SOME TWO-DIMENSIONAL UNITARY GROUPS GENERATED BY THREE REFLECTIONS

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1. Introduction. Shephard and Todd **(5)** give generators for the finite primitive irreducible groups generated by two unitary reflections in $\mathbf{U}_2$. It is the purpose of the present paper to give generating reflections, and defining relations in terms of these reflections, for the seven such groups requiring three generating reflections, that is, for their nos. 7, 11, 12, 13, 15, 19, 22. The reflections are chosen whenever possible so that their product has the property suggested by Theorem 5.4 of **(5)**. That is, except for no. 15, the period of the product of the three generating reflections is $h = m_2 + 1$, and the characteristic roots of this product are $2\pi i m_1/h$ and $2\pi i m_2/h$, where m_1 and m_2 are the "exponents" **(5**, p. 282**)** of the group. The reason for the impossibility of such a choice for no. 15 is given in § 4. In § 5 the homomorphisms between these groups and certain groups of motions in elliptic 3-space are determined.

As in **(5)**, $\omega = \exp 2\pi i/3$, $\epsilon = \exp 2\pi i/8$, and $\eta = \exp 2\pi i/5$. The order of group $\mathfrak{G}$ is $|\mathfrak{G}|$. The identity element of a group, and the 2×2 identity matrix, are both designated by E. The notation $Z \rightleftarrows S, T$ means $ZS = SZ$ and $ZT = TZ$.

2. Groups 7, 11, 19. In terms of the generators

$$S = \omega\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \quad T = \frac{\omega\epsilon^5}{\sqrt{2}}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}, \quad \text{and} \quad Z = -i\omega\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

the defining relations for no. 7 **(5**, pp. 280–1**)** are

2.1 $$S^2 = Z^8,\ T^3 = E,\ (ST)^3 = Z^3,$$
$$Z^{12} = E,\ Z \rightleftarrows S, T.$$

We let

2.2 $$R_1 = SZ^2,\ R_2 = T,\ R_3 = (STZ^3)^{-1}.$$

Then it can be readily verified that R_1, R_2, and R_3 are reflections, and that 2.1 and 2.2 imply

2.3 $$R_1^2 = R_2^3 = R_3^3 = E, \quad (R_1R_3)^3 = (R_3R_1)^3,$$

and

$$R_1R_2R_3 = R_2R_3R_1 = R_3R_1R_2$$
2.4 $$T = R_2,\ S = (R_1R_2R_3)^2R_1,\ Z = (R_3R_1R_2)^{-1}.$$

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Conversely, 2.3 and 2.4 together imply 2.1 and 2.2. First, note that

$$Z^{12} = (R_3R_1R_2)^{-12} = (R_3R_1)^{-12}R_2^{-12} = E,$$

since 2.1 implies $(R_3R_1)^{12} = E$ (**2**, p. 77). Thus

$$S^2 = (R_1R_2R_3)^4R_1^2 = Z^{-4} = Z^8,$$

since $R_1 \rightleftarrows R_1R_2R_3$. Certainly

$$T^3 = R_2^3 = E, \quad \text{and} \quad (ST)^3 = [(R_1R_2R_3)^2R_1R_2]^3 = (R_3R_1R_2)^9 = Z^3,$$

since $R_3 \rightleftarrows R_1R_2$. Finally,

$$R_1 = SZ^2, \quad R_2 = T,$$

and

$$R_3 = R_2^{-1}R_1^{-1}Z^{-1} = T^{-1}S^{-1}Z^{-3} = (STZ^3)^{-1}.$$

In Table I we give the generating reflections and defining relations for the seven groups we consider. The proofs that the given relations for nos. 11 and 19 are equivalent to those given in (**5**) are so similar to the proof just given that they are omitted. In each case $Z = (R_1R_2R_3)^{-1}$. For no. 11, $(R_1R_3)^{24} = E$; for no. 19, $(R_1R_3)^{60} = E$.

If $\mathfrak{G} = \{R_1, R_3\}$ is an arbitrary group defined by relations between its two generators R_1 and R_3, and if the group $\mathfrak{H} = \{R_1, R_2, R_3\}$ is defined by the defining relations for $\mathfrak{G}$ together with $R_2^n = E$ and $R_1R_2R_3 = R_2R_3R_1 = R_3R_1R_2$ then $|\mathfrak{H}| = n|\mathfrak{G}|$. In the present case we can be more specific. Denote by $p_1[2n]p_2$ the group defined by

$$R_1^{p_1} = R_3^{p_2} = E,$$
$$R_1R_3R_1 \ldots = R_3R_1R_3 \ldots \ (2n \text{ factors on each side}) \ (\mathbf{2}, \text{p. } 80).$$

Lemma. *If the period m of $(R_1R_3)^n$ in $p_1[2n]p_2$ is prime to n then the direct product $p_1[2n]p_2 \times \mathfrak{C}_n$ can be presented in the form*

$$R_1^{p_1} = R_2^n = R_3^{p_2} = E, \ (R_1R_3)^n = (R_3R_1)^n, \ R_1R_2R_3 = R_2R_3R_1 = R_3R_1R_2.$$

Proof. We need only show that we can find an element P in $\{R_1, R_2, R_3\}$ but not in $\{R_1, R_3\}$, such that P is of period n and $P \rightleftarrows R_1, R_3$. Since m is prime to n we can find some multiple of m, say r, such that $r \equiv 1 \bmod n$. Let $P = (R_3R_1)^rR_2$. Then $P^n = (R_3R_1)^{rn}R_2^n = E$, since $R_2 \rightleftarrows R_3R_1$. Using the fact that $R_1, R_3 \rightleftarrows (R_3R_1)^{r-1}$ we have $R_1P = R_1(R_3R_1)^rR_2 = (R_3R_1)^{r-1}R_1R_3R_1R_2 = (R_3R_1)^{r-1}R_1R_2R_3R_1 = PR_1$, and $R_3P = R_3(R_3R_1)^rR_2 = (R_3R_1)^{r-1}R_3R_3R_1R_2 = (R_3R_1)^{r-1}R_3R_1R_2R_3 = PR_3$. This completes the proof.

From this we get immediately

Theorem 2.1.

(i) No. $7 \cong 2[6]3 \times \mathfrak{C}_3$
(ii) No. $11 \cong 2[6]4 \times \mathfrak{C}_3$
(iii) No. $19 \cong 2[6]5 \times \mathfrak{C}_3$.

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Proof. The values of n, m, r are as follows (**2**, p. 76):

$$\begin{array}{llll} \text{(i)} & n = 3, & m = r = 4 & \\ \text{(ii)} & n = 3, & m = 8, & r = 16 \\ \text{(iii)} & n = 3, & m = 20, & r = 40. \end{array}$$

3. Groups 12, 13, 22. The appropriate generators and defining relations for nos. 12, 13, and 22 appear in Table I. The given relations for no. 12 imply $(R_1R_3)^3 = (R_2R_3)^6 = E$. If the relation $(R_2R_3)^6 = E$ is replaced by $(R_2R_3)^3 = E$ the resulting group has half the order of no. 12. It is Coxeter's $[1\ 1\ 1]^2 \cong \mathfrak{S}_4$ (**1**, p. 248). The relations for no. 22 imply $(R_1R_3)^5 = E$. If the relation $(R_2R_3)^6 = E$ is replaced by $(R_2R_3)^3 = E$ the resulting group has half the order of no. 22. Slightly extending Coxeter's notation it is $[1\ 1^5\ 1^5]^2$. Although its order is 120 it is not $\mathfrak{S}_5$. By analogy with nos. 12 and 22 it might be expected that no. 13 could be defined by $S_i^2 = (S_1S_2)^4 = (S_1S_3)^4 = (S_2S_3)^6 = S_1S_2S_3S_2S_1S_3S_2S_3 = E$. However, there is no choice of three of the 18 reflections in no. 13 having products of these periods.

It can be verified directly (as was done for no. 7) that the tabulated relations for these three groups are equivalent to those of Shephard and Todd. However, these calculations are tedious and unenlightening. It is more convenient to use the method of enumeration of cosets (**2**, pp. 12–17). Enumeration of the 4 cosets of the subgroup $\{R_2, R_3\}$ (of order $\leqslant 12$) generated by R_2 and R_3 shows that the relations given for no. 12 define a group of order $\leqslant 4.12$. But since the generators S, T, Z are in this group the order is also $\geqslant 48$. Exactly similar arguments apply to nos. 13 and 22. In the former the subgroup $\{R_2, R_3\}$ is of order $\leqslant 16$ and has 6 cosets. In the latter the subgroup $\{R_2, R_3\}$ is of order $\leqslant 12$ and has 20 cosets.

An alternative set of generating reflections for no. 12 is $P_1 = R_1$, $P_2 = R_2$, $P_3 = R_1R_3R_1$. The corresponding defining relations are $P_i^2 = (P_1P_2)^3 = (P_1P_3)^3 = E$, $P_1(P_2P_3)^2 = (P_3P_2)^2P_1$. These imply $(P_2P_3)^4 = E$. Analogous generating reflections for no. 22 are $P_1 = -R_3R_1R_2R_1R_3$, $P_2 = -R_1$, $P_3 = R_2$. Defining relations are $P_i^2 = (P_1P_2)^3 = (P_1P_3)^3 = (P_2P_3)^{10} = E$, $P_1(P_2P_3)^2P_2 = (P_3P_2)^2P_3P_1$. These might be considered analogues of the tabulated definition for no. 13 since the relation $(R_1R_2)^2 = (R_2R_3)^4$ of no. 13 implies both $R_3(R_1R_2)^2 = (R_2R_1)^2R_3$ and $R_1(R_2R_3)^4 = (R_2R_3)^4R_1$.

4. Group 15. Generating reflections and defining relations for no. 15 appear in Table I. The sufficiency of the definition can be verified by enumerating the 6 cosets of $\{R_2, R_3\}$ (of order $\leqslant 48$).

The exponents of this group are 11 and 23. We proceed to show that no matrix in the group has characteristic roots $\exp 2\pi i\, 11/24$, $\exp 2\pi i\, 23/24$. Thus, *a fortiori*, no product of generating reflections has these characteristic roots. We first note that no. 12 is a subgroup of index 6 in no. 15. In fact no. 12 is generated by iS_1 and T_1; its only scalar matrices are $\pm E$. No. 15 is generated by iS_1 and $i\omega T_1$, and contains $Z = -i\omega E$. That is, the elements of no. 15 are of the form MZ^n, $n = 0, 1, \ldots, 5$, where M is a matrix of no. 12. Now suppose $MZ^n = (\alpha_{rs})$ has characteristic roots $\pm \exp 2\pi i\, 11/24$ for some choice of M and n. This implies $\alpha_{11} + \alpha_{22} = m_{11} + m_{22} = 0$, where m_{11} and m_{22} are the diagonal entries of M. The 18 matrices in no. 12 having this property are its 12 reflections and

$$\pm\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \pm\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The former have characteristic roots ± 1, the latter $\pm i$. Since no product of ± 1 or $\pm i$ by a power of $-i\omega = \exp 2\pi i/12$ yields $\pm \exp 2\pi i\, 11/24$, we conclude that no matrix in no. 15 has these characteristic roots.

5. Homomorphisms with Goursat's groups. In this section we assume knowledge of **(3)**. Groups $\mathfrak{L}$, $\mathfrak{R}$, $\mathfrak{l}$, $\mathfrak{r}$ of Clifford translations corresponding to each of the present groups can be determined from quaternion representations of R_1, R_2, R_3. These all appear in Table II. They determine groups 2:1 homomorphic to certain groups of motions in elliptic 3-space given by Goursat **(4)**. In fact, the latter groups are determined by isomorphisms $\mathfrak{L}'/\mathfrak{l}' \cong \mathfrak{R}'/\mathfrak{r}'$ where $\mathfrak{L}'$, $\mathfrak{R}'$, $\mathfrak{l}'$, $\mathfrak{r}'$ are the polyhedral or cyclic groups corresponding to the binary polyhedral or cyclic groups $\mathfrak{L}$, $\mathfrak{R}$, $\mathfrak{l}$, $\mathfrak{r}$ by 2:1 homomorphism.

The subgroups generated by pairs of generating reflections are groups of regular complex polygons. These have been found, after $\mathfrak{L}$ and $\mathfrak{R}$, by reference to the Table in **(3)**. There are some possible ambiguities in this determination, which can all be readily resolved. For example, the subgroup $\{R_2, R_3\}$ of no. 7 has $\mathfrak{L} \cong \mathfrak{C}_6$, $\mathfrak{R} \cong \langle 2, 3, 3\rangle$. Reference to the Table of **(3)** shows that this applies to either 3[4]3 or 3[3]3. But the generators

$$\omega i\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \frac{\omega\epsilon^5}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

(5, p. 281**)** of the larger group 3[4]3 are both in $\{R_2, R_3\}$. Therefore $\{R_2, R_3\}$ is 3[4]3.

We summarize the results of Table II. For a given group $\mathfrak{G}$ let the periods of the tabulated generating reflections R_1, R_2, R_3 be p_1, p_2, p_3. Let the collineation group of $\mathfrak{G}$ be $(2, 3, \nu)$. (For no. 7, $\nu = 3$; for nos. 11, 12, 13, 15, $\nu = 4$; for nos. 19, 22, $\nu = 5$.) Let $\mathfrak{Z}$ be the centre of $\mathfrak{G}$.

THEOREM 5.1. *The group* $\mathfrak{G} = \{R_1, R_2, R_3\}$ *is* 2:1 *homomorphic to the group of motions in elliptic* 3*-space defined by the isomorphism* $\mathfrak{L}'/\mathfrak{l}' \cong \mathfrak{R}'/\mathfrak{r}'$ *where*

(a) $\mathfrak{L}'$ *and* $\mathfrak{l}'$ *are cyclic groups.*

(b) $|\mathfrak{L}'| = \mathfrak{l}.c.m.\ \{p_1, p_2, p_3\}$.

(c) $2|\mathfrak{l}'| = |\mathfrak{Z}|$. *That is, except for no.* 15, $2|\mathfrak{l}'|$ *is the period of the smallest power of* $R_1R_2R_3$ *which is central.*

(d) $\mathfrak{R}'$ *is* $(2, 3, \nu)$.

(e) $\mathfrak{r}'$ *is the unique normal subgroup of* $\mathfrak{R}'$ *such that* $|\mathfrak{L}'|\,|\mathfrak{r}'| = |\mathfrak{R}'|\,|\mathfrak{l}'|$. *In fact, except for no.* 12, $\mathfrak{r}' \cong \mathfrak{R}'$ *and* $\mathfrak{l}' \cong \mathfrak{L}'$.

TABLE I

GENERATING REFLECTIONS AND ABSTRACT DEFINITIONS FOR GROUPS GENERATED BY THREE REFLECTIONS IN U_2

Group	Generating reflections and notations of **(5)**	Defining relations	$R_1R_2R_3$
7	$R_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = SZ^2$ $R_2 = \frac{\omega\epsilon^5}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} = T$ $R_3 = \frac{\omega\epsilon^3}{\sqrt{2}}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} = (STZ^3)^{-1}$	$R_1^{\,2} = R_2^{\,3} = R_3^{\,3} = E$ $(R_1R_3)^3 = (R_3R_1)^3$ $R_1R_2R_3 = R_2R_3R_1 = R_3R_1R_2$	$\exp 2\pi i \frac{11}{12} E$
11	$R_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} = SZ^{12}$ $R_2 = \frac{\omega\epsilon^5}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = TZ^9$ $R_3 = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = (ST)^{-1}Z^2$	$R_1^{\,2} = R_2^{\,3} = R_3^{\,4} = E$ $(R_1R_3)^3 = (R_3R_1)^3$ $R_1R_2R_3 = R_2R_3R_1 = R_3R_1R_2$	$\exp 2\pi i \frac{23}{24} E$
12	$R_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & i \\ -1 & 1 \end{pmatrix} = S$ $R_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = T^2ST$ $R_3 = \epsilon\begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix} = TST^2$	$R_i^{\,2} = (R_1R_2)^3$ $= R_1R_2R_3R_2R_1R_3R_2R_3 = E$	$\frac{i}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$ Char. roots: $\exp 2\pi i \frac{5}{8}$ $\exp 2\pi i \frac{7}{8}$
13	$R_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = T^2STZ$ $R_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & i \\ -i & 1 \end{pmatrix} = S$ $R_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = (TS)^2Z^3$	$R_i^{\,2} = (R_1R_2)^3 = (R_1R_3)^4 = E$ $(R_1R_3)^2 = (R_2R_3)^4$	$\frac{\epsilon^7}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ Char. roots: $\exp 2\pi i \frac{7}{12}$ $\exp 2\pi i \frac{11}{12}$

TABLE 1 (*Cont.*)

Group	Generating reflections and notations of **(5)**	Defining relations	$R_1R_2R_3$
15	$R_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & i \\ -i & 1 \end{pmatrix} = S$	$R_1^2 = R_2^2 = R_3^3 = E$	
	$R_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (ST)^2Z^7$	$(R_2R_3)^3 = (R_3R_2)^3$ $R_2R_1R_3 = R_1R_3R_2$	
	$R_3 = \frac{\omega\epsilon^5}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = TZ^3$	$R_3R_1R_3R_1R_2 = R_1R_2R_3R_1R_3$	
19	$R_1 = \frac{i}{\sqrt{5}}\begin{pmatrix} \eta - \eta^4 & \eta^3 - \eta^2 \\ \eta^3 - \eta^2 & \eta^4 - \eta \end{pmatrix} = SZ^{10}$	$R_1^2 = R_2^3 = R_3^5 = E$	
	$R_2 = \frac{\omega}{\sqrt{5}}\begin{pmatrix} \eta^2 - \eta^4 & \eta^4 - 1 \\ 1 - \eta & \eta^3 - \eta \end{pmatrix} = TZ^{29}$	$(R_1R_3)^3 = (R_3R_1)^3$	$\exp 2\pi i\,\frac{59}{60}\,E$
	$R_3 = \begin{pmatrix} \eta^4 & 0 \\ 0 & 1 \end{pmatrix} = (ST)^{-1}Z^{20}$	$R_1R_2R_3 = R_2R_3R_1 = R_3R_1R_2$	
22	$R_1 = \frac{i}{\sqrt{5}}\begin{pmatrix} \eta^4 - \eta & \eta^2 - \eta^3 \\ \eta^2 - \eta^3 & \eta - \eta^4 \end{pmatrix} = S$		$\frac{i}{\sqrt{5}}\begin{pmatrix} \eta^4 - 1 & 1 - \eta^3 \\ \eta^2 - 1 & \eta - 1 \end{pmatrix}$ Char. roots:
	$R_2 = \frac{i}{\sqrt{5}}\begin{pmatrix} \eta - \eta^4 & \eta^4 - \eta^3 \\ \eta^2 - \eta & \eta^4 - \eta \end{pmatrix} = STST^2SZ^2$	$R_i^2 = (R_1R_2)^5 = (R_2R_3)^6$ $= R_1R_2R_3R_2R_1R_3R_2R_3 = E$	$\exp 2\pi i\,\frac{11}{20}$
	$R_3 = \frac{i}{\sqrt{5}}\begin{pmatrix} \eta - \eta^4 & \eta^2 - \eta \\ \eta^4 - \eta^3 & \eta^4 - \eta \end{pmatrix} = T^2STZ^2$		$\exp 2\pi i\,\frac{19}{20}$

TABLE II*

GOURSAT GROUPS CORRESPONDING TO GROUPS GENERATED BY THREE REFLECTIONS IN U_2

Group		b	$\frac{\mathfrak{L}}{\mathfrak{l}}$	$\frac{\mathfrak{R}}{\mathfrak{r}}$	Subgroups $\{R_1, R_2\}$, $\{R_1, R_3\}$, $\{R_2 R_3\}$, Subgroup	$\mathfrak{L}$	$\mathfrak{R}$	$p_1[q]p_2$
7	R_1:	$-\mathbf{i}$			$\{R_1, R_2\}$	$\mathfrak{C}_{12}$	$\langle 2,3,3\rangle$	3[6]2
	R_2:	$\frac{1}{2}+\frac{\mathbf{i}}{2}-\frac{\mathbf{j}}{2}+\frac{\mathbf{k}}{2}$	$\frac{\mathfrak{C}_{12}}{\mathfrak{C}_{12}}$	$\frac{\langle 2,3,3\rangle}{\langle 2,3,3\rangle}$	$\{R_1, R_3\}$	$\mathfrak{C}_{12}$	$\langle 2,3,3\rangle$	3[6]2
	R_3:	$\frac{1}{2}+\frac{\mathbf{i}}{2}-\frac{\mathbf{j}}{2}-\frac{\mathbf{k}}{2}$			$\{R_2, R_3\}$	$\mathfrak{C}_6$	$\langle 2,3,3\rangle$	3[4]3
11	R_1:	$\frac{\mathbf{i}}{\sqrt{2}}-\frac{\mathbf{j}}{\sqrt{2}}$			$\{R_1, R_2\}$	$\mathfrak{C}_{12}$	$\langle 2,3,4\rangle$	3[8]2
	R_2:	$\frac{1}{2}+\frac{\mathbf{i}}{2}+\frac{\mathbf{j}}{2}+\frac{\mathbf{k}}{2}$	$\frac{\mathfrak{C}_{24}}{\mathfrak{C}_{24}}$	$\frac{\langle 2,3,4\rangle}{\langle 2,3,4\rangle}$	$\{R_1, R_3\}$	C_8	$\langle 2,3,4\rangle$	4[6]2
	R_3:	$\frac{1}{\sqrt{2}}+\frac{\mathbf{i}}{\sqrt{2}}$			$\{R_2, R_3\}$	$\mathfrak{C}_{24}$	$\langle 2,3,4\rangle$	4[4]3
12	R_1:	$\frac{\mathbf{i}}{\sqrt{2}}+\frac{\mathbf{j}}{\sqrt{2}}$			$\{R_1, R_2\}$	$\mathfrak{C}_4$	$\langle 2,2,3\rangle$	2[3]2
	R_2:	$\frac{-\mathbf{i}}{\sqrt{2}}-\frac{\mathbf{k}}{\sqrt{2}}$	$\frac{\mathfrak{C}_4}{\mathfrak{C}_2}$	$\frac{\langle 2,3,4\rangle}{\langle 2,3,3\rangle}$	$\{R_1, R_3\}$	$\mathfrak{C}_4$	$\langle 2,2,3\rangle$	2[3]2
	R_3:	$\frac{-\mathbf{j}}{\sqrt{2}}-\frac{\mathbf{k}}{\sqrt{2}}$			$\{R_2, R_3\}$	$\mathfrak{C}_4$	$\langle 2,2,3\rangle$	2[6]2
13	R_1:	$\frac{-\mathbf{i}}{\sqrt{2}}-\frac{\mathbf{k}}{\sqrt{2}}$			$\{R_1, R_2\}$	$\mathfrak{C}_4$	$\langle 2,2,3\rangle$	2[3]2
	R_2:	$\frac{\mathbf{i}}{\sqrt{2}}+\frac{\mathbf{j}}{\sqrt{2}}$	$\frac{\mathfrak{C}_4}{\mathfrak{C}_4}$	$\frac{\langle 2,3,4\rangle}{\langle 2,3,4\rangle}$	$\{R_1, R_3\}$	$\mathfrak{C}_4$	$\langle 2,2,2\rangle$	2[4]2
	R_3:	$\mathbf{j}$			$\{R_2, R_3\}$	$\mathfrak{C}_4$	$\langle 2,2,4\rangle$	2[8]2

*When a reflection R has period n, its quaternion form is $q' = aqb$, where $a = \exp 2\pi\mathbf{i}/n$ and b is given in the second column below.

TABLE II* (*Cont.*)

Group		b	$\frac{\mathfrak{L}}{\mathfrak{l}}$	$\frac{\mathfrak{R}}{\mathfrak{r}}$	Subgroups $\{R_1, R_2\}$, $\{R_1, R_3\}$, $\{R_2, R_3\}$ Subgroup	$\mathfrak{L}$	$\mathfrak{R}$	$p_1[q]p_2$
15	R_1:	$\frac{\mathbf{i}}{\sqrt{2}} + \frac{\mathbf{j}}{\sqrt{2}}$			$\{R_1, R_2\}$	$\mathfrak{C}_4$	$\langle 2,2,4\rangle$	2[8]2
	R_2:	$-\mathbf{i}$	$\frac{\mathfrak{C}_{12}}{\mathfrak{C}_{12}}$	$\frac{\langle 2,3,4\rangle}{\langle 2,3,4\rangle}$	$\{R_1, R_3\}$	$\mathfrak{C}_{12}$	$\langle 2,3,4\rangle$	3[8]2
	R_3:	$\frac{1}{2} + \frac{\mathbf{i}}{2} + \frac{\mathbf{j}}{2} + \frac{\mathbf{k}}{2}$			$\{R_2, R_3\}$	$\mathfrak{C}_{12}$	$\langle 2,3,3\rangle$	3[6]2
19†	R_1:	$\mathbf{i}\,\frac{(2+\tau)^{\frac{1}{2}}}{\sqrt{5}} - \mathbf{k}\,\frac{(3-\tau)^{\frac{1}{2}}}{\sqrt{5}}$			$\{R_1, R_2\}$	$\mathfrak{C}_{12}$	$\langle 2,3,5\rangle$	3[10]2
	R_2:	$\frac{1}{2} + \mathbf{i}\,\frac{(4\tau+3)^{\frac{1}{2}}}{2\sqrt{5}}$			$\{R_1, R_3\}$	$\mathfrak{C}_{20}$	$\langle 2,3,5\rangle$	5[6]2
		$+\,\mathbf{j}\,\frac{3-\tau}{2\sqrt{5}} + \mathbf{k}\,\frac{(2+\tau)^{\frac{1}{2}}}{2\sqrt{5}}$	$\frac{\mathfrak{C}_{60}}{\mathfrak{C}_{60}}$	$\frac{\langle 2,3,5\rangle}{\langle 2,3,5\rangle}$	$\{R_2, R_3\}$	$\mathfrak{C}_{30}$	$\langle 2,3,5\rangle$	5[4]3
	R_3:	$-\eta^2$						
22†	R_1:	$-\mathbf{i}\,\frac{(2+\tau)^{\frac{1}{2}}}{\sqrt{5}} + \mathbf{k}\,\frac{(3-\tau)^{\frac{1}{2}}}{\sqrt{5}}$			$\{R_1, R_2\}$	$\mathfrak{C}_4$	$\langle 2,2,5\rangle$	2[5]2
	R_2:	$\mathbf{i}\,\frac{(2+\tau)^{\frac{1}{2}}}{\sqrt{5}} + \frac{\mathbf{j}}{2} - \mathbf{k}\,\frac{(7-4\tau)^{\frac{1}{2}}}{2\sqrt{5}}$	$\frac{\mathfrak{C}_4}{\mathfrak{C}_4}$	$\frac{\langle 2,3,5\rangle}{\langle 2,3,5\rangle}$	$\{R_1, R_3\}$	$\mathfrak{C}_4$	$\langle 2,2,5\rangle$	2[5]2
	R_3:	$\mathbf{i}\,\frac{(2+\tau)^{\frac{1}{2}}}{\sqrt{5}} - \frac{\mathbf{j}}{2} - \mathbf{k}\,\frac{(7-4\tau)^{\frac{1}{2}}}{2\sqrt{5}}$			$\{R_2, R_3\}$	$\mathfrak{C}_4$	$\langle 2,2,3\rangle$	2[6]2

*When a reflection R has period n, its quaternion form is $q' = aqb$, where $a = \exp 2\pi\mathbf{i}/n$ and b is given in the second column below.

†Here $\tau = (1 + \sqrt{5})/2$.

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