CONCERNING NON-PLANAR CIRCLE-LIKE CONTINUA

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1. Introduction. In this paper it is proved that if a circle-like continuum M cannot be embedded in the plane, then M is not a continuous image of any plane continuum (Theorem 5).

Suppose that (S, ρ) is a metric space. A finite sequence of domains $L_1, L_2, \ldots, L_n$ is called a *linear chain* provided L_i intersects L_j if and only if $|i - j| \leqslant 1$. If, in addition, there is a positive number ϵ such that, for each i, the diameter of L_i is less than ϵ, then the linear chain is called a linear ϵ*-chain*. If for each positive number ϵ the continuum M can be covered by a linear ϵ-chain, then M is said to be *chainable* (*or snake-like*) **(2)**.

The definition of a *circular chain* (circular ϵ-chain) differs from that of a linear chain (linear ϵ-chain) only in that L_n intersects L_1. The continuum M is said to be a *circle-like* if for each positive number ϵ, M can be irreducibly covered by a circular ϵ-chain.

If the finite sequence of domains $c_1, c_2, \ldots, c_n$ forms a circular chain (similarly, a linear chain) it will be denoted by $C(c_1, c_2, \ldots, c_n)$ and it will be referred to as the circular (linear) chain C, and the domains $c_1, c_2, \ldots, c_n$ will be called links of C.

If C and D are circular chains (linear chains), C is said to be a *refinement* of D if and only if each link of C is contained in some link of D. Further, C is said to be a *strong refinement* of D if and only if the closure of each link of C is contained in some link of D.

If D is a finite collection of point sets, then $\mu(D)$ denotes the largest number which is the diameter of an element of D; $\mu(D)$ is called the mesh of D.

If G is a collection of point sets, the sum of the elements of G is denoted by G^*.

With Bing, we adopt the following convention. If $D(d_1, d_2, \ldots, d_m)$ is a circular chain, then we consider d_{i-1} as the link preceding d_i, and in case $i = 1$ we interpret d_{i-1} to mean d_m. Further, we consider d_{i+1} as the link following d_i, so if $i = m$ we interpret d_{i+1} to mean d_1. Thus, it will be convenient to understand d_0 to be another name of d_m.

In view of the above definitions and the Lebesgue covering lemma **(10**, p. 154**)**, if M is a circle-like continuum, then there exists a simple infinite sequence of circular chains $C_1, C_2, C_3, \ldots$ covering M such that, for each integer n, (1) $\mu(C_n) < 1/n$ and (2) C_{n+1} is a refinement of C_n. Such a sequence of

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circular chains $C_1, C_2, C_3, \ldots$ is said to be a sequence of circular chains *defining* M.

If C is either a circular or linear chain and D is either a circular or linear chain, then C is a *consolidation* of D if (1) each link of C is the sum of a sub-collection of links of D and (2) D is a refinement of C.

The continuum M is said to be *indecomposable* if and only if it is non-degenerate and is not the sum of two continua both distinct from it **(14)**. Further, the continuum M is said to be *hereditarily indecomposable* if and only if each if its non-degenerate subcontinua is indecomposable.

A *pseudo-arc* is a hereditarily indecomposable chainable continuum **(3, 13)**.

2. Definition and some properties of the function of Bing (5) which determines circling. Let the circular chain $C(c_1, c_2, \ldots, c_n)$ be a refinement of the circular chain $D(d_1, d_2, \ldots, d_m)$, $m \geqslant 3$. Let k be an integer, $1 \leqslant k \leqslant n$, and let $f'(c_k)$ be a subscript of one of the links of D which contains c_k. Whenever there is a choice, if possible, $f'(c_k)$ is chosen so that $\bar{c}_k$ is contained in the $f'(c_k)$th link of D.

Let f be a map of the set of integers $\{0, 1, 2, \ldots, n\}$ into a set of integers defined as follows:

$$f(0) = f'(c_n),$$

$$f(i+1) = \begin{cases} f(i) - 1 & \text{if } f'(c_{i+1}) \text{ precedes } f'(c_i), \\ f(i) & \text{if } f'(c_{i+1}) = f'(c_i), \\ f(i) + 1 & \text{if } f'(c_{i+1}) \text{ follows } f'(c_i). \end{cases}$$

Property 1. If i is an integer, $1 \leqslant i \leqslant n$, $f(i) \equiv f'(c_i) \mod m$, and $f(0) = f'(c_n) \equiv f(n) \mod m$ **(5)**.

Definition. The number of times that C circles in D is $|f(n) - f(0)|/m$ **(5)**. (As Bing has noted **(5)** this number is invariant under taking different links of C or D as the first link or in ordering the links in a counter fashion. Nor does it matter which of the two choices for $f'(c_k)$ was made when there was a choice.)

Property 2. If x and y are integers, $1 \leqslant x \leqslant n$, $1 \leqslant y \leqslant n$, such that $f(x) \equiv f(y) \mod m$, then c_x and c_y are contained in the same link of D.

Property 3. If x and y are integers, $1 \leqslant x \leqslant n$, $1 \leqslant y \leqslant n$, such that $f(x) + 1 = f(y)$, then either $f'(c_x) + 1 = f'(c_y)$ or $f'(c_x) = m$ and $f'(c_y) = 1$.

Property 4. If x and y are integers, $1 \leqslant x \leqslant n$, $1 \leqslant y \leqslant n$, such that $f(x) + 1 = f(y)$, then there is an integer z such that (1) either z is x or z is between x and y and (2) if z' is its immediate successor in the order from x to y, then $f(z) = f(x)$ and $f(z') = f(y)$.

Property 5. If x and y are integers $1 \leqslant x \leqslant n$, $1 \leqslant y \leqslant n$, such that $f(x) < f(y) - 1$ and if j is an integer such that $f(x) < j < f(y)$, then there is an integer z between x and y such that $f(z) = j$.

3. Circle-like continua.

THEOREM 1. *Let C and D be circular chains such that C is a refinement of D. If C circles in D zero times, then there is a linear chain E such that* (1) *E is a consolidation of C and* (2) *E is a refinement of D, and, hence, $\mu(E) \leqslant \mu(D)$.*

Proof. Let f be a function which determines the circling of C in D. Since f is bounded, there exist integers M and N each of which is the image under f of an integer between 0 and n (where n is the number of links of C), such that if c_x is a link of C, $M \leqslant f(x) \leqslant N$. If i is an integer, $M \leqslant i \leqslant N$, then denote by L_i the collection to which the link c_x of C belongs if and only if $f(x) = i$. By Property 5, L_i exists for each i. Moreover, by Property 2, for each i, there exists a link of D which contains L_i^*, so the diameter of L_i^* is less than or equal to $\mu(D)$.

Now, L_i^* intersects L_j^* if and only if $|i - j| \leqslant 1$. First, suppose that $|i - j| \leqslant 1$. For convenience assume that $i + 1 = j$ and that c_x belongs to L_i and c_y belongs to L_j. Then $f(x) = i$ and $f(y) = i + 1$; so by Property 4, there exist consecutive integers z and z' such that $f(z) = f(x)$ and $f(z') = f(y)$. Therefore, c_z belongs to L_i, $c_{z'}$ belongs to L_j, and, since c_z intersects $c_{z'}$, L_i^* intersects L_j^*. On the other hand, suppose L_i^* intersects L_j^*, $i \neq j$; then there are links c_x of L_i and c_y of L_j which intersect. Thus, $|x - y| \leqslant 1$; so, by definition of f, $|f(x) - f(y)| \leqslant 1$. However, $f(x) = i$ and $f(y) = j$; consequently, $|i - j| \leqslant 1$. Thus, $E(L_M^*, L_{M+1}^*, \ldots, L_N^*)$ is a linear chain which is a refinement of D, so $\mu(E) \leqslant \mu(D)$.

An immediate corollary to Theorem 1 is the following theorem. Theorem 2 also follows from (**12**, Theorem 4, p. 46).

THEOREM 2. *If M is a circle-like continuum and there is a sequence of circular chains $D_1, D_2, D_3, \ldots$ defining M such that, for each positive integer n, D_{n+1} circles in D_n zero times, then M is chainable.*

Definition. Let δ be a positive real number. A circular or linear chain is *δ-regular* if the distance between any two non-intersecting links of it is greater than or equal to δ.

A proof of a theorem for linear chains similar to the following theorem for circular chains is given by H. Cook in (**6**). Essentially the same proof can be applied to the following theorem.

THEOREM 3. *If C is a circular chain irreducibly covering the continuum M, then there is a positive number δ and a finite sequence $d_1, d_2, \ldots, d_n$ of domains with respect to M such that* (1) *$D(d_1, d_2, \ldots, d_n)$ is a strong refinement of C and* (2) *either D is a δ-regular circular chain or D is a δ-regular linear chain.*

From Theorems 1, 2, and 4 of (**5**) it follows that if M is a circle-like continuum which cannot be embedded in the plane and if $C_1, C_2, C_3, \ldots$ is a sequence of circular chains defining M, then there is a subsequence

$D_1, D_2, D_3, \ldots$ of the sequence $C_1, C_2, C_3, \ldots$ defining M such that D_{n+1} circles at least twice in D_n.

Definitions. Let $X_1, X_2, X_3, \ldots$ be a sequence of topological spaces and $f_1, f_2, f_3, \ldots$ be a sequence of continuous transformations such that f_k throws X_{k+1} onto X_k. Then, if X denotes the sequence $X_1, X_2, \ldots$ and f denotes the sequence $f_1, f_2, \ldots$, the ordered pair (X, f) is said to be an *inverse limit sequence.* Let M denote the subset of the Cartesian product of $X_1, X_2, \ldots$ to which the point $(x_1, x_2, x_3, \ldots)$ belongs if and only if $f_k(x_{k+1}) = x_k$. The space M is the *inverse limit space* of the sequence (X, f).

Denote by f_k^k the identity transformation on X_k. Define f_n^m, $m > n$, by $f_n^m = f_n f_{n+1} \ldots f_{m-1}$ and note that f_n^m throws X_m onto X_n. Further, denote by p_k the projection of M onto X_k.

Definition. Let $n_1, n_2, n_3, \ldots$ be a sequence of positive integers. For each positive integer, k, let C_k denote the unit circle in the plane with centre at the origin, and f_k the transformation throwing C_{k+1} onto C_k, defined by

$$f_k(1, t) = (1, n_k t)$$

(in polar coordinates). If C denotes the sequence $C_1, C_2, C_3, \ldots$ and f denotes the sequence $f_1, f_2, f_3, \ldots$, the inverse limit space of the sequence (C, f) is a *solenoid* **(4)**.

The author wishes to thank the referee for suggestions which shortened the proof of the following theorem.

THEOREM 4. *If M is a circle-like continuum which cannot be embedded in the plane, then there is a continuous transformation throwing M onto a solenoid which cannot be embedded in the plane.*

Proof. Since M cannot be embedded in the plane, there is a positive number ϵ such that M cannot be covered by a linear ϵ-chain and a sequence $D_1, D_2, D_3, \ldots$ of circular chains defining M such that (i) $\mu(D_1) < \epsilon$, (ii) for each n there is a positive number δ_n such that D_n is δ_n-regular, (iii)

$$\mu(D_{n+1}) < 1/4\, \delta_n,$$

and (iv) D_{n+1} is a strong refinement of D_n which circles in D_n at least twice.

Now, we shall define a sequence $E_1, E_2, E_3, \ldots$ of circular chains such that, for each positive integer n, (1) E_n is a consolidation of D_{j_n}, where $j_1 = 1$ and $j_n > j_{n-1}$, (2) E_{n+1} is a strong refinement of E_n such that no link of E_{n+1} intersects more than two links of E_n, and (3) E_{n+1} circles at least twice in E_n and a function which determines the circling is non-decreasing.

Let $E_1 = D_1$ and suppose E_k has been defined. From (iii) it follows that $\mu(E_k) < \epsilon$ since $\mu(E_k) \leqslant \mu(E_1) < \epsilon$. Thus, the circular chain D_{j_k+1} circles at least once in E_k, for, if it does not, then, by Theorem 1, there is a linear chain E which is a consolidation of D_{j_k+1} such that $\mu(E) < \epsilon$, a contradiction to the choice of ϵ. If $D_{j_{k+1}}$ circles in E_k at least twice, then choose j_{k+1} to be $j_k + 1$; but, if $D_{j_{k+1}}$ circles in E_k only once, choose j_{k+1} to be $j_k + 2$. In the latter case D_{j_k+1} circles in E_k at least twice because of Bing's product theorem (**5**, Theorem 1) since D_{j_k+1} circles $D_{j_{k+1}}$ at least twice and $D_{j_{k+1}}$ circles E_k only once.

For convenience, let D be D_{j_k+1} and f be a function which determines the circling of D in E. Suppose D has n links, E has m links, D circles in E p times, and $f(n) > f(0)$. Note that $f(n) - f(0) = mp$. Since E is a consolidation of D_{j_k}, E is δ_{j_k}-regular. Let $\delta = \delta_{j_k}$.

For each positive integer i, $1 \leqslant i \leqslant mp$, denote by $g(i)$ the collection to which the link d_x of D belongs if and only if $f(x) \equiv i \bmod (mp)$. Note that $g(i)^*$ is a subset of e_j, where $i \equiv j \bmod p$. Using properties (1)–(5) of f, each $g(i)$ exists and only adjacent sets $g(i)^*$ intersect. So $G(g(1)^*, \ldots, g(mp)^*)$ is a circular chain consolidating D, strongly refining E, and circling E p times.

Denote by $g(i)_+$ the subcollection of $g(i)$ to which the set d_x belongs if and only if d_x intersects e_{j+1}, and let $g(i)_- = g(i) - g(i)_+$. Since E is δ-regular and $\mu(D) \leqslant \frac{1}{4}\delta$, each $g(i)$ contains links of D which do not intersect e_{j+1}; therefore, the collections of $g(i)_+$ and $g(i)_-$ exist.

Then $G(g(1)_-^*, g(1)_+^*, \ldots, g(mp)_-^*, g(mp)_+^*)$ is the required circular chain E_{k+1}.

Suppose that E_k has m_k links and E_{k+1} circles in E_k n_k times, $n_k > 1$. Then, $m_k = 2^{k-1} n_{k-1}.n_{k-2} \ldots n_1 m_1$.

For convenience, denote the ith link of E_n by $L(i, E_n)$. We note that the closure of $L(j, E_{k+1})$ is a subset of $L(i, E_k)$ if and only if it is true that, if $j = 2(r-1)m_k + s$, where $1 \leqslant r \leqslant n_k$ and $1 \leqslant s \leqslant 2m_k$, then $i = s/2$ if s is even and $i = (s+1)/2$ if s is odd.

For each positive integer k denote by C_k the unit circle in the plane with centre at the origin. Denote by f_k the transformation throwing C_{k+1} onto C_k defined by $f_k(1, t) = (1, n_k t)$ (using polar coordinates). Denote by M' the solenoid which is the inverse limit space of the sequence (C, f) (**4**), and let p_k denote the projection of M' onto C_k.

Denote by U_k the collection to which the arc X of C_k belongs if and only if there exists an integer j, $1 \leqslant j \leqslant m_k$, such that, if $(1, t)$ belongs to X, then for some integer h, $2\pi(j-1)/m_k \leqslant t - 2\pi h \leqslant 2\pi j/m_k$. In this case X will be denoted by $X(j, k)$. Then for each k, $X(i, k)$ intersects $X(j, k)$ if and only if $|i - j| \leqslant 1$ or one of i and j is 1 and the other is m_k.

Denote by V_k the collection to which the subset Y of M' belongs if and only if there is an arc X of U_k such that $Y = p_k^{-1}(X)$. Each element of V_k, for each k, is said to be a *section* of M'. Moreover, if $X = X(j, k)$ and $Y = p_k^{-1}(X)$, then Y is denoted by $Y(j, k)$. Thus, $Y(i, k)$ intersects $Y(j, k)$ if and only if $|i - j| \leqslant 1$ or one of i and j is 1 and the other is m_k.

We wish to show that $V_1, V_2, V_3, \ldots$ is a sequence of sections of M' such that for each k: (1′) V_k has m_k elements; (2′) $Y(i, k)$ intersects $Y(j, k)$ if and only if $|i - j| \leqslant 1$ or one of i and j is 1 and the other is m_k; (3′) $Y(i, k)$ contains

$Y(j, k+1)$ if and only if $L(i, E_k)$ contains $\bar{L}(j, E_{k+1})$; (4′) if ϵ is a positive number, there is an integer N such that if k is an integer, $k \geqslant N$, and i is an integer, $1 \leqslant i \leqslant m_k$, then the diameter of $Y(i, k)$ is less than ϵ; and (5′) each section in V_{k+1} is a subset of some section in V_k.

To prove (3′) let $[a, b]$ denote the interval on the unit circle going counterclockwise from $(1, a)$ to $(1, b)$ (polar coordinates).

Since $m_{k+1} = 2n_k m_k$ and $1 \leqslant j \leqslant m_{k+1}$, there exist integers r and s, $1 \leqslant r \leqslant n_k$, $1 \leqslant s \leqslant 2m_k$, such that $j = 2(r-1)m_k + s$. Now, $Y(i, k)$ contains $Y(j, k+1)$ if and only if $[2\pi(i-1)/m_k, 2\pi i/m_k]$ contains

$$f_k([2\pi(j-1)/m_{k+1}, 2\pi j/m_{k+1}]);$$

thus if and only if $[2\pi(i-1)/m_k, 2\pi i/m_k]$ contains $[2\pi(s-1)/2m_k, 2\pi s/2m_k]$, and thence if and only if

$$i = \begin{cases} s/2 & \text{if } s \text{ is even,} \\ (s+1)/2 & \text{if } s \text{ is odd.} \end{cases}$$

But this is precisely the condition for $L(i, E_k)$ to contain the closure of $L(j, E_{k+1})$.

To prove (4′) let d denote the usual metric on the unit circle. Then an equivalent metric for M' is given by

$$d_1(x, y) = \sum_{k=1}^{\infty} \frac{d(p_k(x), p_k(y))}{2^k}$$

(p_k is the projection of M' onto C_k). There is then a positive integer Q such that

$$\sum_{k=Q+1}^{\infty} \frac{d(p_k(x), p_k(y))}{2^k} < \epsilon/2, \qquad \text{for any } x, y \text{ in } M'.$$

Let x be a point of $Y(i, k) = p_k^{-1}([2\pi(i-1)/m_k, 2\pi i/m_k])$. If n is a positive integer, $n \leqslant k$, then $p_n(x)$ is a point of

$$f_n^k(X(i, k)) = \left[\frac{2\pi(i-1)}{m_n} \cdot \frac{1}{2^{k-n}}, \frac{2\pi i}{m_n} \cdot \frac{1}{2^{k-n}}\right],$$

where $f_n^k = f_n f_{n+1} \ldots f_{k-1}$: $C_k \to C_n$.

For fixed n, the diameter of $f_n^k(X(i, k)) \to 0$ as $k \to \infty$. Hence, there exists an integer N_Q such that if x and y are points of $Y(i, k)$, $k > N_Q$, then $d(p_n(x), p_n(y)) < \epsilon/2$ for $n = 1, 2, \ldots, Q$. So

$$d_1(x, y) = \sum_{k=1}^{Q} \frac{d(p_k(x), p_k(y))}{2^k} + \sum_{k=Q+1}^{\infty} \frac{d(p_k(x), p_k(y))}{2^k} < \epsilon.$$

Hence the diameter of $Y(i, k)$ is less than ϵ.

Now, (5′) follows from the fact that if X' belongs to U_{k+1}, $f_k(X')$ is a subset of an element X of U_k, so that $p_k^{-1}(X)$ contains $p_{k+1}^{-1}(X')$.

Suppose that x is a point of M. Denote by $J_n(x)$, $n = 1, 2, 3, \ldots$, the sum of the collection to which the section in V_n belongs if and only if its subscript

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is the same as that of a link of E_n which contains x. Since x belongs to no more than two links of E_n, $J_n(x)$ is the sum of at most two elements of V_n. Moreover, if a link of E_{n+1} contains x, its closure is contained in one of the links of E_n which contains x. Therefore, by (3′), $J_n(x)$ contains $J_{n+1}(x)$. By (4′), there is only one point common to the terms of the sequence $J_1(x)$, $J_2(x)$, $J_3(x)$, ... ; denote it by $T(x)$. We shall show that the set T of ordered pairs $(x, T(x))$ is a continuous transformation throwing M onto M'.

If y is a point of M', there is a sequence $Y_1, Y_2, Y_3, \ldots$ of sections of M' such that Y_k belongs to V_k and contains y, and Y_{k+1} is a subset of Y_k. Thus, y is the point common to $Y_1, Y_2, Y_3, \ldots$. Correspondingly, there exist links $L_1, L_2, L_3, \ldots$ of $E_1, E_2, E_3, \ldots$, respectively, such that the subscript of L_k (in E_k) is the same as the subscript of Y_k (in V_k); thus $\bar{L}_{k+1}$ is contained in L_k. Therefore, there is a point common to $L_1, L_2, L_3, \ldots$, and, if x is a point of the common part, y belongs to $J_n(x)$ for each n. Thus, $T(x) = y$, so T is a transformation throwing M onto M'.

Furthermore, T is continuous, for suppose that x is a point of M and $T(x) = y$ and R is a region containing y. There is an integer n such that if Y is a section of V_n containing y and Y' intersects Y, then $Y + Y'$ is a subset of R. That such an integer exists follows from (4′). Suppose L is the link of E_n with the same subscript as that of Y. If z belongs to L, either $J_n(z)$ is Y or $J_n(z)$ is the sum of Y and only one other section Y' in V_n which intersects Y. Thus, $J_n(z)$ is a subset of R. However, if $m \geqslant n$, $J_n(z)$ contains $J_m(z)$, so $T(z)$ belongs to R. But L is a domain containing x such that R contains $T(L)$ so T is continuous.

4. Circle-like continua and weakly chainable continua. In **(12)** M. C. McCord generalized the theorem of M. K. Fort **(9)** that the dyadic solenoid is not a continuous image of any plane continuum. McCord proved that each solenoidal continuum is not a continuous image of any plane continuum. Theorem 5 of this paper depends heavily on this result.

Definition. A finite sequence of domains $X_1, X_2, \ldots, X_n$ is said to be a *weak chain* provided that X_i intersects X_j if $|i - j| \leqslant 1$.

Definition. A weak chain $X(X_1, X_2, \ldots, X_n)$ is a *refinement* of a weak chain $X'(X'_1, X'_2, \ldots, X'_m)$ if and only if each link X_i of X is contained in a link X'_{k_i} of X' such that $|k_i - k_j| \leqslant 1$ if $|i - j| \leqslant 1$.

Definition. A continuum M is said to be *weakly chainable* if there exists a sequence $G_1, G_2, G_3, \ldots$ of finite collections of domains covering M such that, for each n, (1) G_n is a weak chain, (2) each link of G_n has diameter less than $1/n$, and (3) G_{n+1} is a refinement of G_n.

THEOREM. *The continuum M is weakly chainable if and only if it is a continuous image of the pseudo-arc.*

Remark. The preceding three definitions and theorem are due to A. Lelek **(11)**. Essentially the same definitions and theorem are presented in an earlier work by Lawrence Fearnley **(7)**; see also **(8)**.

THEOREM 5. *If M is a circle-like continuum which cannot be embedded in the plane, then M is not a continuous image of any plane continuum and is, therefore, not weakly chainable.*

Proof. Let M be a non-planar circle-like continuum and f a continuous transformation throwing the plane continuum K onto M. Then since there is a continuous transformation g throwing M onto a solenoid M', which cannot be embedded in the plane, the continuous transformation gf throws K onto M'. But each solenoid which cannot be embedded in the plane is a solenoidal continuum and, therefore, is not a continuous image of any plane continuum **(12**, Theorem 25**)**.

The following theorem was proved in another way by Fearnley **(7)**.

THEOREM 6. *If H and K are weakly chainable continua with a point in common, then $H + K$ is weakly chainable.*

Proof. Suppose that M_1 and M_2 are pseudo-arcs such that a is the only point of $M_1 \cdot M_2$.

Since H and K are weakly chainable, there exist continuous transformations f and g such that $f(M_1) = H$ and $g(M_2) = K$.

Let x be a point of $H \cdot K$, and suppose that y and z are points of M_1 and M_2, respectively, such that $f(y) = x$ and $g(z) = x$.

Since M_i is homogeneous **(1)**, for $i = 1, 2$, there is a topological transformation T_i such that $T_i(M_i) = M_i$, $i = 1, 2$, and $T_1(a) = y$ and $T_2(a) = z$. So, $fT_1(a) = x$, $gT_2(a) = x$.

Denote by h the transformation throwing $M_1 + M_2$ onto $H + K$ defined by

$$h(t) = \begin{cases} fT_1(t) & \text{if } t \text{ belongs to } M_1, \\ gT_2(t) & \text{if } t \text{ belongs to } M_2. \end{cases}$$

Then h is a continuous transformation such that $h(M_1 + M_2) = H + K$. But $M_1 + M_2$ is chainable, so if M is a pseudo-arc, there is a continuous transformation F such that $F(M) = M_1 + M_2$. Thus $hF(M) = H + K$, and $H + K$ is weakly chainable.

THEOREM 7. *If M is a decomposable circle-like continuum, then M is weakly chainable.*

Proof. There exist proper subcontinua H and K such that $M = H + K$. Since H and K are chainable continua and $H \cdot K$ exists, $H + K$ is weakly chainable.

THEOREM 8. *If M is a circle-like continuum which cannot be embedded in the plane, then M is indecomposable.*

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