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2-LOG-CONCAVITY OF THE BOROS–MOLL POLYNOMIALS

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Abstract The Boros–Moll polynomials $P_m(a)$ arise in the evaluation of a quartic integral. It has been conjectured by Boros and Moll that these polynomials are infinitely log-concave. In this paper, we show that $P_m(a)$ is 2-log-concave for any $m \geqslant 2$. Let $d_i(m)$ be the coefficient of a^i in $P_m(a)$. We also show that the sequence $\{i(i+1)(d_i^2(m) - d_{i-1}(m)d_{i+1}(m))\}_{1\leqslant i\leqslant m}$ is log-concave.

Keywords: 2-log-concavity; Boros–Moll polynomial; quartic integral

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1. Introduction

The objective of this paper is to prove the 2-log-concavity of the Boros–Moll polynomials. Recall that a sequence $\{a_i\}_{0\leqslant i\leqslant n}$ of real numbers is said to be unimodal if there exists an index $0 \leqslant j \leqslant n$ such that

$$a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{j-1} \leqslant a_j \geqslant a_{j+1} \geqslant \cdots \geqslant a_n.$$

Set $a_{-1} = 0$ and $a_{n+1} = 0$. We say that $\{a_i\}_{0\leqslant i\leqslant n}$ is log-concave if

$$a_i^2 - a_{i+1}a_{i-1} \geqslant 0, \quad 1 \leqslant i \leqslant n.$$

A polynomial is said to be unimodal (respectively, log-concave) if the sequence of its coefficients is unimodal (respectively, log-concave). It is easy to see that, for a positive sequence, the log-concavity is stronger than the unimodality. For a sequence $A = \{a_i\}_{0\leqslant i\leqslant n}$, define the operator $\mathcal{L}$ by $\mathcal{L}(A)= \{b_i\}_{0\leqslant i\leqslant n}$, where

$$b_i = a_i^2 - a_{i-1}a_{i+1}, \quad 0 \leqslant i \leqslant n. \tag{1.1}$$

Boros and Moll [**8**] introduced the notion of infinite log-concavity. We say that the sequence $\{a_i\}_{0\leqslant i\leqslant n}$ is k-log-concave if the sequence $\mathcal{L}^j(\{a_i\}_{0\leqslant i\leqslant n})$ is log-concave for every $0 \leqslant j \leqslant k-1$, and we say that $\{a_i\}_{0\leqslant i\leqslant n}$ is ∞-log-concave if $\mathcal{L}^k(\{a_i\}_{0\leqslant i\leqslant n})$ is log-concave for any $k \geqslant 0$.

Boros and Moll [**8**] conjectured that the binomial coefficients $\binom{n}{k}$ are infinitely log-concave for any n. A generalization of this conjecture was given independently by Fisk [**20**], McNamara and Sagan [**23**] and Stanley; see [**9**], which states that if a polynomial $a_0+a_1x+\cdots+a_nx^n$ has only real zeros, then the polynomial $b_0+b_1x+\cdots+b_nx^n$ also has only real zeros, where $b_i = a_i^2 - a_{i-1}a_{i+1}$. This conjecture has been proved by Brändén [**9**]. While Brändén's theorem does not directly apply to the Boros–Moll polynomials, the 2-log-concavity and 3-log-concavity can be recast in terms of the real rootedness of certain polynomials derived from the Boros–Moll polynomials, as conjectured by Brändén. It is worth mentioning that McNamara and Sagan [**23**] conjectured that, for fixed k, the q-Gaussian coefficients $\left[{n \atop k}\right]$ are infinitely q-log-concave. Chen *et al*. [**16**] proved the strong q-log-concavity of the q-Narayana numbers $N_q(n,k)$ for fixed k, which turns out to be equivalent to the 2-fold q-log-concavity of the Gaussian coefficients.

Recall that Boros and Moll [**4**–**8**, **24**] have studied the following quartic integral and have shown that, for any $a > -1$ and any non-negative integer m,

$$\int_0^\infty \frac{1}{(x^4+2ax^2+1)^{m+1}}\,\mathrm{d}x = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}}P_m(a),$$

where

$$P_m(a) = \sum_{j,k}\binom{2m+1}{2j}\binom{m-j}{k}\binom{2k+2j}{k+j}\frac{(a+1)^j(a-1)^k}{2^{3(k+j)}}. \tag{1.2}$$

Using Ramanujan's master theorem, Boros and Moll [**7**, **24**] obtained the following formula for $P_m(a)$:

$$P_m(a) = 2^{-2m}\sum_k 2^k\binom{2m-2k}{m-k}\binom{m+k}{k}(a+1)^k, \tag{1.3}$$

which implies that $P_m(a)$ is a polynomial in a with positive coefficients. Chen *et al*. [**14**] used a combinatorial argument to show that the double sum (1.2) can be reduced to the single sum (1.3). Let $d_i(m)$ be the coefficient of a^i of $P_m(a)$, that is,

$$P_m(a) = \sum_{i=0}^m d_i(m)a^i. \tag{1.4}$$

For any m, $P_m(a)$ is called a Boros–Moll polynomial, and the sequence $\{d_i(m)\}_{0\leqslant i\leqslant m}$ is called a Boros–Moll sequence. From (1.3), we know that $d_i(m)$ can be expressed as

$$d_i(m) = 2^{-2m}\sum_{k=i}^m 2^k\binom{2m-2k}{m-k}\binom{m+k}{k}\binom{k}{i}. \tag{1.5}$$

Several proofs of the above formula can be found in the survey of Amdeberhan and Moll [**1**, **2**].

Boros and Moll [**5**] proved that the sequence $\{d_i(m)\}_{0\leqslant i\leqslant m}$ is unimodal and that the maximum element appears in the middle. In other words,

$$d_0(m) < d_1(m) < \cdots < d_{[m/2]-1}(m) < d_{[m/2]}(m) > d_{[m/2]+1}(m) > \cdots > d_m(m).$$

They also established the unimodality by a different approach [**6**]. Moll [**24**] conjectured that the sequence $\{d_i(m)\}_{0\leqslant i\leqslant m}$ is log-concave. Kauers and Paule [**21**] proved this conjecture based on recurrence relations, which were derived by using a computer algebra approach. Chen *et al*. [**15**] found a combinatorial proof of the log-concavity of $P_m(a)$ by introducing the structure of partially 2-coloured permutations. Chen and Gu [**10**] proved the reverse ultra log-concavity of the sequence $\{d_i(m)\}_{0\leqslant i\leqslant m}$. Amdeberhan *et al*. [**3**] studied the 2-adic valuation of an integer sequence and obtained a combinatorial interpretation of the valuations of the integer sequence, which is related to the Boros–Moll sequences. Chen and Xia [**11**] showed that the sequence $\{d_i(m)\}_{0\leqslant i\leqslant m}$ satisfies the strong ratio monotone property, which implies the log-concavity and the spiral property. Furthermore, Chen *et al*. [**18**] proved that if $f(x)$ is a polynomial with non-decreasing and non-negative coefficients, then $f(1+x)$ is ratio monotone. From (1.3), it is easy to see that the coefficients of $P_n(x-1)$ are non-decreasing and non-negative. Hence, the polynomials $P_n(x)$ are log-concave and ratio monotone. Recently, Chen *et al*. [**17**] introduced the notion of interlacing log-concavity and proved that the Boros–Moll polynomials possess this property.

Boros and Moll [**8**] made the following conjecture.

Conjecture 1.1. *The Boros–Moll sequence* $\{d_i(m)\}_{0\leqslant i\leqslant m}$ *is* ∞*-log-concave.*

As shown by Boros and Moll [**5**], in general, $P_m(a)$ are not polynomials with only real zeros. Thus, the theorem of Brändén [**9**] does not apply to $P_m(a)$. Nevertheless, Brändén made the following conjectures on the real rootedness of polynomials derived from $P_m(a)$. These conjectures imply the 2-log-concavity and the 3-log-concavity of the Boros–Moll polynomials.

Conjecture 1.2 (Brändén). *For each positive integer* m*, the polynomial*

$$Q_m(x) = \sum_{i=0}^{m} \frac{d_i(m)}{i!} x^i$$

has only real zeros.

Conjecture 1.3 (Brändén). *For each positive integer* m*, the polynomial*

$$R_m(x) = \sum_{i=0}^{m} \frac{d_i(m)}{(i+2)!} x^i$$

has only real zeros.

Note that $Q_m(x) = \mathrm{d}(x^2 R_m(x))/\mathrm{d}x^2$. Hence, $Q_m(x)$ has only real zeros if $R_m(x)$ does. This yields that Conjecture 1.3 is stronger than Conjecture 1.2. Based on a result of

Craven and Csordas [**19**], it can be seen that Conjecture 1.2 implies that $P_m(a)$ is 2-log-concave and Conjecture 1.3 implies that $P_m(a)$ is 3-log-concave. Following this, Chen *et al*. [**13**] proved Conjectures 1.2 and 1.3 by showing that both $Q_n(x)$ and $R_n(x)$ form Sturm sequences.

In another direction, Kauers and Paule [**21**] considered using the approach of recurrence relations to prove the 2-log-concavity of $P_m(a)$, and they indicated that there is little hope of success since the recurrence relations are too complicated.

Roughly speaking, the main focus of this paper is to find an intermediate function $f(m, i)$, so we can reduce quartic inequalities for the 2-log-concavity to quadratic inequalities. To be precise, the 2-log-concavity is stated as follows.

Theorem 1.4. *The Boros–Moll sequences are 2-log-concave, that is, for* $1 \leqslant i \leqslant m-1$,

$$\frac{d_{i-1}^2(m) - d_{i-2}(m)d_i(m)}{d_i^2(m) - d_{i-1}(m)d_{i+1}(m)} < \frac{d_i^2(m) - d_{i-1}(m)d_{i+1}(m)}{d_{i+1}^2(m) - d_i(m)d_{i+2}(m)}. \tag{1.6}$$

The intermediate function $f(m, i)$ is given by

$$f(m, i) = \frac{(i+1)(i+2)(m+i+3)^2}{(m+1-i)(m+2-i)(m+i+2)^2}. \tag{1.7}$$

Using this intermediate function, we can divide the 2-log-concavity into two quadratic inequalities, which are stated below.

Theorem 1.5. *For* $1 \leqslant i \leqslant m-1$, *we have that*

$$\frac{(i+1)(i+2)(m+i+3)^2}{(m+1-i)(m+2-i)(m+i+2)^2} < \frac{d_i^2(m) - d_{i-1}(m)d_{i+1}(m)}{d_{i+1}^2(m) - d_i(m)d_{i+2}(m)}. \tag{1.8}$$

Theorem 1.6. *For* $1 \leqslant i \leqslant m-1$, *we have that*

$$\frac{d_{i-1}^2(m) - d_{i-2}(m)d_i(m)}{d_i^2(m) - d_{i-1}(m)d_{i+1}(m)} < \frac{(i+1)(i+2)(m+i+3)^2}{(m+1-i)(m+2-i)(m+i+2)^2}. \tag{1.9}$$

As will be seen, the 2-log-concavity of $P_m(a)$ implies the log-concavity of a sequence considered by Moll [**22, 25**].

Theorem 1.7. *For* $m \geqslant 2$, *the sequence* $\{i(i+1)(d_i^2(m) - d_{i-1}(m)d_{i+1}(m))\}_{1 \leqslant i \leqslant m}$ *is log-concave.*

Since log-concavity implies unimodality, the above property leads to another proof of Moll's minimum conjecture [**25**] for $\{i(i+1)(d_i^2(m) - d_{i-1}(m)d_{i+1}(m))\}_{1 \leqslant i \leqslant m}$. By comparing the first entry with the last entry, we deduce that this sequence attains its minimum at $i = m$, which equals $2^{-2m}m(m+1)\binom{2m}{m}^2$. This conjecture was confirmed by Chen and Xia [**12**] by using a result of Chen and Gu [**10**] and the spiral property of the Boros–Moll sequences [**11**].

2. How to guess the intermediate function $f(m, i)$

In this section, we explain how we found the intermediate function $f(m, i)$. We begin with a brief review of the approach of Kauers and Paule to proving the log-concavity of the Boros–Moll polynomials [**21**], because we need the recurrence relations and an inequality that they established. The four recurrence relations are

$$d_i(m+1) = \frac{m+i}{m+1} d_{i-1}(m) + \frac{(4m+2i+3)}{2(m+1)} d_i(m), \quad 0 \leqslant i \leqslant m+1, \tag{2.1}$$

$$d_i(m+1) = \frac{(4m-2i+3)(m+i+1)}{2(m+1)(m+1-i)} d_i(m) - \frac{i(i+1)}{(m+1)(m+1-i)} d_{i+1}(m), \quad 0 \leqslant i \leqslant m, \tag{2.2}$$

$$d_i(m+2) = \frac{-4i^2+8m^2+24m+19}{2(m+2-i)(m+2)} d_i(m+1) - \frac{(m+i+1)(4m+3)(4m+5)}{4(m+2-i)(m+1)(m+2)} d_i(m), \quad 0 \leqslant i \leqslant m+1, \tag{2.3}$$

and, for $0 \leqslant i \leqslant m+1$,

$$(m+2-i)(m+i-1)d_{i-2}(m) - (i-1)(2m+1)d_{i-1}(m) + i(i-1)d_i(m) = 0. \tag{2.4}$$

These recurrences are derived by Kauers and Paule [**21**]. In fact, (2.3) and (2.4) are derived independently by Moll [**25**] via the WZ-method [**26**], and the other two relations, (2.1) and (2.2), can be easily deduced from (2.3) and (2.4). Based on the four recurrence relations, Kauers and Paule [**21**] proved the following inequality, from which the log-concavity of the Boros–Moll sequences can be deduced.

Theorem 2.1 (see [21]). *Let m, i be integers, with $m \geqslant 2$. For $0 < i < m$, we have that*

$$\frac{d_i(m+1)}{d_i(m)} \geqslant \frac{4m^2+7m+i+3}{2(m+1-i)(m+1)}. \tag{2.5}$$

Chen and Gu [**10**] showed that $\{i! d_i(m)\}_{0 \leqslant i \leqslant m}$ is log-concave and that the sequence $\{d_i(m)\}_{0 \leqslant i \leqslant m}$ is reverse ultra log-concave. They established the following upper bound for $d_i(m+1)/d_i(m)$.

Theorem 2.2 (see [10]). *Let m, i be integers and let $m \geqslant 2$. We have, for $0 \leqslant i \leqslant m$, that*

$$\frac{d_i(m+1)}{d_i(m)} \leqslant \frac{4m^2+7m+3+i\sqrt{4m+4i^2+1}-2i^2}{2(m+1)(m+1-i)}. \tag{2.6}$$

Theorems 2.1 and 2.2 are needed in the proofs of Theorems 1.5 and 1.6, and they are also necessary for us to take a good guess at the intermediate function $f(m, i)$. We start with an approximation of

$$\frac{d_{i-1}^2(m) - d_{i-2}(m)d_i(m)}{d_i^2(m) - d_{i-1}(m)d_{i+1}(m)}.$$

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Recall that the relation

$$\lim_{m\to+\infty}\frac{d_i^2(m)}{(1+(1/i))(1+(1/m-i))d_{i-1}(m)d_{i+1}(m)}=1$$

was proved by Chen and Gu [**10**]. This implies that

$$\frac{d_{i-1}^2(m)-d_{i-2}(m)d_i(m)}{d_i^2(m)-d_{i-1}(m)d_{i+1}(m)}\approx\frac{(i+1)(m+1-i)d_{i-1}^2(m)}{i(m+2-i)d_i^2(m)}. \tag{2.7}$$

Using the recurrence relation (2.1), we find that

$$\frac{d_{i-1}^2(m)}{d_i^2(m)}=\frac{(m+1)^2d_i^2(m+1)}{(m+i)^2d_i^2(m)}-\frac{(4m+2i+3)(m+1)d_i(m+1)}{(m+i)^2d_i(m)}+\frac{(4m+2i+3)^2}{4(m+i)^2}. \tag{2.8}$$

On the other hand, by Theorems 2.1 and 2.2, we get that

$$\lim_{m\to+\infty}\frac{2(m+1)(m+1-i)d_i(m+1)}{(4m^2+7m+i+3)d_i(m)}=1.$$

It follows that

$$\frac{d_i(m+1)}{d_i(m)}\approx\frac{4m^2+7m+i+3}{2(m+1)(m+1-i)}. \tag{2.9}$$

Substituting (2.9) into (2.8) yields that

$$\frac{d_{i-1}^2(m)}{d_i^2(m)}\approx\frac{i^2(i+1+m)^2}{(m+1-i)^2(m+i)^2}. \tag{2.10}$$

Combining (2.7) and (2.10), we deduce that

$$\frac{d_{i-1}^2(m)-d_{i-2}(m)d_i(m)}{d_i^2(m)-d_{i-1}(m)d_{i+1}(m)}\approx\frac{i(i+1)(m+1+i)^2}{(m+1-i)(m+2-i)(m+i)^2}. \tag{2.11}$$

It turns out that the above expression is not the intermediate function that we are looking for; we should try to make it a little bigger. The above expression gives a guideline for a suitable adjustment. We consider the shifts of the factors in (2.11). After a few attempts, we find that the following function serves as the desired intermediate function:

$$\frac{(i+1)(i+2)(m+i+3)^2}{(m+1-i)(m+2-i)(m+i+2)^2}, \tag{2.12}$$

which is the function $f(m,i)$ as given by (1.7).

3. Proof of Theorem 1.5

In this section, we give a proof of Theorem 1.5. This is achieved as follows. We wish to prove an equivalent form of Theorem 1.5, that is, that the difference

$$\begin{aligned}(m+1-i)(m+2-i)(m+i+2)^2(d_i^2(m)-d_{i-1}(m)d_{i+1}(m))\\ -(i+1)(i+2)(m+i+3)^2(d_{i+1}^2(m)-d_i(m)d_{i+2}(m))\end{aligned} \tag{3.1}$$

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is positive. As will be seen, in view of the recurrence relations of $d_i(m)$, (3.1) can be written as

$$A(m,i)d_i^2(m+1)+B(m,i)d_i(m+1)d_i(m)+C(m,i)d_i^2(m), \tag{3.2}$$

where $A(m,i)$, $B(m,i)$ and $C(m,i)$ are given by (3.4)–(3.6). To prove that the quadratic form (3.2) is positive, we consider the quadratic polynomial in $d_i(m+1)/d_i(m)$,

$$A(m,i)\frac{d_i^2(m+1)}{d_i^2(m)}+B(m,i)\frac{d_i(m+1)}{d_i(m)}+C(m,i). \tag{3.3}$$

It will be shown that $A(m,i)<0$ for $1\leqslant i\leqslant m$. Moreover, we shall show that the above polynomial has distinct real roots x_1 and x_2. Assume that $x_1<x_2$. If the relation

$$x_1<\frac{d_i(m+1)}{d_i(m)}<x_2$$

holds, then the quadratic polynomial (3.3) is positive.

Let

$$A(m,i)=-\frac{(m+1)^2(m+1-i)^2D(m,i)}{(m+i)i^2(i+1)}, \tag{3.4}$$

$$B(m,i)=\frac{(i-m-1)(m+1)E(m,i)}{(i+m)i^2(i+1)}, \tag{3.5}$$

$$C(m,i)=\frac{F(m,i)}{4(i+m)i^2(i+1)}, \tag{3.6}$$

$$\begin{aligned}\Delta_1(m,i)&=B^2(m,i)-4A(m,i)C(m,i)\\&=\frac{(m+1-i)^2(m+1)^2(4(m+i)^2G(m,i)+H(m,i))}{i^2(i+m)^2(i+1)^2},\end{aligned} \tag{3.7}$$

where $D(m,i)$, $E(m,i)$, $F(m,i)$, $G(m,i)$ and $H(m,i)$ are given in the appendix.

Theorem 3.1. *For $1\leqslant i\leqslant m-1$ and $m\geqslant 126$, we have that*

$$\frac{-B(m,i)+\sqrt{\Delta_1(m,i)}}{2A(m,i)}<\frac{d_i(m+1)}{d_i(m)}<\frac{-B(m,i)-\sqrt{\Delta_1(m,i)}}{2A(m,i)}. \tag{3.8}$$

In order to prove Theorem 3.1, it is necessary to show that $\Delta_1(m,i)>0$.

Lemma 3.2. *For $1\leqslant i\leqslant m-1$ and $m\geqslant 126$, we have that $\Delta_1(m,i)>0$.*

Proof. In view of definition (3.7) of $\Delta_1(m,i)$ and the fact that $H(m,i)$ is positive, it suffices to show that $G(m,i)>0$ for $1\leqslant i\leqslant m-1$. We consider three cases with respect to the range of i.

Case 1 ($i^3\geqslant\frac{3}{7}m^2$). In this case, we have that

$$m^2(2i^3-m^2)^2\geqslant 0,\qquad 56i^6m-24i^3m^3\geqslant 0,\qquad 20i^5m^2-2i^2m^4>0,$$

and so $G(m,i)>0$.

Case 2 ($\frac{1}{10}m^2 < i^3 < \frac{3}{7}m^2$). In this case, we have that

$$m^2(2i^3 - m^2)^2 \geqslant \tfrac{1}{49}m^6, \qquad 56i^6m - 24i^3m^3 \geqslant -\tfrac{18}{7}m^5, \qquad 20i^5m^2 - 2i^2m^4 > 0.$$

Thus, for $m \geqslant 126$,

$$G(m,i) \geqslant \tfrac{1}{49}m^6 - \tfrac{18}{7}m^5 > 0.$$

Case 3 ($1 \leqslant i^3 \leqslant \frac{1}{10}m^2$). In this case, we have that

$$m^2(2i^3 - m^2)^2 \geqslant \tfrac{16}{25}m^6, \qquad 56i^6m - 24i^3m^3 \geqslant -\tfrac{46}{25}m^5, \qquad 20i^5m^2 - 2i^2m^4 > -2m^{16/3}.$$

It follows that

$$G(m,i) \geqslant \tfrac{16}{25}m^6 - \tfrac{46}{25}m^5 - 2m^{16/3}. \tag{3.9}$$

It is easy to check that the right-hand side of (3.9) is positive for $m \geqslant 10$. This completes the proof. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. We first consider the lower bound of $d_i(m+1)/d_i(m)$, namely, that

$$\frac{d_i(m+1)}{d_i(m)} > \frac{-B(m,i) + \sqrt{\Delta_1(m,i)}}{2A(m,i)}. \tag{3.10}$$

From inequality (2.5) of Kauers and Paule [**21**], we see that (3.10) is a consequence of

$$\frac{4m^2 + 7m + i + 3}{2(m+1)(m+1-i)} > \frac{-B(m,i) + \sqrt{\Delta_1(m,i)}}{2A(m,i)}. \tag{3.11}$$

Since $A(m,i) < 0$ for $1 \leqslant i \leqslant m$, (3.11) can be rewritten as

$$A(m,i)\frac{4m^2 + 7m + i + 3}{(m+1)(m+1-i)} + B(m,i) < \sqrt{\Delta_1(m,i)}. \tag{3.12}$$

To verify (3.12), we calculate the difference of the squares of both sides. It is easy to check that

$$\Delta_1(m,i) - \left(A(m,i)\frac{4m^2 + 7m + i + 3}{(m+1)(m+1-i)} + B(m,i)\right)^2 = \frac{(m+1-i)^2(m+1)^2K(m,i)}{i^2(i+m)^2(i+1)^2},$$

where $K(m,i)$ is given in the appendix. It is easy to check that $K(m,i)$ is positive for $1 \leqslant i \leqslant m-1$. Hence, by Lemma 3.2, we obtain (3.12). This yields (3.10).

It remains to consider the upper bound of $d_i(m+1)/d_i(m)$, namely, that

$$\frac{d_i(m+1)}{d_i(m)} < \frac{-B(m,i) - \sqrt{\Delta_1(m,i)}}{2A(m,i)}. \tag{3.13}$$

By Theorem 2.2 of [**10**], we see that (3.13) is a consequence of

$$\frac{4m^2 + 7m + i\sqrt{4i^2 + 4m + 1} - 2i^2 + 3}{2(m+1)(m+1-i)} < \frac{-B(m,i) - \sqrt{\Delta_1(m,i)}}{2A(m,i)}. \tag{3.14}$$

Since $A(m,i) < 0$ for $1 \leqslant i \leqslant m-1$, (3.14) can be rewritten as

$$A(m,i)\frac{4m^2+7m+i\sqrt{4i^2+4m+1}-2i^2+3}{(m+1)(m+1-i)} + B(m,i) > -\sqrt{\Delta_1(m,i)}. \tag{3.15}$$

As before, we can check (3.15) by computing the difference of the squares of both sides. It is readily seen that

$$\Delta_1(m,i) - \left(A(m,i)\frac{4m^2+7m+i\sqrt{4i^2+4m+1}-2i^2+3}{(m+1)(m+1-i)} + B(m,i)\right)^2 = \frac{(m+1-i)^2(m+1)^2L(m,i)}{i^2(i+m)(i+1)^2},$$

where $L(m,i)$ is given in the appendix. It is easy to verify that

$$\begin{aligned}
&(2i^2m+4im^2+2m^3+i^2+24m+14im+13m^2+6i+9)^2(4i^2+4m+1)\\
&\quad - (-4i^3m-8i^2m^2-4im^3-20i^2m\\
&\qquad -24im^2-4m^3+7i^2-28im-19m^2+20i-20m+7)^2\\
&\qquad\quad = 16i^6m+96i^5m^2+176i^4m^3+128i^3m^4+48i^2m^5+32im^6\\
&\qquad\qquad +16m^7+4i^6+264i^5m+972i^4m^2+1088i^3m^3+492i^2m^4\\
&\qquad\qquad +312im^5+196m^6+48i^5+1456i^4m+3248i^3m^2+2064i^2m^3\\
&\qquad\qquad +1184im^4+960m^5+168i^4+3508i^3m+4368i^2m^2+2372im^3\\
&\qquad\qquad +2384m^4+164i^3+3876i^2m+3036im^2+3196m^3-120i^2\\
&\qquad\qquad +2164im+2404m^2-172i+1036m+32,
\end{aligned}$$

which is positive for $1 \leqslant i \leqslant m-1$. So, we reach the conclusion that $L(m,i) > 0$. Therefore, we obtain (3.15), which implies (3.13). In view of (3.10) and (3.13), we arrive at (3.8). This completes the proof. □

To conclude this section, we present a proof of Theorem 1.5.

Proof of Theorem 1.5. First we show that (3.1) can be represented in terms of $d_i(m)$ and $d_i(m+1)$. In view of (2.1), (2.2) and (2.4), we find that, for $1 \leqslant i \leqslant m-1$,

$$d_{i+1}(m) = \frac{(4m-2i+3)(m+i+1)}{2i(i+1)}d_i(m) - \frac{(m+1-i)(m+1)}{i(i+1)}d_i(m+1), \tag{3.16}$$

$$d_{i+2}(m) = \frac{2m+1}{i+2}d_{i+1}(m) - \frac{(m-i)(m+i+1)}{(i+1)(i+2)}d_i(m), \tag{3.17}$$

$$d_{i-1}(m) = \frac{m+1}{m+i}d_i(m+1) - \frac{4m+2i+3}{2(m+i)}d_i(m). \tag{3.18}$$

Applying the above recurrence relations, we get that

$$\begin{aligned}
&(m+1-i)(m+2-i)(m+i+2)^2(d_i^2(m)-d_{i-1}(m)d_{i+1}(m))\\
&\quad -(i+1)(i+2)(m+i+3)^2(d_{i+1}^2(m)-d_i(m)d_{i+2}(m))\\
&\qquad = A(m,i)d_i^2(m+1)+B(m,i)d_i(m+1)d_i(m)+C(m,i)d_i^2(m).
\end{aligned} \tag{3.19}$$

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It is easy to verify that Theorem 1.5 holds for $2 \leqslant m \leqslant 125$. By Theorem 3.1, we conclude that (3.2) is positive for $m \geqslant 126$ and $1 \leqslant i \leqslant m-1$. This completes the proof. □

4. Proof of Theorem 1.6

In this section, we give a proof of Theorem 1.6. The main steps can be described as follows. To prove the theorem, we show that the difference

$$(i+1)(i+2)(m+i+3)^2(d_i^2(m)-d_{i-1}(m)d_{i+1}(m)) \\ -(m+1-i)(m+2-i)(m+i+2)^2(d_{i-1}^2(m)-d_{i-2}(m)d_i(m)) \quad (4.1)$$

is positive for $1 \leqslant i \leqslant m-1$. By the recurrence relations of $d_i(m)$, (4.1) can be restated as

$$U(m,i)d_i^2(m+1)+V(m,i)d_i(m+1)d_i(m)+W(m,i)d_i^2(m), \quad (4.2)$$

where $U(m,i)$, $V(m,i)$ and $W(m,i)$ are given by (4.3)–(4.5), respectively. We consider five cases for the range of i. The conclusion in each case implies that (4.2) is positive. Note that the definition of $\Delta_2(m,i)$ is given in (4.6), and can be either positive or negative depending on the range of i.

Case 1 ($1 \leqslant i < (m^2/2)^{1/3} - m^{1/3}$). In this case, $\Delta_2(m,i)$ can be either non-negative or negative. We consider the case when $\Delta_2(m,i)$ is non-negative. Theorem 4.1 is established for this purpose.

Case 2 ($(m^2/2)^{1/3} - m^{1/3} \leqslant i \leqslant (m^2/2)^{1/3}$). In this case, we demonstrate that $\Delta_2(m,i) < 0$.

Case 3 ($(m^2/2)^{1/3} < i < m^{2/3}$). In this case, $\Delta_2(m,i)$ can be either non-negative or negative. We establish Theorem 4.3 when $\Delta_2(m,i)$ is non-negative.

Case 4 ($m^{2/3} \leqslant i \leqslant m-4$). We show that $\Delta_2(m,i) > 0$ and give a new lower bound on the ratio $d_i(m+1)/d_i(m)$, which implies that (4.2) is positive.

Case 5 ($m-3 \leqslant i \leqslant m-1$). It can be verified that (4.2) is positive.

The following notation will be used in the statement of Theorem 4.1. Let

$$U(m,i) = \frac{(m+1)^2(m+1-i)R(m,i)}{i(m+i)^2}, \quad (4.3)$$

$$V(m,i) = \frac{(m+1)S(m,i)}{i(m+i-1)(m+i)^2}, \quad (4.4)$$

$$W(m,i) = \frac{T(m,i)}{4i(m+i-1)(m+i)^2}, \quad (4.5)$$

$$\Delta_2(m,i) = V^2(m,i) - 4U(m,i)W(m,i) = \frac{(m+1)^2X(m,i)}{i(m+i)^2(m+i-1)^2}, \quad (4.6)$$

where $R(m,i)$, $S(m,i)$, $T(m,i)$ and $X(m,i)$ are given in the appendix. Obviously, $U(m,i)$ is positive for $1 \leqslant i \leqslant m-1$.

In Case 1, we obtain the following inequality.

Theorem 4.1. *If $\Delta_2(m,i) \geqslant 0$, we have, for $1 \leqslant i \leqslant (m^2/2)^{1/3} - m^{1/3}$ and $m \geqslant 15$, that*

$$\frac{d_i(m+1)}{d_i(m)} < \frac{-V(m,i) - \sqrt{\Delta_2(m,i)}}{2U(m,i)}. \tag{4.7}$$

Proof. From inequality (2.6) of Chen and Gu [**10**], we see that (4.7) can be deduced from

$$\frac{4m^2 + 7m + i\sqrt{4i^2 + 4m + 1} + 3 - 2i^2}{2(m+1)(m+1-i)} < \frac{-V(m,i) - \sqrt{\Delta_2(m,i)}}{2U(m,i)}. \tag{4.8}$$

To prove (4.8), let

$$\begin{aligned} A_1(m,i) &= 2(m+1)(m+1-i), \\ B_1(m,i) &= 4m^2 + 7m + 3 - 2i^2, \\ C_1(m,i) &= 4i^2 + 4m + 1. \end{aligned}$$

Clearly, (4.8) can be restated as

$$D_1(m,i) > A_1(m,i)\sqrt{\Delta_2(m,i)} + 2iU(m,i)\sqrt{C_1(m,i)}, \tag{4.9}$$

where $D_1(m,i)$ is given by

$$\begin{aligned} D_1(m,i) &= -V(m,i)A_1(m,i) - 2U(m,i)B_1(m,i) \\ &= \frac{2(m+1)^2(m+1-i)(2m+1)(i^2 - i + m + m^2)(m+2+i)^2}{(i+m)^2(i+m-1)}. \end{aligned}$$

Hence, $D_1(m,i)$ is positive for $1 \leqslant i \leqslant m$. Since $D_1(m,i)$ is positive, (4.9) follows from

$$D_1^2(m,i) > (A_1(m,i)\sqrt{\Delta_2(m,i)} + 2iU(m,i)\sqrt{C_1(m,i)})^2, \tag{4.10}$$

which can be rewritten as

$$E_1(m,i) > 4iA_1(m,i)U(m,i)\sqrt{\Delta_2(m,i)C_1(m,i)}, \tag{4.11}$$

where $E_1(m,i)$ is given by

$$E_1(m,i) = D_1^2(m,i) - A_1^2(m,i)\Delta_2(m,i) - 4i^2U^2(m,i)C_1(m,i). \tag{4.12}$$

We see that (4.11) is valid if $E_1(m,i)$ is positive and

$$E_1^2(m,i) > 16i^2A_1^2(m,i)U^2(m,i)\Delta_2(m,i)C_1(m,i) \tag{4.13}$$

holds. Given definition (4.12) of $E_1(m,i)$, it is easy to check that

$$E_1(m,i) = -\frac{8(m+1-i)^2(m+1)^4R_1(m,i)S_1(m,i)}{i(m+i-1)(m+i)^3}, \tag{4.14}$$

where $R_1(m,i)$ and $S_1(m,i)$ are given in the appendix. Using expression (4.7) of $E_1(m,i)$, we see that the positivity of $E_1(m,i)$ can be derived from the fact that $S_1(m,i)$ is negative

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for $1 \leqslant i \leqslant (m^2/2)^{1/3} - m^{1/3}$ and $m \geqslant 15$. We now proceed to show that $S_1(m,i)$ is negative. For $15 \leqslant m \leqslant 728$, the claim can be directly verified. Therefore, we may assume that $m \geqslant 729$. By putting the terms of $S_1(m,i)$ into groups, as given in the appendix, one can see that the sum in every pair of parentheses is negative for $1 \leqslant i \leqslant (m^2/2)^{1/3} - m^{1/3}$ and $m \geqslant 729$. Moreover, it is easy to check that

$$8i^5m^2 - 4i^2m^4 < -15m^{11/3}i^2 + 20m^{10/3}i^2 - 8m^3i^2.$$

It follows that

$$\begin{aligned} S_1(m,i) &< -15m^{11/3}i^2 + 20m^{10/3}i^2 - 8m^3i^2 + 36i^4m^2 + 12i^3m^3 \\ &< (-5m^{5/3} + 43m^{4/3})m^2i^2, \end{aligned}$$

which is negative when $m \geqslant 729$. So, we conclude that $E_1(m,i) > 0$ for $1 \leqslant i \leqslant (m^2/2)^{1/3} - m^{1/3}$ and $m \geqslant 15$.

We now turn to the proof of (4.13). Consider the difference of the squares of both sides. It is routine to check that

$$\begin{aligned} F_1(m,i) &= E_1^2(m,i) - 16i^2U^2(m,i)A_1^2(m,i)\Delta_2(m,i)C_1(m,i) \\ &= \frac{-256(m+1-i)^4(m+1)^8M_1^2(m,i)N_1(m,i)}{i^2(i+m-1)^2(i+m)^6}, \end{aligned} \tag{4.15}$$

where $M_1(m,i)$ and $N_1(m,i)$ are given in the appendix. It is now easy to see that $N_1(m,i) < 0$ for $1 \leqslant i < (m^2/2)^{1/3} - m^{1/3}$ and $m \geqslant 15$. So, we have that $F_1(m,i) > 0$ for $1 \leqslant i < (m^2/2)^{1/3} - m^{1/3}$ and $m \geqslant 15$. Hence, (4.13) holds. This completes the proof. □

For Case 2, the following lemma asserts that $\Delta_2(m,i)$ is negative.

Lemma 4.2. *For $(m^2/2)^{1/3} - m^{1/3} \leqslant i \leqslant (m^2/2)^{1/3}$ and $m \geqslant 50$, we have that $\Delta_2(m,i) < 0$.*

Proof. By the definition (4.6) of $\Delta_2(m,i)$, it suffices to show that $X(m,i)$ is negative for $(m^2/2)^{1/3} - m^{1/3} \leqslant i \leqslant (m^2/2)^{1/3}$ and $m \geqslant 50$. For $50 \leqslant m \leqslant 2743$, the lemma can be directly verified. Hence, we may assume that $m \geqslant 2744$. Note that the expression in every pair of parentheses is negative for $(m^2/2)^{1/3} - m^{1/3} \leqslant i \leqslant (m^2/2)^{1/3}$ and $m \geqslant 2744$. On the other hand, one can check that

$$16i^7m^4 - 16i^4m^6 + 4im^8 = 4im^4(2i^3 - m^2)^2 < 58im^{22/3} \leqslant 47m^8,$$
$$64i^8m^3 - 24i^2m^7 + 16i^{11} + 64i^{10}m + 96i^9m^2 \leqslant -8i^2m^7 + 176i^9m^2 \leqslant -5m^{25/3} + 22m^8.$$

This yields that

$$X(m,i) < -5m^{25/3} + 69m^8,$$

where $X(m,i)$ is given in the appendix. But, the right-hand side of the above inequality is negative when $m \geqslant 2744$. This completes the proof. □

As will be seen, Theorems 4.3 and 4.4 have the same expression of the lower bound for $d_i(m+1)/d_i(m)$. This expression will be needed in the proof of Theorem 1.6. It should be noted that, for the case of Theorem 4.3, we show that this lower bound can be derived from the lower bound of Kauers and Paule [**21**]. Numerical evidence shows that the bound in Theorem 4.3 seems sharper than the bound of Kauers and Paule when i is large.

For Case 3, we have the following inequality. It should remarked that, in this case, $\Delta_2(m,i)$ can be either positive or negative, and there is no need to specify the range of i for which $\Delta_2(m,i)$ is positive.

Theorem 4.3. *If $\Delta_2(m,i) \geqslant 0$, we have, for $(m^2/2)^{1/3} \leqslant i \leqslant m^{2/3}$ and $m \geqslant 2$, that*

$$\frac{d_i(m+1)}{d_i(m)} > \frac{-V(m,i) + \sqrt{\Delta_2(m,i)}}{2U(m,i)}. \tag{4.16}$$

Proof. By the lower bound of $d_i(m+1)/d_i(m)$, as given in (2.5), we see that (4.16) can be obtained from

$$\frac{4m^2+7m+i+3}{2(m+1)(m+1-i)} > \frac{-V(m,i) + \sqrt{\Delta_2(m,i)}}{2U(m,i)}, \tag{4.17}$$

which can be rewritten as

$$U(m,i)\frac{4m^2+7m+i+3}{(m+1)(m+1-i)} + V(m,i) > \sqrt{\Delta_2(m,i)}. \tag{4.18}$$

In order to prove (4.18), we show that, for $(m^2/2)^{1/3} \leqslant i \leqslant m^{2/3}$ and $m \geqslant 2$,

$$U(m,i)\frac{4m^2+7m+i+3}{(m+1)(m+1-i)} + V(m,i) > 0 \tag{4.19}$$

and

$$\left(U(m,i)\frac{4m^2+7m+i+3}{(m+1)(m+1-i)} + V(m,i)\right)^2 - \Delta_2(m,i) > 0. \tag{4.20}$$

We first deal with (4.19). It is easy to check that

$$U(m,i)\frac{4m^2+7m+i+3}{(m+1)(m+1-i)} + V(m,i) = \frac{(m+1)P(m,i)}{(m+i)^2(m+i-1)},$$

where $P(m,i)$ is given in the appendix. Since the sum in every pair of parentheses in the expression of $P(m,i)$ is non-negative for $(m^2/2)^{1/3} \leqslant i \leqslant m^{2/3}$ and $m \geqslant 2$, it follows that $P(m,i) > 0$. Thus, we obtain (4.19).

We still need to consider (4.20). Clearly,

$$\left(U(m,i)\frac{4m^2+7m+i+3}{(m+1)(m+1-i)} + V(m,i)\right)^2 - \Delta_2(m,i) = \frac{4(m+1)^2 G_1(m,i) H_1(m,i)}{(m+i)^4(i+m-1)i},$$

where $G_1(m,i)$ and $H_1(m,i)$ are given in the appendix. We see that $G_1(m,i) > 0$ and $H_1(m,i) > 0$ for $(m^2/2)^{1/3} \leqslant i \leqslant m^{2/3}$ and $m \geqslant 2$. Hence, (4.20) holds. This completes the proof. □

For Case 4, we give a lower bound for $d_i(m+1)/d_i(m)$ that takes the same form as the lower bound in Case 3.

Theorem 4.4. *For $m \geqslant 273$ and $m^{2/3} \leqslant i \leqslant m-4$, we have that*

$$\frac{d_i(m+1)}{d_i(m)} > \frac{-V(m,i)+\sqrt{\Delta_2(m,i)}}{2U(m,i)}. \tag{4.21}$$

For the clarity of presentation, we establish two lemmas for the proof of Theorem 4.4. First, we prove that $\Delta_2(m,i)$ is positive.

Lemma 4.5. *For $m^{2/3} \leqslant i \leqslant m-1$ and $m \geqslant 19$, we have that $\Delta_2(m,i) > 0$.*

Proof. By definition (4.6) of $\Delta_2(m,i)$, it suffices to show that $X(m,i)$ is positive for $m^{2/3} \leqslant i \leqslant m-1$ and $m \geqslant 19$. By direct computation we find that the lemma holds for $19 \leqslant m \leqslant 132$. Moreover, for $m \geqslant 133$ and $m^{2/3} \leqslant i \leqslant m-1$, it can be seen that $X(m,i) > 0$; see the appendix. This completes the proof. □

The proof of Theorem 4.4 is by induction on m. The inductive argument requires an inequality concerning the desired lower bound. We present this inequality in Lemma 4.6. Let

$$Y_1(m,i) = \frac{(m+i+1)(4m+3)(4m+5)}{4(m+2-i)(m+1)(m+2)},$$

$$Y_2(m,i) = \frac{-4i^2+8m^2+24m+19}{2(m+2-i)(m+2)},$$

$$Y_3(m,i) = 2U(m+1,i)Y_2(m,i)+V(m+1,i) = \frac{(m+2)Y_5(m,i)}{(m+i)i(m+i+1)},$$

$$Y_4(m,i) = Y_3^2(m,i)-\Delta_2(m+1,i) = \frac{(m+2)^2Y_6(m,i)}{(m+1+i)^2i^2(m+i)},$$

where the explicit expressions for $Y_5(m,i)$ and $Y_6(m,i)$ are given in the appendix. One can easily see that $Y_1(m,i)$, $Y_2(m,i)$, $Y_3(m,i)$ and $Y_4(m,i)$ are all positive for $1 \leqslant i \leqslant m-1$ and $m \geqslant 2$.

Lemma 4.6. *For $m^{2/3} \leqslant i \leqslant m-4$ and $m \geqslant 273$, we have that*

$$\frac{-V(m,i)+\sqrt{\Delta_2(m,i)}}{2U(m,i)} > \frac{Y_1(m,i)}{Y_2(m,i)-(-V(m+1,i)+\sqrt{\Delta_2(m+1,i)})/2U(m+1,i)}. \tag{4.22}$$

Proof. We rewrite (4.22) as

$$\frac{-V(m,i)+\sqrt{\Delta_2(m,i)}}{2U(m,i)} > \frac{2U(m+1,i)Y_1(m,i)}{Y_3(m,i)-\sqrt{\Delta_2(m+1,i)}}. \tag{4.23}$$

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Since $Y_3(m,i) > 0$ and $Y_4(m,i) > 0$ for $m^{2/3} \leqslant i \leqslant m-4$ and $m \geqslant 273$, (4.23) follows from

$$V(m,i)\sqrt{\Delta_2(m+1,i)} + Y_3\sqrt{\Delta_2(m,i)} > Z_1(m,i) + \sqrt{\Delta_2(m,i)\Delta_2(m+1,i)}, \tag{4.24}$$

where $Z_1(m,i)$ is given by

$$Z_1(m,i) = 4U(m,i)U(m+1,i)Y_1(m,i) + V(m,i)Y_3(m,i). \tag{4.25}$$

Clearly, $Z_1(m,i) < 0$ for $m^{2/3} \leqslant i \leqslant m-4$ and $m \geqslant 273$. To prove (4.24), we shall show that

$$Z_1(m,i) + \sqrt{\Delta_2(m,i)\Delta_2(m+1,i)} < 0, \tag{4.26}$$

$$V(m,i)\sqrt{\Delta_2(m+1,i)} + Y_3(m,i)\sqrt{\Delta_2(m,i)} < 0 \tag{4.27}$$

and

$$(V(m,i)\sqrt{\Delta_2(m+1,i)} + Y_3(m,i)\sqrt{\Delta_2(m,i)})^2 < (Z_1(m,i) + \sqrt{\Delta_2(m,i)\Delta_2(m+1,i)})^2 \tag{4.28}$$

hold. We first consider (4.26). Let

$$Z_2(m,i) = \Delta_2(m,i)\Delta_2(m+1,i) - Z_1^2(m,i). \tag{4.29}$$

Employing the same argument as in the proofs of Lemmas 3.2, 4.2 and 4.5, we find that $Z_2(m,i) < 0$ for $m^{2/3} \leqslant i \leqslant m-4$ and $m \geqslant 273$. The detailed verification is omitted since the expansion of $Z_2(m,i)$ is a little lengthy. Thus, we obtain (4.26) since both $Z_1(m,i)$ and $Z_2(m,i)$ are negative for $m^{2/3} \leqslant i \leqslant m-4$ and $m \geqslant 273$.

We now turn to the proof of (4.27). Note that $V(m,i) < 0$ for $1 \leqslant i \leqslant m-1$. Let

$$Z_3(m,i) = Y_3^2(m,i)\Delta_2(m,i) - V^2(m,i)\Delta_2(m+1,i). \tag{4.30}$$

It is not difficult to show that $Z_3(m,i) < 0$ for $m^{2/3} \leqslant i \leqslant m-4$ and $m \geqslant 273$. The detailed proof is omitted, as before. Since $Z_3(m,i)$ and $V(m,i)$ are negative and $Y_3(m,i)$ and $\Delta_2(m,i)$ are positive for $m^{2/3} \leqslant i \leqslant m-4$ and $m \geqslant 273$, we arrive at (4.27).

It remains to prove (4.28), which can be restated as

$$Z_4(m,i) > Z_5(m,i)\sqrt{\Delta_2(m,i)\Delta_2(m+1,i)}, \tag{4.31}$$

where $Z_4(m,i)$ and $Z_5(m,i)$ are given by

$$\begin{aligned} Z_4(m,i) &= V^2(m,i)\Delta_2(m+1,i) + Y_3^2(m,i)\Delta_2(m,i) \\ &\quad - Z_1^2(m,i) - \Delta_2(m,i)\Delta_2(m+1,i), \end{aligned} \tag{4.32}$$

$$Z_5(m,i) = 2Z_1(m,i) - 2V(m,i)Y_3(m,i). \tag{4.33}$$

Using the same argument as in the proofs of Lemmas 3.2, 4.2 and 4.5, we can deduce that $Z_4(m,i)$ and $Z_5(m,i)$ are positive for $m^{2/3} \leqslant i \leqslant m-4$ and $m \geqslant 273$. Therefore, (4.31) is a consequence of the fact that

$$Z_6(m,i) = Z_5^2(m,i)\Delta_2(m,i)\Delta_2(m+1,i) - Z_4^2(m,i) \tag{4.34}$$

is positive for $m^{2/3} \leqslant i \leqslant m-4$ and $m \geqslant 273$, which is not difficult to prove although the working for $Z_6(m,i)$ is rather tedious. This completes the proof. □

We are now in a position to prove Theorem 4.4.

Proof of Theorem 4.4. We proceed by induction on m. It is easy to check that the theorem holds for $m = 273$. We assume that the theorem is true for $n \geqslant 273$, that is,

$$d_i(n+1) \geqslant \frac{-V(n,i)+\sqrt{\Delta_2(n,i)}}{2U(n,i)} d_i(n), \quad n^{2/3} \leqslant i \leqslant n-4. \tag{4.35}$$

We show that (4.21) holds for $m = n+1$, that is,

$$d_i(n+2) \geqslant \frac{-V(n+1,i)+\sqrt{\Delta_2(n+1,i)}}{2U(n+1,i)} d_i(n+1), \quad (n+1)^{2/3} \leqslant i \leqslant n-3. \tag{4.36}$$

In view of Lemma 4.6 and (4.35), we find that

$$d_i(n+1) > \frac{Y_1(n,i)}{Y_2(n,i) - (-V(n+1,i)+\sqrt{\Delta_2(n+1,i)})/2U(n+1,i)} d_i(n).$$

It follows that, for $n^{2/3} \leqslant i \leqslant n-4$,

$$Y_2(n,i)d_i(n+1) - Y_1(n,i)d_i(n) > \frac{-V(n+1,i)+\sqrt{\Delta_2(n+1,i)}}{2U(n+1,i)} d_i(n+1). \tag{4.37}$$

By recurrence relation (2.3), the left-hand side of (4.37) equals $d_i(n+2)$. Thus, we have verified (4.36) for $(n+1)^{2/3} \leqslant i \leqslant n-4$. It is still necessary to show that (4.36) is true for $i = n-3$, that is,

$$d_{n-3}(n+2) > \frac{-V(n+1,n-3)+\sqrt{\Delta_2(n+1,n-3)}}{2U(n+1,n-3)} d_{n-3}(n+1). \tag{4.38}$$

Let

$$\begin{aligned} f(n) = {} & 256n^{11} - 4608n^{10} + 36\,544n^9 - 177\,920n^8 + 572\,592n^7 - 1\,218\,432n^6 \\ & + 1\,573\,768n^5 - 940\,352n^4 - 66\,903n^3 - 65\,525n^2 - 3657n - 963. \end{aligned}$$

By expression (1.5) of $d_i(m)$, we have that

$$\begin{aligned} \frac{d_{n-3}(n+2)}{d_{n-3}(n+1)} &= \frac{(2n+5)(16n^4+80n^3+180n^2+240n+189)(2n-1)}{10(n+2)(45+72n+68n^2+48n^3+16n^4)} \\ &> \frac{12-65n+14n^2+3108n^4-3041n^3-1020n^5+136n^6+16n^7}{10(n+2)(2n-3)(1+2n+33n^2+4n^4-16n^3)} \\ &\quad + \frac{(n-1)\sqrt{(n-3)f(n)}}{10(n+2)(2n-3)(1+2n+33n^2+4n^4-16n^3)} \\ &= \frac{-V(n+1,n-3)+\sqrt{\Delta_2(n+1,n-3)}}{2U(n+1,n-3)}. \end{aligned}$$

Hence, the proof is complete by induction. □

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Finally, we are ready to complete the proof of Theorem 1.6.

Proof of Theorem 1.6. For $2 \leqslant m \leqslant 272$, the theorem can be easily verified. So, we may assume that $m \geqslant 273$. Difference (4.1) can be represented in terms of $d_i(m+1)$ and $d_i(m)$. From (2.4), it follows that

$$d_{i-2}(m) = \frac{(i-1)(2m+1)}{(m+2-i)(m+i-1)} d_{i-1}(m) - \frac{i(i-1)}{(m+2-i)(m+i-1)} d_i(m). \tag{4.39}$$

Using recurrence relations (3.16), (3.18) and (4.39), we find that

$$\begin{aligned}(i+1)&(i+2)(m+i+3)^2(d_i^2(m) - d_{i-1}(m)d_{i+1}(m)) \\ &- (m+1-i)(m+2-i)(m+i+2)^2(d_{i-1}^2(m) - d_{i-2}(m)d_i(m)) \\ &= U(m,i)d_i^2(m+1) + V(m,i)d_i(m+1)d_i(m) + W(m,i)d_i^2(m).\end{aligned} \tag{4.40}$$

Hence, the theorem shows that (4.2) is positive. If $\Delta_2(m,i) < 0$, it is obvious that (4.2) is positive, since $U(m,i) > 0$ for $1 \leqslant i \leqslant m-1$. We now assume that $\Delta_2(m,i) \geqslant 0$.

Recall the five cases for the range of i, as given before.

Case 1 ($1 \leqslant i < (m^2/2)^{1/3} - m^{1/3}$). By Theorem 4.1, we see that (4.2) is positive.

Case 2 ($(m^2/2)^{1/3} - m^{1/3} \leqslant i \leqslant (m^2/2)^{1/3}$). Note that, in this case, by Lemma 4.2, we have that $\Delta_2(m,i) < 0$, which corresponds to the case that we have already considered.

Case 3 ($(m^2/2)^{1/3} < i < m^{2/3}$). It follows from Theorem 4.3 that (4.2) is positive.

Case 4 ($m^{2/3} \leqslant i \leqslant m-4$). The lower bound given in Theorem 4.4 ensures that (4.2) is positive.

It remains to consider the cases when $i = m-3$, $m-2$, $m-1$. Here, we only verify the statement for $i = m-3$. The other two cases can be justified analogously. By (1.5), we see that

$$\begin{aligned}U(m,m-3)d_{m-3}^2(m+1) &+ V(m,m-3)d_{m-3}(m+1)d_{m-3}(m) + W(m,m-3)d_{m-3}^2(m) \\ &= \frac{(m+1)^2(m-2)g(m)}{9216(2m+1)^2(2m-1)^2(2m-3)^2} 2^{-2m} \binom{2m+2}{m+1}^2,\end{aligned}$$

where $g(m)$ is given by

$$\begin{aligned}g(m) = 2048m^{12} &- 10\,240m^{11} + 16\,512m^{10} - 3456m^9 - 35\,232m^8 \\ &+ 99\,120m^7 + 44\,488m^6 - 375\,620m^5 + 431\,652m^4 \\ &- 182\,601m^3 + 7362m^2 + 13797m - 2430,\end{aligned}$$

which is positive for $m \geqslant 273$. This completes the proof. □

To conclude this paper, we show that the 2-log-concavity of the Boros–Moll polynomials implies the log-concavity of the sequence $\{i(i+1)(d_i^2(m) - d_{i-1}(m)d_{i+1}(m))\}_{1\leqslant i\leqslant m}$, as stated in Theorem 1.7.

Clearly, for $i \geqslant 2$, we have that

$$\frac{i(i+1)}{(i-1)(i+2)} > 1. \tag{4.41}$$

By Theorem 1.4 and (4.41), we obtain that, for $2 \leqslant i \leqslant m-1$,

$$\frac{d_{i-1}^2(m) - d_{i-2}(m)d_i(m)}{d_i^2(m) - d_{i-1}(m)d_{i+1}(m)} < \frac{i(i+1)}{(i-1)(i+2)} \frac{d_i^2(m) - d_{i-1}(m)d_{i+1}(m)}{d_{i+1}^2(m) - d_i(m)d_{i+2}(m)}.$$

Replacing i by $i+1$, we find that, for $1 \leqslant i \leqslant m-2$,

$$\frac{d_i^2(m) - d_{i-1}(m)d_{i+1}(m)}{d_{i+1}^2(m) - d_i(m)d_{i+2}(m)} < \frac{(i+1)(i+2)}{i(i+3)} \frac{(d_{i+1}^2(m) - d_i(m)d_{i+2}(m))}{(d_{i+2}^2(m) - d_{i+1}(m)d_{i+3}(m))},$$

which can be written as

$$\frac{i(i+1)(d_i^2(m) - d_{i-1}(m)d_{i+1}(m))}{(i+1)(i+2)(d_{i+1}^2(m) - d_i(m)d_{i+2}(m))} < \frac{(i+1)(i+2)(d_{i+1}^2(m) - d_i(m)d_{i+2}(m))}{(i+2)(i+3)(d_{i+2}^2(m) - d_{i+1}(m)d_{i+3}(m))}.$$

Thus, the sequence $\{i(i+1)(d_i^2(m) - d_{i-1}(m)d_{i+1}(m))\}_{1\leqslant i\leqslant m}$ is log-concave.

Appendix A.

In the statement of Theorem 3.1, the polynomials $D(m,i)$, $E(m,i)$, $F(m,i)$, $G(m,i)$ and $H(m,i)$ are given by

$$\begin{aligned}
D(m,i) &= 6m^2i + 2m^2i^2 + 21mi + 14mi^2 + 4mi^3 + 10i \\
&\quad + 17i^2 + 10i^3 + 2i^4 + 2m^3 + 12m^2 + 18m, \\
E(m,i) &= 4i^2(i^2 - 2m^2)(i+m)^2 + 2(i+m)(10i^4 - 4m^4 - 9im^3 - 27i^2m^2 - 4i^3m) \\
&\quad + 27i^4 - 55i^3m - 175i^2m^2 - 139im^3 - 62m^4 - 16i^3 - 155i^2m \\
&\quad - 229im^2 - 162m^3 - 60i^2 - 142im - 162m^2 - 30i - 54m, \\
F(m,i) &= 32i^2m^2(i-m)(i+m)^3 + 16m(4i^4 + 10i^3m - 14i^2m^2 - 3im^3 - 2m^4)(i+m)^2 \\
&\quad + 2(i+m)(-152m^5 - 250im^4 - 377i^2m^3 + 111i^3m^2 + 181i^4m + 15i^5) \\
&\quad + 168i^5 + 694i^4m - 280i^3m^2 - 2052i^2m^3 - 2160im^4 - 1106m^5 + 273i^4 \\
&\quad - i^3m - 1809i^2m^2 - 2831im^3 - 1968m^4 + 18i^3 - 898i^2m - 1936im^2 \\
&\quad - 1836m^3 - 207i^2 - 663im - 864m^2 - 90i - 162m, \\
G(m,i) &= m^2(2i^3 - m^2)^2 + (56i^6m - 24i^3m^3) + (20i^5m^2 - 2i^2m^4) \\
&\quad + 4i^8 + 8i^7m + 40i^7 + 169i^6 + 166i^5m + 70i^4m^2,
\end{aligned}$$

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$$\begin{aligned}H(m,i) &= 1588i^7+4440i^6m+4768i^5m^2+2148i^4m^3+324i^3m^4+144i^2m^5\\&\quad+104im^6+52m^7+2345i^6+6666i^5m+6991i^4m^2+3624i^3m^3+1567i^2m^4\\&\quad+646im^5+289m^6+2418i^5+7232i^4m+8044i^3m^2+5340i^2m^3+2234im^4\\&\quad+892m^5+1903i^4+5810i^3m+7225i^2m^2+4104im^3+1618m^4+1086i^3\\&\quad+3332i^2m+3470im^2+1608m^3+321i^2+914im+657m^2.\end{aligned}$$

In the proof of Theorem 3.1, the polynomials $K(m,i)$ and $L(m,i)$ are given by

$$\begin{aligned}K(m,i) &= 4(2i^4+4i^3m+2i^2m^2+10i^3+14i^2m+6im^2\\&\qquad+2m^3+17i^2+21im+12m^2+10i+18m)\\&\quad\times(2i^3m^2+2i^2m^3-2i^5-2i^4m-9i^4+2i^3m+16i^2m^2+6im^3\\&\qquad+m^4-7i^3+23i^2m+23im^2+9m^3+12i^2+16im+20m^2+8i+8m),\\L(m,i) &= 2(2i^4+4i^3m+2i^2m^2+10i^3+14i^2m+6im^2\\&\qquad+2m^3+17i^2+21im+12m^2+10i+18m)\\&\quad\times(-4i^3m-8i^2m^2-4im^3-20i^2m-24im^2-4m^3+7i^2-28im-19m^2\\&\qquad+20i-20m+7+(2i^2m+4im^2+2m^3+i^2+24m\\&\qquad\qquad+14im+13m^2+6i+9)\sqrt{4i^2+4m+1}).\end{aligned}$$

In the statement of Theorem 4.1, the polynomials $R(m,i)$, $S(m,i)$, $T(m,i)$ and $X(m,i)$ are given by

$$\begin{aligned}R(m,i) &= 2i^2m^2+4mi^3+6im^2+14mi^2+2i^4+10i^3\\&\quad+21mi+17i^2+2m^3+12m^2+18m+10i,\\S(m,i) &= 4i^2(i^2-2m^2)(i+m)^3+2(8i^4-4i^3m-21i^2m^2-9im^3-4m^4)(i+m)^2\\&\quad+(i+m)(-54m^4-121im^3-99i^2m^2-41i^3m+7i^4)-41i^4\\&\quad-98i^3m-187i^2m^2-262im^3-100m^4-41i^3-51i^2m\\&\quad-106im^2+25i^2+45im+108m^2+30i+54m,\\T(m,i) &= 32i^2m^2(i+m)^4+16m(4i^4+18i^3m+18i^2m^2+7im^3+2m^4)(i+m)^2\\&\quad+2(i+m)(120m^5+414im^4+601i^2m^3+523i^3m^2+199i^4m+15i^5)\\&\quad+132i^5+850i^4m+1912i^3m^2+2652i^2m^3+2084im^4+562m^5+153i^4\\&\quad+417i^3m+983i^2m^2+1307im^3+300m^4-48i^3-328i^2m\\&\quad-248im^2-432m^3-177i^2-405im-540m^2-90i-162m,\\X(m,i) &= 16i^7m^4-16i^4m^6+4im^8+64i^8m^3-24i^2m^7+16i^{11}+64i^{10}m+96i^9m^2\\&\quad+(128i^{10}+448i^9m+624i^8m^2+448i^7m^3+160i^6m^4-100i^3m^6)\\&\quad+(372i^9+1280i^8m+1868i^7m^2+1256i^6m^3+128i^5m^4-240i^4m^5)\\&\quad+(340i^8+1712i^7m+2520i^6m^2\\&\qquad+620i^5m^3-1132i^4m^4-1096i^3m^5-528i^2m^6)\end{aligned}$$

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$$
\begin{aligned}
&+ (3692i^2m - 52im^7 - 16m^8 - 523i^7 - 2i^6m - 509i^5m^2\\
&\quad - 2584i^4m^3 - 3749i^3m^4 - 2910i^2m^5 - 635im^6 - 176m^7 - 1416i^6\\
&\quad - 5048i^3m^3 - 5940i^2m^4 - 1810im^5 - 656m^6 - 586i^5 - 3890i^4m\\
&\quad - 3588i^2m^3 - 667im^4 - 688m^5 + 1240i^4 + 1054i^3m + 2274i^2m^2\\
&\quad + 3216im^3 + 1104m^4 + 1221i^3 + 2896im^2 + 2160m^3 - 3550i^5m\\
&\quad - 4508i^4m^2 - 268i^2 - 2525i^3m^2 + 488im - 432m^2 - 524i - 1296m).
\end{aligned}
$$

In the proof of Theorem 4.1, the polynomials $R_1(m,i)$, $S_1(m,i)$, $M_1(m,i)$ and $N_1(m,i)$ are given by

$$
\begin{aligned}
R_1(m,i) &= 2i^2m^2 + 4mi^3 + 6im^2 + 14mi^2 + 2i^4\\
&\quad + 10i^3 + 21mi + 17i^2 + 2m^3 + 12m^2 + 18m + 10i,\\
S_1(m,i) &= 8i^5m^2 - 4i^2m^4 + 36i^4m^2 + 12i^3m^3 + (16i^6m - 4m^5) + (8i^7 - 2im^4)\\
&\quad + (32i^6 + 52i^5m + 30i^5 + 88i^4m + 66i^3m^2 - 28i^2m^3 - 6im^4)\\
&\quad + (36m - 27i^4 + 55i^3m - 65i^2m^2 - 23im^3 - 24m^4 - 56i^3\\
&\qquad - 101i^2m - 9im^2 - 32m^3 - 9i^2 - 20im + 24m^2 + 22i),\\
M_1(m,i) &= 2i^4 + 4i^3m + 2i^2m^2 + 10i^3 + 14i^2m + 6im^2\\
&\quad + 2m^3 + 17i^2 + 21im + 12m^2 + 10i + 18m,\\
N_1(m,i) &= 4i^{10}m - 40i^8m^3 - 96i^7m^4 - 128i^6m^5 - 128i^5m^6 - 88i^4m^7 - 32i^3m^8\\
&\quad - 4i^2m^9 + i^{10} + 12i^9m - 92i^8m^2 - 400i^7m^3 - 774i^6m^4\\
&\quad - 1100i^5m^5 - 1072i^4m^6 - 592i^3m^7 - 171i^2m^8 - 32im^9 - 4m^{10} + 6i^9\\
&\quad - 58i^8m - 556i^7m^2 - 1602i^6m^3 - 3236i^5m^4 - 4334i^4m^5 - 3204i^3m^6\\
&\quad - 1270i^2m^7 - 322im^8 - 48m^9 - 3i^8 - 351i^7m - 1487i^6m^2\\
&\quad - 4194i^5m^3 - 7663i^4m^4 - 7213i^3m^5 - 3519i^2m^6 - 1122im^7 - 208m^8\\
&\quad - 87i^7 - 695i^6m - 2422i^5m^2 - 5984i^4m^3 - 6495i^3m^4 - 3165i^2m^5\\
&\quad - 1272im^6 - 336m^7 - 161i^6 - 399i^5m - 1212i^4m^2 - 107i^3m^3 + 2447i^2m^4\\
&\quad + 1012im^5 + 104m^6 + 87i^5 + 839i^4m + 3175i^3m^2 + 6101i^2m^3 + 2902im^4\\
&\quad + 816m^5 + 377i^4 + 1388i^3m + 3137i^2m^2 + 862im^3 + 432m^4 + 32i^3\\
&\quad - 20i^2m - 1308im^2 - 432m^3 - 252i^2 - 720im - 324m^2.
\end{aligned}
$$

In the proof of Theorem 4.3, the polynomials $P(m,i)$, $G_1(m,i)$ and $H_1(m,i)$ are given by

$$
\begin{aligned}
P(m,i) &= 4i^6 + (4i^3m^3 - 2m^5) + (38i^3m^2 - 9m^4) + (14i^2m^3 - 11m^3) + 12i^5m\\
&\quad + 18i^5 + 44i^4m + (21i^4 - 10i^3) + 60i^3m + (35i^2m - 21im) + 12i^4m^2\\
&\quad + (64i^2m^2 - 10m^2 - 22m) + 16im^3 + (34im^2 - 27i^2 - 6i),\\
G_1(m,i) &= 2i^4 + 4i^3m + 2i^2m^2 + 10i^3 + 14i^2m + 6im^2\\
&\quad + 2m^3 + 17i^2 + 21im + 12m^2 + 10i + 18m,
\end{aligned}
$$

$$\begin{aligned} H_1(m,i) = {} & 2i^7 + 4i^6m + 7i^6 + 11i^5m + 8i^4m^2 + 14i^3m^3 + 15i^2m^4 + 3i^4 \\ & + (7im^5 - 4i^4m^3) + (2m^6 - 2i^3m^4) + 7i^5 + 34i^4m + 68i^3m^2 + 58i^2m^3 \\ & + (29im^4 - 10i^2m) + (12m^5 - 12m^3) + (61i^3m - 14i^2 - 40im) \\ & + (63i^2m^2 - 25im^2 - 18m^2) + 21im^3 + 16m^4 - 5i^3. \end{aligned}$$

In the proof of Lemma 4.6, the polynomials $Y_5(m,i)$ and $Y_6(m,i)$ are given by

$$\begin{aligned} Y_5(m,i) = {} & 4i^2(2m^2 - i^2)(i+m)^2 + 2(i+m)(4m^4 + 7im^3 + 31i^2m^2 + 4i^3m - 12i^4) \\ & - 35i^4 + 59i^3m + 199i^2m^2 + 151im^3 + 82m^4 + 16i^3 + 181i^2m + 321im^2 \\ & + 282m^3 + 70i^2 + 294im + 368m^2 + 106i + 160m, \end{aligned}$$

$$\begin{aligned} Y_6(m,i) = {} & (2i^4 + 4i^3m + 2i^2m^2 + 14i^3 + 18i^2m + 6im^2 \\ & \quad + 2m^3 + 33i^2 + 33im + 18m^2 + 37i + 48m + 32) \\ & \times (32i^2m^2(m-i)(i+m)^2 \\ & \quad + 16m(i+m)(2m^4 + im^3 + 16i^2m^2 - 11i^3m - 4i^4) - 30i^5 \\ & \quad - 394i^4m - 110i^3m^2 + 762i^2m^3 + 300im^4 + 368m^5 - 168i^4 \\ & \quad - 338i^3m + 1154i^2m^2 + 558im^3 + 1538m^4 + 1028i^2m - 141i^3 \\ & \quad + 631im^2 + 2882m^3 + 391i^2 + 639im + 2480m^2 + 260i + 800m). \end{aligned}$$

In the proof of Lemma 4.5, we need to check that, for $m \geqslant 133$ and $m^{2/3} \leqslant i \leqslant m-1$, $X(m,i) > 0$. Indeed, we have that

$$\begin{aligned} X(m,i) \geqslant {} & 16i^{11} + 64i^{10}m + 96i^9m^2 + 40i^8m^3 + 24(i^8m^3 - i^2m^7) + 16(i^7m^4 - i^4m^6) \\ & + 128i^{10} + (448i^9m - 176m^7) + 624i^8m^2 + (292i^7m^3 - 240i^4m^5 - 52im^7) \\ & + (116i^6m^4 - 100i^3m^6 - 16m^8) + (1868i^7m^2 - 1132i^4m^4 - 635im^6) \\ & + 1096(i^6m^3 - i^3m^5) + (160m^{2/3} - 2910)i^2m^5 + (620m^{2/3} - 2584)i^5m^3 \\ & + (128m^{4/3} - 3749)i^3m^4 + (4m - 528)m^6i^2 + (340m^{2/3} - 523)i^7 \\ & + 1712i^7m + (2520i^6m^2 - 2i^6m - 509i^5m^2) + 372i^9 + 1280i^8m \\ & - 22\,928m^6 - 11\,944m^5 \\ \geqslant {} & 96m^8 - 22928m^6 - 11944m^5, \end{aligned}$$

which is positive for $m \geqslant 133$.

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