_1, \ldots, w_N]$ defining Y ($\mathcal{I}_Y = 1$ for the case $Y = \mathbb{P}^N(\mathbb{C})$), and set $\mathbb{C}[w_0, w_1, \ldots, w_N]_m$ to represent the vector space of homogenous polynomials of degree m (including 0) in $\mathbb{C}[w_0, w_1, \ldots, w_N]$. For $\mathcal{I}_m := \mathcal{I}_Y \cap \mathbb{C}[w_0, w_1, \ldots, w_N]_m$, the Hilbert function of Y is defined by

$$H_Y(m) := \dim_{\mathbb{C}}(\mathbb{C}[w_0, w_1, \ldots, w_N]_m / \mathcal{I}_m); \tag{5.6}$$

in other words, it is the dimension of the quotient vector space $\mathbb{C}[w_0, w_1, \ldots, w_N]_m / \mathcal{I}_m$ over $\mathbb{C}$. Then, from the general theory of Hilbert polynomials, it follows that

$$H_Y(m) = \Delta \frac{m^n}{n!} + O(m^{n-1}). \tag{5.7}$$

On the other hand, we define the mth Hilbert weight of Y with respect to a tuple of non-negative real numbers $\boldsymbol{c} = (c_0, c_1, \ldots, c_N) \in \mathbb{R}^{N+1}_{\geqslant 0}$ by

$$S_Y(m, \boldsymbol{c}) := \max\left\{ \sum_{l=1}^{H_Y(m)} \boldsymbol{a}_l \boldsymbol{c} = \sum_{l=1}^{H_Y(m)} \sum_{j=0}^{N} a_{lj} c_j \right\}, \tag{5.8}$$

where the maximum is taken over all sets, with cardinality $H_Y(m)$, of tuples of non-negative integers $\boldsymbol{a}_l = (a_{l0}, a_{l1}, \ldots, a_{lN}) \in \mathbb{N}^{N+1}_{\geqslant 0}$ with $\sum_{j=0}^{N} a_{lj} = m$, such that, for each corresponding set of monomials

$$\left\{ \boldsymbol{w}^{\boldsymbol{a}_l} := \sum_{j=0}^{N} w_j^{a_{lj}} \;\middle|\; l = 1, 2, \ldots, H_Y(m) \right\},$$

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its generated set of residue classes, module $\mathcal{I}_m$, forms a basis of $\mathbb{C}[w_0, w_1, \ldots, w_N]_m/\mathcal{I}_m$. Then, a result of Mumford (see [**11**, Proposition 2.11]) gives us

$$S_Y(m, \boldsymbol{c}) = e_Y(\boldsymbol{c})\frac{m^{n+1}}{(n+1)!} + O(m^n). \tag{5.9}$$

Now, combining (5.7) and (5.9) yields

$$\lim_{m\to\infty} \frac{1}{mH_Y(m)} S_Y(m, \boldsymbol{c}) = \frac{1}{\Delta(n+1)} e_Y(\boldsymbol{c}), \tag{5.10}$$

where the two terms

$$\frac{S_Y(m, \boldsymbol{c})}{mH_Y(m)} \quad \text{and} \quad \frac{e_Y(\boldsymbol{c})}{\Delta(n+1)}$$

are called the mth normalized Hilbert weight and the normalized Chow weight of Y with respect to $\boldsymbol{c}$, respectively.

Finally, we list two results, each giving a lower bound for the Chow weight and the Hilbert weight, respectively, that suffice for our purpose.

Proposition 5.1 (Ru [14, Theorem 2.1]; Evertse and Ferretti [6, Theorem 4.1]). *Let $Y \subseteq \mathbb{P}^N(\mathbb{C})$ be a smooth projective algebraic variety of dimension n and degree Δ. Then, for any positive integer m ($> \Delta$) and any $\boldsymbol{c} = (c_0, c_1, \ldots, c_N) \in \mathbb{R}^{N+1}_{\geqslant 0}$, we have*

$$\frac{1}{mH_Y(m)} S_Y(m, \boldsymbol{c}) \geqslant \frac{1}{\Delta(n+1)} e_Y(\boldsymbol{c}) - \frac{1}{m}\Delta(2n+1) \max_{0\leqslant j\leqslant N}\{c_j\}. \tag{5.11}$$

Proposition 5.2 (Ru [14, Lemma 3.2]). *Under the hypothesis of Proposition 5.1, for any distinct subset $\{j_0, j_1, \ldots, j_n\} \subseteq \{0, 1, \ldots, N\}$ and any tuple of non-negative real numbers $\boldsymbol{c} = (c_0, c_1, \ldots, c_N) \in \mathbb{R}^{N+1}_{\geqslant 0}$, and for the coordinate $\boldsymbol{w} = (w_0, w_1, \ldots, w_N)$ of $\mathbb{C}^{N+1}$, the condition $Y \cap \{w_{j_0} = w_{j_1} = \cdots = w_{j_n} = 0\} = \emptyset$ implies that*

$$\frac{1}{\Delta} e_Y(\boldsymbol{c}) \geqslant \sum_{k=0}^{n} c_{j_k}. \tag{5.12}$$

6. Second main theorem and defect relation

Finally, in this last section, we shall prove our main result. Conventionally, we first need to establish a second main theorem. The essential idea for its proof is that we can apply the algebraic-geometric method developed earlier to construct an isomorphism $\mathfrak{h}$ between some smooth complex projective algebraic variety Y that is a surjective finite morphism image of V, and a subvariety of some higher-dimensional complex projective space $\mathbb{P}^{n^Y}(\mathbb{C})$, so that we can take advantage of the value distribution properties of the associated linearly non-degenerate meromorphic map F.

Theorem 6.1 (second main theorem). *Let $f\colon M \to V \subseteq \mathbb{P}^N(\mathbb{C})$ be an algebraically non-degenerate meromorphic map from a generalized p-parabolic manifold M*

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that satisfies the general conditions $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$ *to a smooth projective algebraic variety* V *such that* $\dim V = n \geqslant 1$, *and let* $D_1, D_2, \dots, D_q \in \mathbb{P}^N(\mathbb{C})$ *be* q $(> n)$ *hypersurfaces, located in general position and of degrees* $d_1, d_2, \dots, d_q$, *respectively. Then, for any* $\varepsilon > 0$ *and* $r > s > 0$, *we have*

$$\sum_{j=1}^{q} \frac{1}{d_j} m_f(D_j; r) \mathrel{.\leqslant.} (n+1+\varepsilon) T_f(r,s) \\ + c_m(m_0(\mathfrak{L}; r, s) + \mathrm{Ric}_p(r,s) + \kappa \log^+ Y(r^2) + \kappa \log^+ r), \quad (6.1)$$

where the term 'in general position' is defined in Theorem 1.1, κ *is given by (2.5),* $Y(\tau)$ *by (2.8),* $\mathrm{Ric}_p(r,s)$ *by (3.11),* $m_0(\mathfrak{L}; r, s)$ *by (3.19) and* $c_m \gg 1$ *is a constant.*

Proof. Without loss of generality, we assume that all the q hypersurfaces D_j are of the same degree d, since otherwise replacing $\tilde{D}_j$ by $\tilde{D}_j^{d/d_j}$ would achieve this aim, where d is the least common multiplier of the d_j and $\tilde{D}_j$ is the homogenous form of degree d_j that defines D_j for $j = 1, 2, \dots, q$. Also, the 'in general position' hypothesis of the q hypersurfaces D_j on V shows that at any point $z \notin \mathfrak{I}_f$, the indeterminacy of f, $\|f(z); D_j\| = 0$ (see (3.4)) for at most n D_j values, while $\|f(z); D_j\| > 0$ for all the remaining ones, which implies that, by the continuity of f and the compactness of V, there exists a positive constant $C > 0$ such that $\|f; D_j\| > C$, except for at most n D_j values, on $M \setminus \mathfrak{I}_f$. As an immediate consequence, it yields that

$$\sum_{j=1}^{q} m_f(D_j; r) = \int_{M\langle r\rangle} \log \prod_{j=1}^{q} \frac{1}{\|f; D_j\|} \sigma \\ \leqslant \int_{M\langle r\rangle} \max_{\{j_0, j_1, \dots, j_n\} \subseteq \{1,2,\dots,q\}} \left\{ \log \prod_{k=0}^{n} \frac{1}{\|f; D_{j_k}\|} \right\} \sigma + O(1). \quad (6.2)$$

Now, we define a map

$$\psi \colon V \subseteq \mathbb{P}^N(\mathbb{C}) \to \mathbb{P}^{q-1}(\mathbb{C}) \quad \text{by} \quad \psi([\boldsymbol{x}]) =: [\tilde{D}_1(\boldsymbol{x}) \colon \tilde{D}_2(\boldsymbol{x}) \colon \cdots \colon \tilde{D}_q(\boldsymbol{x})] \quad (6.3)$$

for the coordinate $\boldsymbol{x} \in \mathbb{C}^{N+1}$, and write $Y := \psi(V)$ such that, from the 'in general position' assumption of the q hypersurfaces D_j on V, ψ is a well-defined surjective (and injective if $q = n+1$ and $d = 1$) finite morphism from V to Y, a smooth complex projective algebraic variety of $\mathbb{P}^{q-1}(\mathbb{C})$. Then we have

$$\dim Y = \dim V = n \quad \text{and} \quad \varDelta := \deg Y \leqslant d^n \deg V. \quad (6.4)$$

Our next step is to construct an isomorphism between Y and a subvariety in some higher-dimensional projective space. For this purpose, for a positive integer m, we use the Hilbert function for $Y \subseteq \mathbb{P}^{q-1}(\mathbb{C})$ to define

$$n_m^Y := H_Y(m) - 1 \quad \text{and} \quad q_m^Y := \binom{q+m-1}{m} - 1. \quad (6.5)$$

Now, for q_m^Y+1 tuples of non-negative integers $\boldsymbol{a}_l=(a_{l1},a_{l2},\dots,a_{lq})\in\mathbb{N}^q_{\geqslant 0}$ with $\sum_{j=1}^q a_{lj}=m$, and the coordinate $\boldsymbol{y}=(y_1,y_2,\dots,y_q)$ of $\mathbb{C}^q$, we write

$$\boldsymbol{y}^{\boldsymbol{a}_l}:=\sum_{j=1}^q y_j^{a_{lj}}\quad\text{for } l=0,1,\dots,q_m^Y.$$

Note that when the subscript l of $\boldsymbol{a}_l$ runs through $\{0,1,\dots,q_m^Y\}$, $\boldsymbol{y}^{\boldsymbol{a}_l}$ runs through the set containing all the possible homogenous monomials of degree m in the variables $\{y_1,y_2,\dots,y_q\}$. Then, the Veronese embedding theorem shows

$$\psi_m^Y\colon\mathbb{P}^{q-1}(\mathbb{C})\hookrightarrow\mathbb{P}^{q_m^Y}(\mathbb{C})\quad\text{such that } \psi_m^Y([\boldsymbol{y}])=[\boldsymbol{y}^{\boldsymbol{a}_0}\colon\boldsymbol{y}^{\boldsymbol{a}_1}\colon\ \cdots\colon\boldsymbol{y}^{\boldsymbol{a}_{q_m^Y}}],\tag{6.6}$$

where $\boldsymbol{y}^{\boldsymbol{a}_0},\boldsymbol{y}^{\boldsymbol{a}_1},\dots,\boldsymbol{y}^{\boldsymbol{a}_{q_m^Y}}$ are all the homogenous monomials of degree m in some order. Also, we set Y_m to be the smallest linear subvariety of $\mathbb{P}^{q_m^Y}(\mathbb{C})$ containing $\psi_m^Y(Y)$. It is straightforward to see that a linear form $\sum_{l=0}^{q_m^Y}\beta_l v_l$ vanishes identically on Y_m (and thus defining a hyperplane), for the coordinate $\boldsymbol{v}=(v_0,v_1,\dots,v_{q_m^Y})$ of $\mathbb{C}^{q_m^Y+1}$ if and only if $\sum_{l=0}^{q_m^Y}\beta_l\boldsymbol{y}^{\boldsymbol{a}_l}$, as a homogenous polynomial of degree m, vanishes identically on Y (and thus defining a hypersurface). In other words, there exists an isomorphism

$$\mathbb{C}[y_1,y_2,\dots,y_q]_m/(\mathcal{I}_Y)_m\simeq\tilde{Y}_m\quad\text{such that } \boldsymbol{y}^{\boldsymbol{a}_l}\leftrightarrow v_l \text{ for } l=0,1,\dots,q_m^Y,\tag{6.7}$$

where $\mathcal{I}_Y$ is the prime ideal in $\mathbb{C}[y_1,y_2,\dots,y_q]$ defining Y, $(\mathcal{I}_Y)_m$ is the vector space of homogenous polynomials of degree m in $\mathcal{I}_Y$ and $\tilde{Y}_m:=\mathbb{C}[v_0,v_1,\dots,v_{q_m^Y}]_1/\mathcal{I}_{Y_m}$ is the quotient vector space of the linear forms in $\mathbb{C}[v_0,v_1,\dots,v_{q_m^Y}]$ modulo the prime ideal $\mathcal{I}_{Y_m}$ in $\mathbb{C}[v_0,v_1,\dots,v_{q_m^Y}]_1$ generating Y_m. Hence, we have $\dim Y_m=\dim\tilde{Y}_m-1$ since Y_m is a linear subvariety of $\mathbb{P}^{q_m^Y}(\mathbb{C})$, which, together with the definition of $H_Y(m)$, (6.5) and (6.7), shows that Y_m is an n_m^Y-dimensional linear subspace of $\mathbb{P}^{q_m^Y}(\mathbb{C})$, so that there exist q_m^Y+1 linear forms $\tilde{L}_0,\tilde{L}_1,\dots,\tilde{L}_{q_m^Y}$ in $\mathbb{C}[w_0,w_1,\dots,w_{n_m^Y}]$ such that

$$\varphi_m^Y\colon\mathbb{P}^{n_m^Y}(\mathbb{C})\to Y_m\subseteq\mathbb{P}^{q_m^Y}\quad\text{for } \varphi_m^Y([\boldsymbol{w}]):=[\tilde{L}_0(\boldsymbol{w})\colon\tilde{L}_1(\boldsymbol{w})\colon\ \cdots\colon\tilde{L}_{q_m^Y}(\boldsymbol{w})]\tag{6.8}$$

is a linear isomorphism between $\mathbb{P}^{n_m^Y}(\mathbb{C})$ and Y_m, where $\boldsymbol{w}=(w_0,w_1,\dots,w_{n_m^Y})$ is the coordinate of $\mathbb{C}^{n_m^Y+1}$. Now, (6.6) and (6.8) yield an isomorphism between the smooth projective algebraic variety Y and a subvariety of $\mathbb{P}^{n_m^Y}(\mathbb{C})$ such as

$$\mathfrak{h}:=(\varphi_m^Y)^{-1}\circ\psi_m^Y\colon Y\to\mathbb{P}^{n_m^Y}(\mathbb{C}).\tag{6.9}$$

As a consequence, together with our assumptions on f and the q hypersurfaces D_j, (6.3) and (6.9) imply that

$$F:=\mathfrak{h}\circ\psi\circ f\colon M\to\mathbb{P}^{n_m^Y}(\mathbb{C})\tag{6.10}$$

is a linearly non-degenerate meromorphic map defined on the generalized p-parabolic manifold M that satisfies the general conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$.

In the following, for fixed $z\in M\setminus\mathfrak{I}_f$, we set $\boldsymbol{c}^z=(c_1^z,c_2^z,\dots,c_q^z)\in\mathbb{R}^q_{\geqslant 0}$ by

$$c_j^z:=\log\frac{1}{\|f(z);D_j\|}\geqslant 0\quad\text{for } j=1,2,\dots,q.\tag{6.11}$$

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By definitions of the Hilbert function and the Hilbert weight, and that of q_m^Y, there exists a subset $\mathcal{L}_z$, dependent on z, of $\{0,1,\dots,q_m^Y\}$ with cardinality $H_Y(m)=n_m^Y+1$ such that $\{\boldsymbol{y}^{\boldsymbol{a}_l}: l\in\mathcal{L}_z\}$ is a basis of $\mathbb{C}[y_1,y_2,\dots,y_q]_m/(\mathcal{I}_Y)_m$ and such that

$$S_Y(m,\boldsymbol{c}^z)=\sum_{l\in\mathcal{L}_z}\boldsymbol{a}_l\boldsymbol{c}^z. \tag{6.12}$$

Note that, for the coordinate $\boldsymbol{w}\in\mathbb{C}^{H_m(Y)}$, $\tilde{L}_l(\boldsymbol{w})=\boldsymbol{y}^{\boldsymbol{a}_l}$ follows by (6.6) and (6.8), which implies that, for $l=0,1,\dots,q_m^Y$, if L_l is the hyperplane generated by $\tilde{L}_l$, then, for a reduced representation $\tilde{f}$ of f, and the one $\tilde{F}$ of F, the identity

$$\langle F,L_l\rangle=\tilde{L}_l(\tilde{F})=\prod_{j=1}^{q}(\tilde{D}_j(\tilde{f}))^{a_{lj}}=\prod_{j=1}^{q}\langle f,D_j\rangle^{a_{lj}} \tag{6.13}$$

holds by (6.3) and (6.10). Hence, from (6.11), (6.12) and (6.13), and noting that $\#\mathcal{L}_z=H_Y(m)$ and

$$\sum_{j=1}^{q}a_{lj}=m\quad\text{for } l=0,1,\dots,q_m^Y,$$

we have

$$\begin{aligned}\log\prod_{l\in\mathcal{L}_z}\frac{\|L_l\|}{|\langle F(z),L_l\rangle|}&=\log\prod_{l\in\mathcal{L}_z}\prod_{j=1}^{q}\frac{1}{|\langle f(z),D_j\rangle|^{a_{lj}}}+O(H_Y(m))\\&=\log\prod_{l\in\mathcal{L}_z}\prod_{j=1}^{q}\left\{\frac{1}{\|f(z);D_j\|^{a_{lj}}}\left(\frac{1}{\|f(z)\|^d\|D_j\|}\right)^{a_{lj}}\right\}+O(H_Y(m))\\&=\sum_{l\in\mathcal{L}_z}\sum_{j=1}^{q}a_{lj}\left(\log\frac{1}{\|f(z);D_j\|}+\log\frac{1}{\|f(z)\|^d\|D_j\|}\right)+O(H_Y(m))\\&=\sum_{l\in\mathcal{L}_z}\boldsymbol{a}_l\boldsymbol{c}^z-d\log\|f(z)\|\left(\sum_{l\in\mathcal{L}_z}\sum_{j=1}^{q}a_{lj}\right)+O(H_Y(m))\\&=S_Y(m,\boldsymbol{c}^z)-dmH_Y(m)\log\|f(z)\|+O(H_Y(m)),\end{aligned}$$

which gives us the following estimate:

$$\begin{aligned}S_Y(m,\boldsymbol{c}^z)&\leqslant\max_{\mathcal{L}}\log\prod_{l\in\mathcal{L}}\frac{\|L_l\|}{|\langle F(z),L_l\rangle|}+dmH_Y(m)\log\|f(z)\|+O(H_Y(m))\\&=\max_{\mathcal{L}}\log\prod_{l\in\mathcal{L}}\frac{1}{\|F(z);L_l\|}-H_Y(m)\log\|F(z)\|\\&\quad+dmH_Y(m)\log\|f(z)\|+O(H_Y(m)),\end{aligned}\tag{6.14}$$

where the maximum is taken over all subsets $\mathcal{L}\subseteq\{0,1,\dots,q_m^Y\}$ with $\#\mathcal{L}=n_Y(m)+1$ such that $\tilde{L}_l$, $l\in\mathcal{L}$, are linearly independent.

Then, combining the conclusions of Proposition 5.1 with (6.14) yields

$$\begin{aligned}\frac{1}{\Delta(n+1)}e_Y(\boldsymbol{c}^z) &\leqslant \frac{1}{mH_Y(m)}S_Y(m,\boldsymbol{c}^z)+\frac{\Delta(2n+1)}{m}\max_{1\leqslant j\leqslant q}\{c_j^z\}\\ &\leqslant \frac{1}{mH_Y(m)}\biggl(\max_{\mathcal{L}}\log\prod_{l\in\mathcal{L}}\frac{1}{\|F(z);L_l\|}-H_Y(m)\log\|F(z)\|\biggr)\\ &\qquad + d\log\|f(z)\|+\frac{\Delta(2n+1)}{m}\max_{1\leqslant j\leqslant q}\{c_j^z\}+O\biggl(\frac{1}{m}\biggr). \end{aligned} \tag{6.15}$$

Also, the 'in general position' assumption of the q hypersurfaces D_j on V implies that the conclusions of Proposition 5.2 hold for any distinct subset $\{j_0,j_1,\dots,j_n\}$ of $\{1,2,\dots,q\}$, which together with (6.11) and (6.15) shows that

$$\begin{aligned}\log\prod_{k=0}^{n}&\frac{1}{\|f(z);D_{j_k}\|}\\ &= c_{j_0}^z+c_{j_1}^z+\cdots+c_{j_n}^z\\ &\leqslant \frac{1}{\Delta}e_Y(\boldsymbol{c}^z)\\ &\leqslant \frac{n+1}{mH_Y(m)}\biggl(\max_{\mathcal{L}}\log\prod_{l\in\mathcal{L}}\frac{1}{\|F(z);L_l\|}-H_Y(m)\log\|F(z)\|\biggr)\\ &\qquad + d(n+1)\log\|f(z)\|+\frac{\Delta(n+1)(2n+1)}{m}\max_{1\leqslant j\leqslant q}\{c_j^z\}+O\biggl(\frac{1}{m}\biggr)\\ &\leqslant \frac{n+1}{mH_Y(m)}\biggl(\max_{\mathcal{L}}\log\prod_{l\in\mathcal{L}}\frac{1}{\|F(z);L_l\|}-H_Y(m)\log\|F(z)\|\biggr)+O\biggl(\frac{1}{m}\biggr)\\ &\qquad + d(n+1)\log\|f(z)\|+\frac{\Delta(n+1)(2n+1)}{m}\max_{1\leqslant j\leqslant q}\biggl\{\log\frac{1}{\|f(z);D_j\|}\biggr\}. \end{aligned} \tag{6.16}$$

However, we now encounter a problem, that is, the functions $\log\|F(z)\|$ and $\log\|f(z)\|$ are usually not globally defined. In order to take integration over $M\langle r\rangle$, we need to avoid this difficulty. Hence, we resort to the concept of 'reduced representation sections' of F and f (see [**17**]). We only do this for F in detail, as the case for f is similar.

Set $\{\tilde F_\alpha, U_\alpha\}$ to be a system of local reduced representations of F such that, on $U_\alpha\cap U_\beta(\neq\emptyset)$, we have

$$\tilde F_\alpha = h_{\alpha\beta}\tilde F_\beta$$

for a non-vanishing holomorphic function $h_{\alpha\beta}\colon U_\alpha\cap U_\beta\to\mathbb{C}^*$. Then, $\{h_{\alpha\beta}\}$ forms a basic cocycle so that there exists a holomorphic line bundle $\mathcal{H}_F$ on M, with a holomorphic frame atlas $\{s_\alpha^F, U_\alpha\}$ such that, on $U_\alpha\cap U_\beta$, we have

$$s_\alpha^F = h_{\beta\alpha}s_\beta^F,$$

which is called the hyperplane section bundle of F. Now, define a holomorphic section

$$\tilde F_\alpha^*(z) := (z,\tilde F_\alpha(z))\in\Gamma(U_\alpha, M\times\mathbb{C}^{n_m^Y+1})$$

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such that, on $U_\alpha \cap U_\beta$, it satisfies

$$\tilde{F}_\alpha^* \otimes s_\alpha^F = h_{\alpha\beta} \tilde{F}_\beta^* \otimes s_\alpha^F = \tilde{F}_\beta^* \otimes s_\beta^F. \tag{6.17}$$

Hence, there is a global holomorphic section $\chi \in \Gamma(M, (M \times \mathbb{C}^{n_m^Y+1}) \otimes \mathcal{H}_F)$, called the standard reduced representation section of F, such that $\chi|_{U_\alpha} = \tilde{F}_\alpha^* \otimes s_\alpha^F$.

Set ℓ_1 to be the standard Hermitian metric along the fibres of the trivial bundle $M \times \mathbb{C}^{n_m^Y+1}$ and $\wp_1$ to be a Hermitian metric along the fibres of $\mathcal{H}_F$. Then, we can apply our Green–Jensen formula to the function $\log \|\chi\|_{\ell_1 \otimes \wp_1}$ to get

$$\begin{aligned}\int_s^r \frac{\mathrm{d}t}{t^{2p-1}} \int_{M[t]} dd^c \log \|\chi\|^2_{\ell_1 \otimes \wp_1} \wedge (dd^c\tau)^{p-1} \wedge \omega^{m-p} \\ = \frac{1}{2} \int_{M\langle r\rangle} \log \|\chi\|^2_{\ell_1 \otimes \wp_1} \sigma - \frac{1}{2} \int_{M\langle s\rangle} \log \|\chi\|^2_{\ell_1 \otimes \wp_1} \sigma,\end{aligned}$$

which can be rewritten as

$$T_F(r,s) - T_{\mathcal{H}_F}(r,s) = \int_{M\langle r\rangle} \log \|F\|_{\ell_1} \otimes \|s^F\|_{\wp_1} \sigma - \int_{M\langle s\rangle} \log \|F\|_{\ell_1} \otimes \|s^F\|_{\wp_1} \sigma, \tag{6.18}$$

where $T_{\mathcal{H}_F}(r,s)$ is defined via the pull-back of the first Chern form on $(\mathcal{H}_F, \wp_1)$.

Analogously, we derive that

$$T_f(r,s) - T_{\mathcal{H}_f}(r,s) = \int_{M\langle r\rangle} \log \|f\|_{\ell_2} \otimes \|s^f\|_{\wp_2} \sigma - \int_{M\langle s\rangle} \log \|f\|_{\ell_2} \otimes \|s^f\|_{\wp_2} \sigma, \tag{6.19}$$

where each term is self-evident from the context.

Then, from the constructions of ψ, ψ_m^Y, φ_m^Y and F, this leads to

$$(\|F\|_{\ell_1})|_{U_\alpha} = (\|f\|_{\ell_2})|_{U_\alpha}^{dm},$$

which implies that $(\|s^F\|_{\wp_1})|_{U_\alpha} = (\|s^f\|_{\wp_2})|_{U_\alpha}^{dm}$ and $\mathcal{H}_F = \mathcal{H}_f^{dm}$ (from the transition functions $h_{\alpha\beta}$), so that, by definition, we derive the following estimate:

$$T_{\mathcal{H}_F}(r,s) = dm T_{\mathcal{H}_f}(r,s), \tag{6.20}$$

which, together with (6.18) and (6.19), yields

$$T_F(r,s) = dm T_f(r,s). \tag{6.21}$$

Now, (6.16) turns out to be

$$\begin{aligned}\log \prod_{k=0}^{n} \frac{1}{\|f(z); D_{j_k}\|} \leqslant \frac{n+1}{mH_Y(m)} \max_{\mathcal{L}} \log \prod_{l \in \mathcal{L}} \frac{1}{\|F(z); L_l\|} \\ + \frac{\Delta(n+1)(2n+1)}{m} \max_{1 \leqslant j \leqslant q} \left\{ \log \frac{1}{\|f(z); D_j\|} \right\} + O\left(\frac{1}{m}\right),\end{aligned} \tag{6.22}$$

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which, by application of (3.5) and noting (6.2), implies that

$$\begin{aligned}\sum_{j=1}^{q} m_f(D_j;r) &\leqslant \int_{M\langle r\rangle} \max_{\{j_0,j_1,\dots,j_n\}\subseteq\{1,2,\dots,q\}} \left\{\log\prod_{k=0}^{n}\frac{1}{\|f;D_{j_k}\|}\right\}\sigma + O(1)\\
&\leqslant \frac{n+1}{mH_Y(m)}\int_{M\langle r\rangle}\max_{\mathcal{L}}\log\prod_{l\in\mathcal{L}}\frac{1}{\|F;L_l\|}\sigma\\
&\quad + \frac{\Delta(n+1)(2n+1)}{2m}\sum_{j=1}^{q}\int_{M\langle r\rangle}\log\frac{1}{\|f;D_j\|^2}\sigma + O\left(\frac{1}{m}\right)\\
&= \frac{n+1}{mH_Y(m)}\int_{M\langle r\rangle}\max_{\mathcal{L}}\log\prod_{l\in\mathcal{L}}\frac{1}{\|F;L_l\|}\sigma\\
&\quad + \frac{\Delta(n+1)(2n+1)}{m}\sum_{j=1}^{q}(dT_f(r,s) - m_f(D_j;s)) + O\left(\frac{1}{m}\right)\\
&\leqslant \frac{n+1}{mH_Y(m)}\int_{M\langle r\rangle}\max_{\mathcal{L}}\log\prod_{l\in\mathcal{L}}\frac{1}{\|F;L_l\|}\sigma\\
&\quad + dq\frac{\Delta(n+1)(2n+1)}{m}T_f(r,s) + O\left(\frac{1}{m}\right),\end{aligned}\tag{6.23}$$

where the term $O(1/m)$ depends only on the q hypersurfaces D_j and thus is very small for sufficiently large m.

Our final step is to apply the second main theorem proved in § 4 for the linearly non-degenerate meromorphic map $F\colon M \to \mathbb{P}^{n_m^Y}(\mathbb{C})$ and the $q_m^Y + 1$ hyperplanes' L_l values. Now, noting that $N_{\mathrm{Ram}\,F}(r,s) \geqslant 0$ (see the note immediately after Proposition 3.5) and $n_m^Y = H_Y(m) - 1$, the estimate (4.1) yields that, for any $\varepsilon > 0$,

$$\begin{aligned}\int_{M\langle r\rangle}\max_{\mathcal{L}}\log\prod_{l\in\mathcal{L}}\frac{1}{\|F;L_l\|}\sigma \mathrel{.{\leqslant}.} (H_Y(m)+\varepsilon)T_F(r,s) + \{\tfrac{1}{2}H_Y(m)(H_Y(m)-1)+\varepsilon\}\\
\cdot(m_0(\mathfrak{L};r,s) + \mathrm{Ric}_p(r,s) + \kappa\log^+ Y(r^2) + \kappa\log^+ r),\end{aligned}\tag{6.24}$$

which combined with (5.7), (6.21) and (6.23) shows that

$$\begin{aligned}\sum_{j=1}^{q} m_f(D_j;r) \mathrel{.{\leqslant}.} d\left((n+1) + \frac{n+1}{H_Y(m)}\varepsilon + q\frac{\Delta(n+1)(2n+1)}{m}\right)T_f(r,s)\\
+ C_m(m_0(\mathfrak{L};r,s) + \mathrm{Ric}_p(r,s) + \kappa\log^+ Y(r^2) + \kappa\log^+ r),\end{aligned}\tag{6.25}$$

where

$$C_m := \Delta\frac{n+1}{2n!}m^{n-1} + O(m^{n-2})$$

is a constant dependent upon m.

Now, for m sufficiently large, we can suppose

$$\frac{n+1}{H_Y(m)} \leqslant \frac{1}{2} \quad\text{and}\quad q\frac{\Delta(n+1)(2n+1)}{m} \leqslant \frac{\varepsilon}{2},\tag{6.26}$$

which, combined with (6.25), yields the desired estimate (6.1) for $c_m = C_m/d$. (In general, c_m depends on both m and the degrees of the hypersurfaces involved.) $\square$

6.1. Defect relation

Now, when M is assumed to be either an affine algebraic variety or an algebraic vector bundle over an affine algebraic variety or its projectivization, via Remarks 1.3 and 2.2, (3.11) and (3.19), it follows that

$$m_0(\mathfrak{L}; r, s) + \mathrm{Ric}_p(r, s) + \kappa \log^+ Y(r^2) = O(\log^+ r),$$

so that we naturally have a stronger estimate

$$\liminf_{r\to+\infty} \frac{m_0(\mathfrak{L}; r, s) + \mathrm{Ric}_p(r, s) + \kappa \log^+ Y(r^2)}{T_f(r, s)} = 0, \tag{6.27}$$

and hence, from (6.1), by letting $\varepsilon \to 0$, this yields immediately that

$$\sum_{j=1}^{q} \delta(D, f) \leqslant n + 1. \tag{6.28}$$

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