23
Non-Abelian fields

Positive and negative electrical charges label the different kinds of matter that respond to the electromagnetic field; there is the gravitational mass, for example, which only seems to have positive sign and labels matter which responds to the gravitational field. More kinds of charge are required to label particles which respond to the nuclear forces. With more kinds of charge, there are many more possibilities for conservation than simply that the sum of all positive and negative charges is constant. Non-Abelian gauge theories are physical models analogous to electromagnetism, but with more general ideas of charge. Some have three kinds of charge: red, green and blue (named whimsically after the primary colours); other theories have more kinds with very complicated rules about how the different charges are conserved. This chapter is about such theories.

23.1 Lie groups and algebras

In chapter 9 it was noted that the gauge invariance of matter and electromagnetic radiation could be thought of as a symmetry group called $U(1)$: the group of phase transformations on matter fields:

$$\Phi \to e^{i\theta(x)}\Phi, \tag{23.1}$$

for some scalar function $\theta(x)$. Since phase factors of this type commute with one another,

$$[e^{i\theta(x)}, e^{i\theta'(x)}] = 0 \tag{23.2}$$

such a symmetry group is called commutative or *Abelian*. The symmetry group was identified from the anti-symmetry properties of the curls in Maxwell's equations, but the full beauty of the symmetry only became apparent in the covariant formulation of field theory.

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In the study of angular momentum in chapter 9, it was noted that the symmetry group of rotations in two spatial dimensions was $U(1)$, but that in three spatial dimensions it was $O(3)$. The latter is a *non-Abelian* group, i.e. its generators do not commute; instead they have a commutator which satisfies a relation called a Lie algebra.

Non-Abelian gauge theories have, for the most part, been the domain of particle physicists trying to explain the elementary nature of the nuclear forces in collision experiments. In recent times, some non-Abelian field theories have also been used by condensed matter physicists. In the latter case, it is not fundamental fields which satisfy the exotic symmetry properties, but composite excitations in matter referred to as quasi-particles.

The motivation for non-Abelian field theory is the existence of families of excitations which are related to one another by the fact that they share and respond to a common form of charge. Each so-called flavour of excitation is represented by an individual field which satisfies an equation of motion. The fields are grouped together so that they form the components of a column vector, and *matrices*, which multiply these vectors exact symmetry transformations on them – precisely analogous to the phase transformations of electromagnetism, but now with more components. The local form of the symmetry requires the existence of a non-Abelian gauge field, A_μ, which is matrix-valued.

Thus one asks the question: what happens if fields are grouped into multiplets (analogous to the components of angular momentum) by postulating hidden symmetries, based on non-Abelian groups.

This idea was first used by Yang and Mills in 1954 to develop the isospin $SU(2)$ model for the nuclear force [141]. The unfolding of the experimental evidence surrounding nucleons led to a series of deductions about conservation from observed particle lifetimes. Charge labels such as baryon number, isospin and strangeness were invented to give a name to these, and the supposition that conserved charges are associated with symmetries led to the development of non-Abelian symmetry models. For a summary of the particle physics, see, for example, refs. [34, 108].

Non-Abelian models have been used in condensed matter physics, where quasi-fields for mean-field spin systems have been formulated as field theories with $SU(N)$ symmetry [1, 54].

23.2 Construction

We can now extend the formalism in the remainder of this book to encompass non-Abelian fields. To do this, we have to treat the fields as multi-component vectors on the abstract internal space of the symmetry group, since the transformations which act on the fields are now matrices. The dimension of the matrices which act on matter fields (Klein–Gordon or Dirac) does not have to be the same as those which attach to the gauge field A_μ – the only requirement is that both

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sets of matrices satisfy the same algebra. This will become clearer when we examine the nature of gauge transformations for non-Abelian groups.

We begin with some notation. Let $\{T^a_{\mathrm{R}}\}$, where $a = 1, \ldots, d_G$, denote a set of matrices which acts as the generators of the simple Lie algebra for the group G. These matrices satisfy the Lie algebra

$$[T^a, T^b] = -\mathrm{i} f^{ab}{}_c T^c \,. \tag{23.3}$$

T^a are chosen here to be Hermitian. This makes the structure constants real numbers. It is also possible to find an anti-Hermitian representation by multiplying all of the T^a by a factor of $\mathrm{i} = \sqrt{-1}$, but we shall not use this convention here. With anti-Hermitian conventions, the Abelian limit leads to an imaginary electric charge which does not agree with the conventions used in other chapters.

The T^a are $d_{\mathrm{R}} \times d_{\mathrm{R}}$ matrices. In component form, one may therefore write them explicitly $(T^a)_{AB}$, where $A, B = 1, \ldots, d_{\mathrm{R}}$, but normally the explicit components of T^a are suppressed and a matrix multiplication is understood. We denote the group which is obtained from these by G_{R}, which means the representation R of the group G. The normalization of the generators is fixed by defining

$$\mathrm{Tr}\left(T^a_{\mathrm{R}} T^b_{\mathrm{R}}\right) = I_2(G_{\mathrm{R}})\delta^{ab}, \tag{23.4}$$

where I_2 is called the Dynkin index for the representation G_{R}. The Dynkin index may also be written

$$I_2(G_{\mathrm{R}}) = \frac{d_{\mathrm{R}}}{d_G} C_2(G_{\mathrm{R}}), \tag{23.5}$$

where d_R is the dimension (number of rows/columns) of the generators in the representation G_R and d_G is the dimension of the group. $C_2(G_R)$ is the quadratic Casimir invariant for the group in the representation G_R: $C_2(G_R)$ and $I_2(G_R)$ are constants which are listed in tables for various representations of Lie groups. d_G is the same as the dimension of the adjoint representation of the algebra G_{adj}, by definition of the adjoint representation. Note therefore that $I_2(G_{\mathrm{adj}}) = C_2(G_{\mathrm{adj}})$.

In many texts, authors make the arbitrary choice of replacing the right hand side of eqn. (23.4) with $\frac{1}{2}\delta^{ab}$. This practice simplifies formulae in a small number of special cases, but can lead to confusion later. Also, it makes the identification of group constants (for arbitrary groups) impossible and leads therefore to expressions which are not covariant with respect to changes of symmetry group.

To construct a physical theory with such an internal symmetry group we must look to the behaviour of the fields under a symmetry transformation. We

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require the analogue of a gauge transformation in the Abelian case. We begin by assuming that the form of a symmetry transformation on matter fields is

$$\Phi \rightarrow U\Phi, \tag{23.6}$$

for some matter field Φ and some matrix

$$U = \exp\left(\mathrm{i}\theta^a(x)T^a\right), \tag{23.7}$$

which is the element of some Lie group, with an algebra generated by T^a, $(a = 1, \ldots, d_G)$. Eqn. (23.6) contains an implicit matrix multiplication: the components are normally suppressed; if we write them explicitly, eqn. (23.6) has the appearance:

$$\Phi^A \rightarrow U^A_{\ B}\Phi^B. \tag{23.8}$$

Since the generators do not commute with one another, and since U is a combination of these generators, T^a and U cannot commute; moreover, consecutive gauge transformations do not commute,

$$[U, U'] \neq 0, \tag{23.9}$$

in general. The exception to this statement is if the group element U lies in the centre of the group (i.e. the group's Abelian sub-group) which is generated purely by the Cartan sub-algebra:

$$U_{\mathrm{c}} = \exp\left(\mathrm{i}\theta^i(x)H^i\right), \qquad (i = 1, \ldots, \mathrm{rank}\, G) \tag{23.10}$$

$$0 = [U_{\mathrm{c}}, U]. \tag{23.11}$$

Under such a transformation, the spacetime-covariant derivative is not gauge-covariant:

$$\partial_\mu(U\Phi) \neq U(\partial_\mu\Phi). \tag{23.12}$$

We must therefore follow the analogue of the procedure in chapter 10 to define a covariant derivative for the non-Abelian symmetry. We do this in the usual way, by introducing a gauge connection, or vector potential

$$A_\mu = A^a_\mu(x)T^a, \tag{23.13}$$

which is a linear combination of all the generators. The basis components $A^a_\mu(x)$ are now the physical fields, which are to be varied in the action. There is one such field for each generator, i.e. the total number of fields is equal to the dimension of the group d_G. In terms of this new field, we write the covariant derivative

$$D_\mu = \partial_\mu + \mathrm{i}\frac{g}{\hbar}A_\mu, \tag{23.14}$$

where g is a new charge for the non-Abelian symmetry. As in the Abelian case, D_μ will only satisfy

$$D_\mu(U\Phi) = U(D_\mu\Phi), \tag{23.15}$$

if Φ and A_μ both transform together. We can determine the way in which A_μ must transform by writing

$$\begin{aligned} D_\mu(U\Phi) &= (\partial_\mu U)\Phi + U(\partial_\mu\Phi) + \mathrm{i}\frac{g}{\hbar}A_\mu U\Phi \\ &= U\left(\partial_\mu\Phi + U^{-1}(\partial_\mu U)\Phi + \mathrm{i}\frac{g}{\hbar}U^{-1}A_\mu U\Phi\right). \end{aligned} \tag{23.16}$$

From this, we deduce that

$$\mathrm{i}\frac{g}{\hbar}A'_\mu\Phi = \mathrm{i}\frac{g}{\hbar}U^{-1}A_\mu U\Phi + U^{-1}(\partial_\mu U)\Phi, \tag{23.17}$$

so that the complete non-Abelian gauge transformation has the form

$$\begin{aligned} \Phi' &= U\Phi \\ A'_\mu &= U^{-1}A_\mu U - \frac{\mathrm{i}\hbar}{g}U^{-1}(\partial_\mu U). \end{aligned} \tag{23.18}$$

The transformation of the field strength tensor in a non-Abelian field theory can be derived from its definition:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}\frac{g}{\hbar}[A_\mu, A_\nu], \tag{23.19}$$

and has the form

$$F_{\mu\nu} \to U^{-1}F_{\mu\nu}U. \tag{23.20}$$

Note that the field strength is not gauge-invariant: it transforms in a non-trivial way. This means that $F_{\mu\nu}$ is not an observable in non-Abelian field theory. The field strength tensor can also be expressed directly in terms of the covariant derivative by the formula

$$[D_\mu, D_\nu] = \mathrm{i}\frac{g}{\hbar}F_{\mu\nu}, \tag{23.21}$$

or

$$F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu - \mathrm{i}\frac{g}{\hbar}[A_\mu, A_\nu]. \tag{23.22}$$

The field strength can also be expressed as a linear combination of the generators of the Lie algebra, and we define the physical components relative to a given basis set T^a by

$$F_{\mu\nu} = F^a_{\mu\nu}T^a. \tag{23.23}$$

Using the algebra relation (23.3), these components can be expressed in the form

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^a_{\ bc}A^b_\mu A^c_\nu. \tag{23.24}$$

23.3 The action

We are now in a position to postulate a form for the action of a non-Abelian gauge theory. We have no way of knowing what the 'correct' action for such a theory is (nor any way of knowing if such a theory is relevant to nature), so we allow ourselves to be guided by the invariant quantities which can be formed from the non-Abelian fields. For free scalar matter fields, it is natural to write

$$S_{\mathrm{M}} = \int (\mathrm{d}x) \left\{ \hbar^2 c^2 (D^\mu \Phi)^\dagger (D_\mu \Phi) + m^2 c^4 \Phi^\dagger \Phi \right\}, \qquad (23.25)$$

where

$$D_\mu \Phi = \partial_\mu \Phi + \mathrm{i} g A_\mu \Phi \qquad (23.26)$$

which has the form of a matrix acting on a vector. Clearly, the number of components in the vector Φ must be the same as the number of rows and columns in the matrix A_μ in order for this to make sense. The dagger symbol implies complex conjugation and transposition.

For the non-Abelian Yang–Mills field the action analogous to the Maxwell action is $S_{\mathrm{YM}}[\overline{A} + A]$, where

$$S_{\mathrm{YM}}[A] = \frac{1}{4\mu_{\mathrm{NA}} I_2(G_{\mathrm{adj}})} \int (\mathrm{d}x) \mathrm{Tr}\Big(F^{\mu\nu} F_{\mu\nu} \Big), \qquad (23.27)$$

where μ_{NA} is analogous to the permeability in electromagnetism. The trace in eqn. (23.27) refers to the trace over implicit matrix components of the generators. The cyclic property of the trace ensures that this quantity is gauge-invariant. Under a gauge transformation, one has

$$\mathrm{Tr}\left(F^{\mu\nu} F_{\mu\nu}\right) \to \mathrm{Tr}\left(U^{-1} F^{\mu\nu} F_{\mu\nu} U\right) = \mathrm{Tr}\left(F^{\mu\nu} F_{\mu\nu}\right). \qquad (23.28)$$

23.4 Equations of motion and continuity

The Wong equations describe classical point particles coupled to a non-Abelian gauge field [140]:

$$\begin{aligned} m\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} &= p^\mu \\ m\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} &= g Q^a F^{a\mu\nu} p_\nu \\ m\frac{\mathrm{d}Q}{\mathrm{d}\tau} &= -g f^{abc} p^\mu A_\mu^b Q^c. \end{aligned} \qquad (23.29)$$

23.5 Multiple representations

The gauge field A_μ appears several times in the action: both in connection with the covariant derivative acting on matter fields, and in connection with the Yang–Mills term. The dimension d_R of the matrix representation used in the different parts of the action does not have to be equal throughout. Indeed, the number of components in the matter vector is chosen on 'phenomenological' grounds to match the number of particles known to exist in a multiplet. A *common* choice is:

- the fundamental representation for matter fields, i.e. $A_\mu = A^a_\mu T^a_f$ in D_μ;
- the adjoint representation for the Yang–Mills terms, i.e. $A_\mu = A^a_\mu T^a_{\text{adj}}$ in $\text{Tr}F^2$. *common* situation is to choose

Although this is a common situation, it is not a necessity. The choice of representation for the matter fields should be motivated by phenomenology. In the classical theory, there seems to be no good reason for choosing the adjoint matrices for gauge fields. It is always true that the components of the field transform in the adjoint representation regardless of the matrices used to define the action.

23.6 The adjoint representation

One commonly held belief is that the gauge field, A_μ, must be constructed from the generators of the adjoint representation. The components of the gauge field A^a_μ transform like a vector in the adjoint representation, regardless of the matrix representations used to define the gauge fields in the action. This follows simply from the fact that A_μ is a linear combination of all the generators of the algebra [21]. To show this, we begin by noting that, in a given representation, the structure constants which are identical for any matrix representation form the components of a matrix representation for the adjoint representation, by virtue of the Jacobi identity (see section 8.5.2).

Consider an arbitrary field Λ, with components θ^a relative to a set of basis generators T^a in an arbitrary representation, defined by

$$\Lambda = T^a \lambda^a. \tag{23.30}$$

The generator matrices may be in a representation with arbitrary dimension d_R. Under a gauge transformation, we shall assume that the field transforms like

$$\Lambda' = U^{-1} \Lambda U, \tag{23.31}$$

where U is in the same matrix representation as T^a and may be written

$$U = \exp(\mathrm{i}\theta^a T^a). \tag{23.32}$$

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Using the matrix identity,

$$\exp(A)B\exp(-A) = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots, \tag{23.33}$$

it is straightforward to show that

$$\Lambda' = \lambda^a \Big\{ \delta^a{}_r - \theta_b f^{ab}{}_r + \frac{1}{2}\theta_b\theta_c f^{ca}{}_s f^{bs}{}_r - \frac{1}{3!}\theta_b\theta_c\theta_d f^{da}{}_q f^{cq}{}_p f^{bp}{}_r + \ldots \Big\} T^r, \tag{23.34}$$

where the algebra commutation relation has been used. In our notation, the generators of the adjoint representation may be written

$$(T^a_{\text{adj}})^b{}_c = \mathrm{i} f^{ab}{}_c, \tag{23.35}$$

and the structure constants are real. Eqn. (23.34) may therefore be identified as

$$\Lambda' = \lambda^a (U_{\text{adj}})^a{}_b T^b, \tag{23.36}$$

where

$$U_{\text{adj}} = \exp(\mathrm{i}\theta^a T^a_{\text{adj}}). \tag{23.37}$$

If we now define the components of the transformed field by

$$\Lambda' = \lambda'^a T^a, \tag{23.38}$$

in terms of the original generators, then it follows that

$$\lambda'^a = (U_{\text{adj}})^a{}_b \lambda^b. \tag{23.39}$$

We can now think of the set of components λ^a and λ'^a as being grouped into d_G component column *vectors* λ and λ', so that

$$\lambda' = U_{\text{adj}}\lambda. \tag{23.40}$$

In matrix notation, the covariant derivative of the matrix-valued field Λ is

$$D_\mu \Lambda = \partial_\mu \Lambda + \mathrm{i}g[A_\mu, \Lambda], \tag{23.41}$$

for any representation. Using the algebra commutation relation this becomes

$$D_\mu \Lambda = \partial_\mu \Lambda + \mathrm{i}g A^{\text{adj}}_\mu \Lambda, \tag{23.42}$$

where $A^{\text{adj}}_\mu = A^a_\mu T^a_{\text{adj}}$. We have therefore shown that the vectorial components of the gauge field transform according to the adjoint representation, regardless of the matrices which are used in the matrix form.

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23.7 Field equations and continuity

$$S = \int (\mathrm{d}x) \left\{ (D^\mu \Phi)^\dagger (D_\mu \Phi) + m^2 \Phi^\dagger \Phi + \frac{1}{4 I_2(G_{\mathrm{adj}})} \mathrm{Tr}\Big(F^{\mu\nu} F_{\mu\nu} \Big) \right\}. \tag{23.43}$$

The variation of the action with respect to $\Phi^\dagger$ yields the equation of motion for Φ:

$$\begin{aligned} \delta S &= \int (\mathrm{d}x) \left\{ (D^\mu \delta\Phi)^\dagger (D_\mu \Phi) + m^2 \delta\Phi^\dagger \Phi \right\} \\ &= \int (\mathrm{d}x) \delta\Phi^\dagger \left\{ -D^2 \Phi + m^2 \Phi \right\} \\ &+ \int \mathrm{d}\sigma^\mu \left\{ \delta\Phi^\dagger (D_\mu \Phi) \right\}. \end{aligned} \tag{23.44}$$

The gauge-fixing term is

$$S_{\mathrm{GF}} = \frac{1}{2\alpha \mu_{\mathrm{NA}} I_2(G_{\mathrm{adj}})} \int \mathrm{d}v_x \mathrm{Tr}\Big(D_\mu A^\mu \Big)^2. \tag{23.45}$$

23.8 Commonly used generators

It is useful to have explicit forms for the generators in the fundamental and adjoint representations for the two most commonly discussed groups. For $SU(N)$, the matrices of the fundamental representation have dimension N.

23.8.1 $SU(2)$ Hermitian fundamental representation

Here, the generators are simply one-half the Pauli matrices in the usual basis:

$$\begin{aligned} T^1 &= \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ T^2 &= \frac{1}{2} \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \\ T^3 &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{23.46}$$

In the Cartan–Weyl basis, we construct

$$H = T^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$E_\alpha = \frac{1}{\sqrt{2}} (T^1 + \mathrm{i} T^2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

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$$E_{-\alpha} = \frac{1}{\sqrt{2}}(T^1 - \mathrm{i}T^2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{23.47}$$

where the eigenvalue $\alpha = 1$ and

$$\begin{aligned} [H, E_\alpha] &= \alpha E_\alpha \\ [E_\alpha, E_{-\alpha}] &= \alpha H. \end{aligned} \tag{23.48}$$

The diagonal components of H are the weights of the representation.

23.8.2 $SU(2)$ Hermitian adjoint representation

In the adjoint representation, the generators are simply the components of the structure constants in the regular basis:

$$T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}$$

$$T^2 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}$$

$$T^3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{23.49}$$

To find a Cartan–Weyl basis, in which the Cartan sub-algebra matrices are diagonal, we explicitly look for a transformation which diagonalizes one of the matrices. The same transformation will diagonalize the entire Cartan sub-algebra. Pick arbitrarily T^1 to diagonalize. The self-inverse matrix of eigenvectors for T^1 is easily found. It is given by

$$\Lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-\mathrm{i}}{\sqrt{2}} \\ 0 & \frac{\mathrm{i}}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}. \tag{23.50}$$

Constructing the matrices $\Lambda^{-1}T^a\Lambda$, one finds a new set of generators,

$$T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$T^2 = \begin{pmatrix} 0 & 1 & \mathrm{i} \\ 1 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}$$

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$$T^3 = \begin{pmatrix} 0 & \mathrm{i} & 1 \\ -\mathrm{i} & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{23.51}$$

The Cartan–Weyl basis is obtained from these by constructing the combinations

$$\begin{aligned} E_{\pm 1} &= \frac{1}{\sqrt{2}}(T^3 \mp \mathrm{i}T^2) \\ H &= T^1. \end{aligned} \tag{23.52}$$

Explicitly,

$$\begin{aligned} E_1 &= \begin{pmatrix} 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ E_{-1} &= \begin{pmatrix} 0 & 0 & 1 \\ -\mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{23.53}$$

It may be verified that

$$[H, E_\alpha] = \alpha E_\alpha \tag{23.54}$$

for $\alpha = \pm 1$. The diagonal values of H are the roots of the Lie algebra. It is interesting to note that the footprint of $SU(2)$ crops up often in the generators of other groups. This is because $SU(2)$ sub-groups are a basic entity where the roots show the simplest reflection symmetry. Since roots occur in signed pairs, $SU(2)$ is associated with root pairs.

23.8.3 $SU(3)$ Hermitian fundamental representation

The generators of $SU(3)$'s fundamental representation are the Gell-Mann matrices:

$$\begin{aligned} T_1 &= \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ T_2 &= \frac{1}{2}\begin{pmatrix} 0 & \mathrm{i} & 0 \\ -\mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ T_3 &= \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

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$$T_4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$T_5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}$$

$$T_6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$T_7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{i} \\ 0 & -\mathrm{i} & 0 \end{pmatrix}$$

$$T_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{23.55}$$

The generators of the Cartan sub-algebra T^3 and T^8 are already diagonal in this representation. Forming a matrix which is an explicit linear combination of these generators $\theta^a T^a$, the following linear combinations are seen to parametrize the algebra naturally:

$$\begin{aligned} E_{\mp 1} &= \frac{\mathrm{i}}{\sqrt{2}}(T^1 \pm \mathrm{i}T^2) \\ E_{\mp 2} &= \frac{\mathrm{i}}{\sqrt{2}}(T^4 \pm \mathrm{i}T^5) \\ E_{\mp 3} &= \frac{\mathrm{i}}{\sqrt{2}}(T^6 \pm \mathrm{i}T^7) \\ H^1 &= T^3 \\ H^2 &= T^8. \end{aligned} \tag{23.56}$$

These matrices satisfy the Cartan–Weyl relations

$$\begin{aligned} \left[H^i, E_\alpha\right] &= \alpha^i E_\alpha \\ [E_\alpha, E_{-\alpha}] &= \alpha^i H_i, \end{aligned} \tag{23.57}$$

where i is summed over the elements of the Cartan sub-algebra. This last relation tells us that the commutator of the generators for equal and opposite roots always generates an element of the centre of the group. The coefficients α^i are the components of the root vectors on the sub-space spanned by the Cartan

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sub-algebra. Explicitly,

$$
\begin{aligned}
E_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
E_{-1} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
E_2 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
E_{-2} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
E_3 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \\
E_{-3} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
\tag{23.58}
$$

Constructing all of the opposite combinations in the second relation of eqn. (23.57), one finds the root vectors in the Cartan–Weyl basis,

$$
\begin{aligned}
\alpha_{\pm 1} &= \pm(1, 0) \\
\alpha_{\pm 2} &= \pm\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \\
\alpha_{\pm 3} &= \pm\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right).
\end{aligned}
\tag{23.59}
$$

23.8.4 $SU(3)$ Hermitian adjoint representation

The generators in the adjoint representation are obtained from the observation in eqn. (23.35) that the structure constants form a representation of the Lie algebra with the same dimension as the group:

$$
(T^a)^b{}_c = \mathrm{i} f^{ab}{}_c, \tag{23.60}
$$

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where $a, b, c = 1, \ldots, 8$. The structure constants are

$$\begin{aligned} f_{123} &= 1 \\ f_{147} &= -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2} \\ f_{458} &= f_{678} = \frac{\sqrt{3}}{2}, \end{aligned} \tag{23.61}$$

together with anti-symmetric permutations. In explicit form, we have

$$T^1 = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T^2 = \mathrm{i}\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T^3 = \mathrm{i}\begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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$$T^4 = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 \end{pmatrix}$$

$$T^5 = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T^6 = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}$$

$$T^7 = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 \end{pmatrix}$$

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$$T^8 = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{23.62}$$

The anti-Hermitian form of these matrices is obtained by dropping the leading factor of i. The Cartan–Weyl basis for the adjoint representation is obtained by diagonalizing one (and thereby several) of the generators. We choose to diagonalize T^8 because of its simple form. This matrix has four zero eigenvalues representing an invariant sub-space, so eigenvectors must be constructed for these manually. A set of normalized eigenvectors can be formed into a matrix which will diagonalize the generators of the Cartan sub-algebra:

$$\Lambda = \begin{pmatrix} \frac{\mathrm{i}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{\mathrm{i}}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\mathrm{i}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{\mathrm{i}}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\mathrm{i}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{\mathrm{i}}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{23.63}$$

The inverse of this is simply the complex conjugate. The new basis is now constructed by forming $\Lambda^{-1}T^a\Lambda$:

$$T^1 = \begin{pmatrix} 0 & 0 & \frac{\mathrm{i}}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ -\frac{\mathrm{i}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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$$T^2 = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2\sqrt{2}} & 0 & 0 & 0 & 0 & -\frac{i\sqrt{3}}{2\sqrt{2}} \\ 0 & 0 & -\frac{1}{2\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2\sqrt{2}} \\ 0 & -\frac{i}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{i}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & 0 & 0 & 0 \end{pmatrix}$$

$$T^5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\sqrt{2}} & \frac{i}{2\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2\sqrt{2}} & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2\sqrt{2}} \\ 0 & 0 & -\frac{i}{2\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{i\sqrt{3}}{2\sqrt{2}} \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2\sqrt{2}} & -\frac{i\sqrt{3}}{2\sqrt{2}} & 0 & 0 & 0 \end{pmatrix}$$

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$$T^6 = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\mathrm{i}}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{i}\frac{\mathrm{i}}{2\sqrt{2}} & 0 & 0 & 0 & 0 & -\frac{\mathrm{i}\sqrt{3}}{2\sqrt{2}} \\ 0 & 0 & \frac{1}{2\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & \frac{\mathrm{i}\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & 0 \end{pmatrix}$$

$$T^7 = \begin{pmatrix} 0 & 0 & 0 & -\frac{\mathrm{i}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\mathrm{i}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{\mathrm{i}}{2\sqrt{2}} & 0 \\ \frac{\mathrm{i}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\mathrm{i}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2\sqrt{2}} & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2\sqrt{2}} \\ 0 & 0 & \frac{\mathrm{i}}{2\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{\mathrm{i}\sqrt{3}}{2\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2\sqrt{2}} & -\frac{\mathrm{i}\sqrt{3}}{2\sqrt{2}} & 0 \end{pmatrix}$$

$$T^8 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{23.64}$$

The Cartan–Weyl basis is now obtained by constructing the linear combinations

$$\begin{aligned} E_{\mp 1} &= \frac{1}{\sqrt{2}}(T^1 \pm \mathrm{i}T^2) \\ E_{\mp 2} &= \frac{1}{\sqrt{2}}(T^4 \pm \mathrm{i}T^5) \\ E_{\mp 3} &= \frac{1}{\sqrt{2}}(T^6 \pm \mathrm{i}T^7). \end{aligned} \tag{23.65}$$

Explicitly,

$$E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E_{-1} = \begin{pmatrix} 0 & 0 & \mathrm{i} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & -\frac{\mathrm{i}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & -\mathrm{i}\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\mathrm{i}}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{i}\frac{\sqrt{3}}{2} & 0 & 0 & 0 \end{pmatrix}$$

$$E_{-2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{\mathrm{i}}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{\mathrm{i}}{2} & 0 & 0 & 0 & 0 & -\mathrm{i}\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\mathrm{i}}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & 0 & i\frac{\sqrt{3}}{2} & 0 & 0 \end{pmatrix}$$

$$E_{-3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{i}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}. \tag{23.66}$$

These generators satisfy the relations in eqns. (23.57) and define the components of the root vectors in two ways. The diagonal components of the generators spanning the Cartan sub-algebra are the components of the root vectors. We define

$$\begin{aligned} H_1 &= T_3 \\ H_2 &= T_8. \end{aligned} \tag{23.67}$$

The commutators in eqns. (23.57) may now be calculated, and one identifies

$$\begin{aligned} \alpha_{\pm 1} &= \mp(1, 0) \\ \alpha_{\pm 2} &= \mp\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \\ \alpha_{\pm 3} &= \mp\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right). \end{aligned} \tag{23.68}$$

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