Proceedings of the Edinburgh Mathematical Society (2014) **57**, 505–519
DOI:10.1017/S001309151300045X

THE OPERATOR SPACE PROJECTIVE TENSOR PRODUCT: EMBEDDING INTO THE SECOND DUAL AND IDEAL STRUCTURE

RANJANA JAIN[1] AND AJAY KUMAR[2]

[1] *Department of Mathematics, Lady Shri Ram College for Women, Lajpat Nagar IV, New Delhi 110024, India* (ranjanaj_81@rediffmail.com)
[2] *Department of Mathematics, University of Delhi, Delhi 110007, India* (akumar@maths.du.ac.in)

(Received 7 April 2011)

Abstract We prove that, for operator spaces V and W, the operator space $V^{**} \otimes_{\mathrm{h}} W^{**}$ can be completely isometrically embedded into $(V \otimes_{\mathrm{h}} W)^{**}$, $\otimes_{\mathrm{h}}$ being the Haagerup tensor product. We also show that, for exact operator spaces V and W, a jointly completely bounded bilinear form on $V \times W$ can be extended uniquely to a separately w^*-continuous jointly completely bounded bilinear form on $V^{**} \times W^{**}$. This paves the way to obtaining a canonical embedding of $V^{**} \widehat{\otimes} W^{**}$ into $(V \widehat{\otimes} W)^{**}$ with a continuous inverse, where $\widehat{\otimes}$ is the operator space projective tensor product. Further, for C^*-algebras A and B, we study the (closed) ideal structure of $A \widehat{\otimes} B$, which, in particular, determines the lattice of closed ideals of $\mathcal{B}(H) \widehat{\otimes} \mathcal{B}(H)$ completely.

Keywords: operator space; C^*-algebras; operator space projective norm; Haagerup norm

2010 *Mathematics subject classification:* Primary 46L06; 46L07; 47L25

1. Introduction

The operator space projective tensor product serves as an analogue to the Banach space projective tensor product in the category of operator spaces. It is used to linearize the jointly (matricially) completely bounded bilinear maps in the same way as the Banach space projective tensor product linearizes the bounded bilinear maps. If E and F are operator spaces, then their *operator space projective tensor product*, denoted by $E \widehat{\otimes} F$, is the completion of the algebraic tensor product $E \otimes F$ under the norm

$$\|u\|_{\wedge} = \inf\{\|\alpha\| \, \|v\| \, \|w\| \, \|\beta\| \colon u = \alpha(v \otimes w)\beta\}, \quad u \in M_n(E \otimes F),$$

where the infimum runs over arbitrary decompositions with $v \in M_p(E)$, $w \in M_q(F)$, $\alpha \in \mathbb{M}_{n,pq}$, $\beta \in \mathbb{M}_{pq,n}$ and $p, q \in \mathbb{N}$ arbitrary, $\mathbb{M}_{k,l}$ being the space of $k \times l$ matrices over $\mathbb{C}$. The theory of operator space tensor products was developed independently by Blecher and Paulsen [**5**] and Effros and Ruan [**8, 9**].

For C^*-algebras A and B, it is known that $A^{**} \otimes_{\mathrm{h}} B^{**}$ can be completely isometrically embedded into the bidual $(A \otimes_{\mathrm{h}} B)^{**}$ [**18**, Theorem 4.1]. In § 2, we prove that the same is true, in general, for operator spaces. Haagerup [**12**] proved that a bounded bilinear form on $A \times B$ has a unique norm-preserving extension to a separately normal bounded bilinear form on $A^{**} \times B^{**}$. We prove an analogous result for a jointly completely bounded bilinear form on exact operator spaces and for arbitrary C^*-algebras. Kumar and Sinclair [**18**] proved, by using the above extension result of Haagerup, that for C^*-algebras A and B there is a canonical bi-continuous embedding of $A^{**} \otimes_\gamma B^{**}$ into $(A \otimes_\gamma B)^{**}$, where $\otimes_\gamma$ denotes the Banach space projective tensor product. Its analogue for the operator space projective tensor product was left open. In this paper, we present an affirmative answer in the cases of exact operator spaces and arbitrary C^*-algebras. As an application of this result, we re-establish the fact that for C^*-algebras A and B the Haagerup norm and the operator space projective norm are equivalent on $A \otimes B$ if and only if A and B are subhomogeneous, which was proved in [**18**]. Note that Sanchez and Garcia [**21**] also studied the relationship between the tensor product of the biduals and the bidual of the tensor product for the Banach space projective tensor norm. In particular, they proved that, for a Banach space X of type 2 such that X^* is of cotype 2, the embedding $X^{**} \otimes_\gamma X^{**} \hookrightarrow (X \otimes_\gamma X)^{**}$ is bi-continuous.

The closed ideal structures of $A \otimes_h B$, $A \otimes_{\min} B$ and $A \otimes_{\max} B$, A and B being C^*-algebras, have been investigated by Allen *et al*. [**1**], Archbold *et al*. [**2**], Takesaki [**23**] and Wassermann [**24**], respectively. For the commutative case, the closed ideals of $A \otimes_\gamma B$ have been discussed in [**11**]. However, the analysis of the (closed) ideal structure of the Banach $*$-algebra $A \widehat{\otimes} B$ requires further attention. We present some results in this direction in § 3. We prove that the sum of two product ideals in $A \widehat{\otimes} B$ is closed and the same technique leads to a shorter proof of [**1**, Theorem 3.8]. We further show that the minimal and maximal ideals in A and B generate their counterparts in $A \widehat{\otimes} B$. As a consequence, we obtain the lattice of closed ideals of $\mathcal{B}(H) \widehat{\otimes} \mathcal{B}(H)$.

2. Embedding operator space projective tensor product into second duals

For operator spaces V and W, a *jointly completely bounded bilinear map* (j.c.b.) is a bilinear map $\phi\colon V \times W \to \mathbb{C}$ such that the maps $\phi_n\colon M_n(V) \times M_n(W) \to \mathbb{M}_{n^2}$ given by

$$\phi_n((a_{ij}), (b_{kl})) = (\phi(a_{ij}, b_{kl})), \quad n \in \mathbb{N},$$

are uniformly bounded, and in this case we write $\|\phi\|_{\text{jcb}} := \sup\{\|\phi_n\|\colon n \in \mathbb{N}\}$ [**5**]. Also, a map $\phi\colon V \times W \to \mathbb{C}$ is said to be *completely bounded* (c.b.) if the maps $\phi_n\colon M_n(V) \times M_n(W) \to \mathbb{M}_n$ given by

$$\phi_n((a_{ij}), (b_{kl})) = \left(\sum_{k=1}^{n} \phi(a_{ik}, b_{kj}) \right), \quad n \in \mathbb{N},$$

are uniformly bounded, and then we write $\|\phi\|_{\text{cb}} := \sup\{\|\phi_n\|\colon n \in \mathbb{N}\}$. It is well known that $(V \widehat{\otimes} W)^*$ and $(V \otimes_{\text{h}} W)^*$ are completely isometrically isomorphic to $\mathcal{JCB}(V \times W, \mathbb{C})$ and $\mathcal{CB}(V \times W, \mathbb{C})$, respectively, where $\mathcal{JCB}(V \times W, \mathbb{C})$ (respectively, $\mathcal{CB}(V \times W, \mathbb{C})$) denotes the space of j.c.b. (respectively, c.b.) bilinear maps [**5**, **10**]. Every completely bounded map ϕ is jointly completely bounded with $\|\phi\|_{\text{jcb}} \leqslant \|\phi\|_{\text{cb}}$.

Recall that, for operator spaces V and W, the *Haagerup norm* of an element $u \in M_n(V \otimes W)$, $n \in \mathbb{N}$, is defined by

$$\|u\|_{\mathrm{h}} = \inf\{\|v\|\,\|w\| \colon u = v \odot w,\ v \in M_{n,p}(V),\ w \in M_{p,n}(W),\ p \in \mathbb{N}\},$$

where

$$v \odot w = \left(\sum_{k=1}^{p} v_{ik} \otimes w_{kj} \right)_{ij}.$$

The norms $\|\cdot\|_{\mathrm{h}}$, $\|\cdot\|_{\wedge}$ and $\|\cdot\|_{\gamma}$ on the tensor product $A \otimes B$ of two C^*-algebras A and B satisfy

$$\|\cdot\|_{\mathrm{h}} \leqslant \|\cdot\|_{\wedge} \leqslant \|\cdot\|_{\gamma}.$$

We first state an important result, whose proof can be found in [**4**, § 1.6.7].

Proposition 2.1. *Let V and W be operator spaces, let E be a dual operator space and let $u \colon V \times W \to E$ be a completely bounded bilinear map. Then u admits a unique separately w^*-continuous extension $\tilde{u} \colon V^{**} \times W^{**} \to E$, which is completely bounded with $\|u\|_{\mathrm{cb}} = \|\tilde{u}\|_{\mathrm{cb}}$.*

We now prove an embedding result for the Haagerup tensor product of operator spaces. It turns out that the operator space version is much easier than the C^*-algebra case [**18**, Theorem 4.1], as observed below. Note that if either V or W is finite dimensional, then $V^{**} \otimes_{\mathrm{h}} W^{**}$ is completely isometrically isomorphic to $(V \otimes_{\mathrm{h}} W)^{**}$ [**10**, Corollary 9.4.8].

Theorem 2.2. *For operator spaces V and W, there is a canonical embedding of $V^{**} \otimes_{\mathrm{h}} W^{**}$ into $(V \otimes_{\mathrm{h}} W)^{**}$ that is a complete isometry.*

Proof. For the operator spaces V^{**} and W^{**}, recall that $(V^{**} \otimes_{\mathrm{h}} W^{**})^*_\sigma$ denotes the subspace of $(V^{**} \otimes_{\mathrm{h}} W^{**})^*$ containing all the separately w^*-continuous completely bounded bilinear forms on $V^{**} \times W^{**}$. By Proposition 2.1, taking the map $u \to \tilde{u}$, and E as $\mathbb{M}_n$, one easily sees that $(V \otimes_{\mathrm{h}} W)^*$ is completely isometrically isomorphic to $(V^{**} \otimes_{\mathrm{h}} W^{**})^*_\sigma$. In particular, the normal Haagerup tensor product $V^{**} \otimes_{\sigma\mathrm{h}} W^{**}$, which is defined as the operator space dual of $(V^{**} \otimes_{\mathrm{h}} W^{**})^*_\sigma$, is completely isometrically isomorphic to $(V \otimes_{\mathrm{h}} W)^{**}$. Recall that there also exists a completely isometric embedding [**4**, § 1.6.8] $V^{**} \otimes_{\mathrm{h}} W^{**} \hookrightarrow V^{**} \otimes_{\sigma\mathrm{h}} W^{**}$. Hence, there is a completely isometric embedding of $V^{**} \otimes_{\mathrm{h}} W^{**}$ into $(V \otimes_{\mathrm{h}} W)^{**}$. □

We now move on to analyse the embedding of biduals for operator space projective tensor product. This will need some preparation. Recall that an operator space V is said to be *exact* if there exists a constant K such that for any finite-dimensional subspace $G \subset V$ there exist an integer n, a subspace $\tilde{G} \subset \mathbb{M}_n$ and an isomorphism $u \colon G \to \tilde{G}$ such that $\|u\|_{\mathrm{cb}} \|u^{-1}\|_{\mathrm{cb}} \leqslant K$. The smallest such constant is the exactness constant and is denoted by $\mathrm{ex}(V)$. The matrix algebra $\mathbb{M}_n$ and, in general, all nuclear C^*-algebras are simple examples of exact operator spaces with exactness constant 1.

Proposition 2.3. *Let V and W be exact operator spaces. Then every j.c.b. bilinear map $u \colon V \times W \to \mathbb{C}$ can be extended uniquely to a separately w^*-continuous j.c.b. bilinear map $\tilde{u} \colon V^{**} \times W^{**} \to \mathbb{C}$ such that $\|\tilde{u}\|_{\mathrm{jcb}} \leqslant 2K \|u\|_{\mathrm{jcb}}$, where $K = 2\sqrt{2}\,\mathrm{ex}(V)\,\mathrm{ex}(W)$.*

Proof. Since V and W are both exact, by [**19**, Theorem 0.4] there exist completely bounded bilinear forms u_1 and u_2^t on $V \times W$ and $W \times V$, respectively, such that $u = u_1 + u_2$ with $\|u_1\|_{\mathrm{cb}} + \|u_2^t\|_{\mathrm{cb}} \leqslant 2K\|u\|_{\mathrm{jcb}}$, where $K = 2\sqrt{2}\,\mathrm{ex}(V)\,\mathrm{ex}(W)$ and $u_2^t(w, v) = u_2(v, w)$. Using Proposition 2.1, there exist unique separately w^*-continuous extensions $\tilde{u}_1 \colon V^{**} \times W^{**} \to \mathbb{C}$ and $\tilde{u}_2^t \colon W^{**} \times V^{**} \to \mathbb{C}$ of u_1 and u_2^t, which are completely bounded with $\|u_1\|_{\mathrm{cb}} = \|\tilde{u}_1\|_{\mathrm{cb}}$ and $\|u_2^t\|_{\mathrm{cb}} = \|\tilde{u}_2^t\|_{\mathrm{cb}}$. Note that $\tilde{u}_2^t$ is j.c.b, being c.b., so $\tilde{u}_2$ is also a j.c.b. bilinear form with

$$\|\tilde{u}_2\|_{\mathrm{jcb}} = \|\tilde{u}_2^t\|_{\mathrm{jcb}} \leqslant \|\tilde{u}_2^t\|_{\mathrm{cb}} = \|u_2^t\|_{\mathrm{cb}}.$$

Also, it can be easily seen that $\tilde{u}_2$ is separately w^*-continuous, $\tilde{u}_2^t$ being separately w^*-continuous. Set $\tilde{u} = \tilde{u}_1 + \tilde{u}_2$. Then $\tilde{u}$ is a separately w^*-continuous j.c.b. bilinear form on $V^{**} \times W^{**}$ with $\|\tilde{u}\|_{\mathrm{jcb}} \leqslant 2K\|u\|_{\mathrm{jcb}}$. Finally, some routine calculations show that $\tilde{u}$ is the unique extension of u. □

In the case of C^*-algebras A and B, using the same techniques as in Proposition 2.3 and [**13**, Lemma 3.1], one can easily prove that every j.c.b. bilinear map $u \colon A \times B \to \mathbb{C}$ can be extended uniquely to a separately normal j.c.b. bilinear map $\tilde{u} \colon A^{**} \times B^{**} \to \mathbb{C}$ such that $\|\tilde{u}\|_{\mathrm{jcb}} \leqslant 2\|u\|_{\mathrm{jcb}}$. *A priori*, it is not clear why this extension should be norm preserving. However, using a completely different approach, we establish that for C^*-algebras there exists a unique norm-preserving separately normal extension.

Lemma 2.4. *Let A and B be von Neumann algebras and let $T \colon A \times B \to \mathbb{C}$ be a separately normal bilinear form. Then, for each n, the map $T_n \colon M_n(A) \times M_n(B) \to \mathbb{M}_{n^2}$ defined by*

$$T_n((a_{ij}), (b_{kl})) = (T(a_{ij}, b_{kl}))$$

is separately normal.

Proof. For any $a = (a_{ij}) \in M_n(A)$ and for a fixed $b = (b_{ij}) \in M_n(B)$ we can write

$$T_n((a_{ij}), (b_{kl})) = \begin{pmatrix} T_{11}(a) & T_{12}(a) & \cdots & T_{1n}(a) \\ T_{21}(a) & T_{22}(a) & \cdots & T_{2n}(a) \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1}(a) & T_{n2}(a) & \cdots & T_{nn}(a) \end{pmatrix},$$

where $T_{kl} \colon M_n(A) \to \mathbb{M}_n$ maps $(a_{ij}) \to (T(a_{ij}, b_{kl}))$. In order to show that the map T_n is separately normal, we can equivalently show that the map

$$a \to \sum_{i,j=1}^{n} e_{ij} \otimes T_{ij}(a)$$

from $M_n(A)$ into $\mathbb{M}_n \otimes \mathbb{M}_n$ is normal. Let (a_λ) be an increasing net of positive elements in $M_n(A)$ such that $a_\lambda \xrightarrow{w^*} a$. Clearly, each T_{ij} is normal, so $T_{ij}(a)$ is a weak limit of the net $(T_{ij}(a_\lambda))$ and thus $\sum e_{ij} \otimes T_{ij}(a)$ is a weak limit of $\sum e_{ij} \otimes T_{ij}(a_\lambda)$. Hence, the result follows. □

Proposition 2.5. *Let A and B be C^*-algebras and let $\phi\colon A \times B \to \mathbb{C}$ be a j.c.b. bilinear form. Then ϕ admits a unique separately normal j.c.b. bilinear extension $\tilde{\phi}\colon A^{**} \times B^{**} \to \mathbb{C}$ such that $\|\tilde{\phi}\|_{\mathrm{jcb}} = \|\phi\|_{\mathrm{jcb}}$.*

Proof. Since $\phi\colon A \times B \to \mathbb{C}$ is a continuous bilinear form, there exists a unique separately normal bilinear form $\tilde{\phi}\colon A^{**} \times B^{**} \to \mathbb{C}$ with $\|\tilde{\phi}\| = \|\phi\|$ [**12**, Corollary 2.4]. Let n be a positive integer. Consider the map $\tilde{\phi}_n\colon M_n(A^{**}) \times M_n(B^{**}) \to \mathbb{M}_{n^2}$ defined by

$$\tilde{\phi}_n((a_{ij}), (b_{kl})) = (\tilde{\phi}(a_{ij}, b_{kl})).$$

We claim that $\|\tilde{\phi}_n\| \leqslant \|\phi\|_{\mathrm{jcb}}$. Consider any $a \in M_n(A^{**})$, $b \in M_n(B^{**})$ with $\|a\| = \|b\| = 1$. Using the fact that the unit ball of $M_n(A)$ is w^*-dense in the unit ball of $M_n(A^{**})$, we obtain a net $(a_\lambda) \in M_n(A)$ that is w^*-convergent to a with $\|a_\lambda\| \leqslant 1$ and a net $(b_\mu) \in M_n(B)$ that is w^*-convergent to b with $\|b_\mu\| \leqslant 1$. By Lemma 2.4, $\tilde{\phi}_n$ is separately normal, so, for a fixed $k \in \mathbb{N}$,

$$\|\tilde{\phi}_n(a, b)\| \leqslant (1 + 1/k)^2 \|\tilde{\phi}_n(a_\lambda, b_\mu)\|,$$

for some λ and μ. This further gives

$$\|\tilde{\phi}_n(a, b)\| \leqslant (1 + 1/k)^2 \|\phi_n\|, \quad \forall k \in \mathbb{N}.$$

Thus, $\|\tilde{\phi}_n\| \leqslant \|\phi_n\| \leqslant \|\phi\|_{\mathrm{jcb}}$, and this is true for all $n \in \mathbb{N}$, giving that $\tilde{\phi}$ is j.c.b. with $\|\tilde{\phi}\|_{\mathrm{jcb}} \leqslant \|\phi\|_{\mathrm{jcb}}$. Also $\|\phi\|_{\mathrm{jcb}} \leqslant \|\tilde{\phi}\|_{\mathrm{jcb}}$, ϕ being the restriction of $\tilde{\phi}$, and hence the result follows. □

We next prove a result that is an operator space version of [**18**, Lemma 5.3], and whose proof is largely inspired by the same lemma.

Lemma 2.6. *Let V and W be operator spaces with $V \subset \mathcal{B}(H)$ and $W \subset \mathcal{B}(K)$. Then the unit ball of $\mathcal{CB}_\sigma(V \times W, \mathbb{C})$ is w^*-dense in the unit ball of $\mathcal{CB}(V \times W, \mathbb{C})$, where $\mathcal{CB}_\sigma(V \times W, \mathbb{C})$ denotes the space of all separately w^*-continuous c.b. bilinear forms on $V \times W$.*

Proof. Let B_1 and B_2 denote the unit balls of $\mathcal{CB}_\sigma(V \times W, \mathbb{C})$ and $\mathcal{CB}(V \times W, \mathbb{C})$, respectively. Suppose that there exists a ϕ in B_2 such that $\phi \notin \bar{B}_1^{w^*}$, where $\bar{B}_1^{w^*}$ denotes the w^*-closure of B_1 in B_2. By a corollary of the Hahn–Banach separation theorem [**20**, Theorem 3.7], we obtain a w^*-continuous linear functional $\Phi\colon \mathcal{CB}(V \times W, \mathbb{C}) \to \mathbb{C}$ such that $|\Phi(\psi)| \leqslant 1$ for all $\psi \in B_1$ and $\Phi(\phi) > 1$. Now Φ can be identified with an element u of $V \otimes_{\mathrm{h}} W$, being a w^*-continuous functional on $(V \otimes_{\mathrm{h}} W)^*$. Therefore, there exists $u \in V \otimes_{\mathrm{h}} W$, $\|u\|_{\mathrm{h}} > 1$, such that $|\psi(u)| \leqslant 1$ for all $\psi \in B_1$, and $\phi(u) > 1$.

It is well known that there is an isometric embedding of $\mathcal{B}(H) \otimes_{\mathrm{h}} \mathcal{B}(K)$ into $\mathcal{CB}(\mathcal{B}(K, H))$ [**22**, Theorem 4.3]. Using the injectivity of the Haagerup tensor product, we obtain an isometric embedding (that need not be algebraic), say θ, of $V \otimes_{\mathrm{h}} W$ into $\mathcal{CB}(\mathcal{B}(K, H))$, given by $\theta(v \otimes w)(T) = vTw$. Since $\|\theta(u)\|_{\mathrm{cb}} > 1$, for some $n \in \mathbb{N}$, $\|(\theta(u))_n\| > 1$. So there exists $(x_{ij}) \in M_n(\mathcal{B}(K, H))$ with $\|(x_{ij})\| = 1$ such that

$$\|(\theta(u)x_{ij})\| = \|(\theta(u))_n(x_{ij})\| > 1.$$

Now, we can choose unit vectors $\xi \in K^n$ and $\eta \in H^n$ such that

$$|\langle(\theta(u)x_{ij})\xi, \eta\rangle| > 1.$$

Define $\psi\colon V \times W \to \mathbb{C}$ as

$$\psi(v, w) = \langle(\theta(v \otimes w)x_{ij})\xi, \eta\rangle.$$

Clearly, ψ belongs to B_1, which, together with the relation

$$|\psi(u)| = |\langle(\theta(u)x_{ij})\xi, \eta\rangle| > 1,$$

gives a contradiction. Hence, B_1 is w^*-dense in B_2. □

Now, with all the necessary ingredients at our disposal, we are ready to prove the main result of this section, an operator space analogue of [**18**, Theorem 5.1], and the proof presented here borrows ideas from the same theorem. Let us first define the required embedding. For exact operator spaces V and W, by Proposition 2.3, we have a map

$$\chi\colon (V \widehat{\otimes} W)^* \to (V^{**} \widehat{\otimes} W^{**})^*$$

with $\|\chi\| \leqslant 2K$. Define

$$\mu := \chi^* \circ i\colon V^{**} \widehat{\otimes} W^{**} \to (V \widehat{\otimes} W)^{**},$$

where $i\colon V^{**} \widehat{\otimes} W^{**} \to (V^{**} \widehat{\otimes} W^{**})^{**}$ is the canonical completely isometric embedding. Then we have the following.

Theorem 2.7. *For exact operator spaces V and W, the embedding μ of $V^{**} \widehat{\otimes} W^{**}$ into $(V \widehat{\otimes} W)^{**}$ satisfies*

$$\frac{1}{2K}\|u\| \leqslant \|\mu(u)\| \leqslant 2K\|u\|, \quad \forall u \in V^{**} \widehat{\otimes} W^{**},$$

where $K = 2\sqrt{2}\,\mathrm{ex}(V)\,\mathrm{ex}(W)$. In particular, μ has a continuous inverse.

Proof. By the definition of μ, the inequality on the right-hand side is obvious. For the other inequality, consider any $u \in V^{**} \otimes W^{**}$ with $\|u\|_\wedge = 1$. By the Hahn–Banach theorem, there exists a j.c.b. bilinear map $\phi\colon V^{**} \times W^{**} \to \mathbb{C}$ such that $\|\phi\|_{\mathrm{jcb}} = 1$ and $\phi(u) = 1$. By [**19**, Theorem 0.4], ϕ can be decomposed as $\phi = \phi_1 + \phi_2$, where ϕ_1 and ϕ_2 are bounded bilinear forms with $\|\phi_1\|_{\mathrm{cb}} \leqslant K$ and $\|\phi_2^t\|_{\mathrm{cb}} \leqslant K$, where $\phi_2^t(b, a) = \phi_2(a, b)$. Now consider any $\epsilon > 0$. For the w^*-open sets

$$\{\theta \in B_1(\mathcal{CB}(V^{**} \times W^{**}, \mathbb{C}))\colon |(\theta - \phi_1/K)(u)| < \epsilon\}$$

and

$$\{\zeta \in B_1(\mathcal{CB}(W^{**} \times V^{**}, \mathbb{C}))\colon |(\zeta - \phi_2^t/K)(u^t)| < \epsilon\},$$

using Lemma 2.6 we get $\Phi_1 \in \mathcal{CB}_\sigma(V^{**} \times W^{**}, \mathbb{C})$ and $\Phi_2 \in \mathcal{CB}_\sigma(W^{**} \times V^{**}, \mathbb{C})$ with $\|\Phi_j\|_{\mathrm{cb}} \leqslant 1$, $j = 1, 2$, such that

$$|\phi_1(u) - K\Phi_1(u)| < K\epsilon, \quad |\phi_2^t(u^t) - K\Phi_2(u^t)| < K\epsilon, \tag{2.1}$$

which further give

$$|\phi_1(u) - K\Phi_1(u)| < K\epsilon, \qquad |\phi_2(u) - K\Phi_2^t(u)| < K\epsilon. \tag{2.2}$$

Now Φ_1 and Φ_2 both are j.c.b., being c.b.. Also Φ_2^t is a separately w^*-continuous j.c.b. form on $V^{**} \times W^{**}$ with $\|\Phi_2^t\|_{\mathrm{jcb}} = \|\Phi_2\|_{\mathrm{jcb}}$, so $\Phi = \Phi_1 + \Phi_2^t$ is a j.c.b. map. Let ψ_1, ψ_2 be the restrictions of Φ_1 and Φ_2^t to $V \times W$, then these are j.c.b. bilinear maps. Thus, by the definition of χ, $\Phi_1 = \chi(\psi_1)$, $\Phi_2^t = \chi(\psi_2)$. Set $\psi = \psi_1 + \psi_2$. Then ψ is a j.c.b. bilinear map and thus it is a continuous linear functional on $V \widehat{\otimes} W$ with $\|\psi\| \leqslant 2$. Further,

$$\mu(u)(\psi) = \chi^* i(u)(\psi) = i(u)(\chi\psi) = (\chi\psi)(u) = \Phi(u)$$

which, along with (2.2), give $\|\mu(u)\| \geqslant 1/2K$. □

Remark 2.8. If the extension of [**19**, Conjecture 0.2′] is true for $\mathbb{M}_n$-valued bilinear functions, which is not known to us, then we can prove that μ is completely bounded. Indeed, if the conjecture were true, then, using the same argument as that in the proof of Proposition 2.3, one can prove that for exact operator spaces V and W every j.c.b. bilinear map $u\colon V \times W \to \mathbb{M}_n$ can be extended uniquely to a separately w^*-continuous j.c.b. bilinear map $\tilde{u}\colon V^{**} \times W^{**} \to \mathbb{M}_n$ such that $\|\tilde{u}\|_{\mathrm{jcb}} \leqslant 2K\|u\|_{\mathrm{jcb}}$, for some constant K independent of n. Now, to show that μ is completely bounded, it is sufficient to show that χ is completely bounded, and in that case $\|\mu\|_{\mathrm{cb}} \leqslant \|\chi\|_{\mathrm{cb}}$. Note that, for $n \in \mathbb{N}$, we have the following commutative diagram:

$$\begin{array}{ccc} M_n(\mathcal{JCB}(V \times W, \mathbb{C})) & \xrightarrow{\chi_n} & M_n(\mathcal{JCB}(V^{**} \times W^{**}, \mathbb{C})) \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle i'} \\ \mathcal{JCB}(V \times W, \mathbb{M}_n) & \xrightarrow{\chi'} & \mathcal{JCB}(V^{**} \times W^{**}, \mathbb{M}_n) \end{array}$$

In the above diagram, $\|\chi'\| \leqslant 2K$, K being independent of n, and both i and i' are complete isometric isomorphisms. So $\|\chi_n\| \leqslant 2K$, and this is true for all $n \in \mathbb{N}$. Thus, χ is completely bounded with $\|\chi\|_{\mathrm{cb}} \leqslant 2K$.

For C^*-algebras A and B, using Proposition 2.5 and the techniques of Theorem 2.7, one can prove the following.

Theorem 2.9. *For C^*-algebras A and B, there is a canonical bi-continuous embedding μ of $A^{**} \widehat{\otimes} B^{**}$ into $(A \widehat{\otimes} B)^{**}$ that satisfies*

$$\tfrac{1}{2}\|u\| \leqslant \|\mu(u)\| \leqslant \|u\|, \quad \forall u \in A^{**} \widehat{\otimes} B^{**}.$$

https://doi.org/10.1017/S001309151300045X Published online by Cambridge University Press

As an application of the above result, we prove an equivalence between the Haagerup norm and the operator space projective norm for tensor product of C^*-algebras. This result has already been proved by Kumar and Sinclair [**18**, Theorem 7.4]. However, we use a different and rather simple technique to prove the same result. We first need the following easy result dealing with the injectivity of the projective norm.

Lemma 2.10. *If A_0 and B_0 are both finite-dimensional C^*-subalgebras of the C^*-algebras A and B, then $A_0 \widehat{\otimes} B_0$ is a closed $*$-subalgebra of $A \widehat{\otimes} B$.*

Proof. Since A_0 and B_0 are both finite-dimensional C^*-subalgebras of A and B, there are conditional expectations P_1 and P_2 from A and B onto A_0 and B_0, respectively, with $\|P_1\|_{\mathrm{cb}} = \|P_2\|_{\mathrm{cb}} = 1$ (see [**3**, Example II.6.10.4]). It can be easily seen that $\|P_1 \widehat{\otimes} P_2\| \leqslant 1$ and for the inclusion map $i\colon A_0 \widehat{\otimes} B_0 \to A \widehat{\otimes} B$, the composition $(P_1 \widehat{\otimes} P_2) \circ i$ agrees with the identity map on $A_0 \otimes B_0$. So for any element $x \in A_0 \otimes B_0$,

$$\begin{aligned} \|x\|_{A_0 \widehat{\otimes} B_0} &= \|((P_1 \widehat{\otimes} P_2) \circ i)(x)\|_{A_0 \widehat{\otimes} B_0} \\ &\leqslant \|i(x)\|_{A \widehat{\otimes} B} \\ &\leqslant \|x\|_{A_0 \widehat{\otimes} B_0}. \end{aligned}$$

Hence, i is an isometry, giving $A_0 \widehat{\otimes} B_0$ as a closed subalgebra of $A \widehat{\otimes} B$. □

Recall that a C^*-algebra A is said to be *n-subhomogeneous* if each irreducible representation of A has dimension less than or equal to n, and *subhomogeneous* if it is n-subhomogeneous for some $n \in \mathbb{N}$. It is known that a C^*-algebra A is n-subhomogeneous if and only if A^{**} does not contain a C^*-subalgebra isomorphic to $\mathbb{M}_{n+1}$.

Theorem 2.11. *For C^*-algebras A and B, the Haagerup norm $\|\cdot\|_{\mathrm{h}}$ is equivalent to the operator space projective tensor norm $\|\cdot\|_{\wedge}$ on $A \otimes B$ if and only if either A or B is finite dimensional, or A and B both are infinite-dimensional and subhomogeneous.*

Proof. Let us assume that A and B are both infinite dimensional, and

$$\|x\|_{\wedge} \leqslant c\|x\|_{\mathrm{h}}, \quad \forall x \in A \otimes B,$$

for some constant c, that is, the canonical map $j\colon A \otimes_{\mathrm{h}} B \to A \widehat{\otimes} B$ is continuous with $\|j\| \leqslant c$. We first claim that

$$\|x^{**}\|_{\wedge} \leqslant 2c\|x^{**}\|_{\mathrm{h}}, \quad \forall x^{**} \in A^{**} \otimes B^{**}.$$

In other words, the identity map $J\colon A^{**} \otimes B^{**} \to A^{**} \widehat{\otimes} B^{**}$ is continuous with respect to the Haagerup norm. Using Theorems 2.9 and 2.2, we have a bi-continuous canonical embedding $\mu\colon A^{**} \widehat{\otimes} B^{**} \to (A \widehat{\otimes} B)^{**}$ and a canonical completely isometric embedding $\zeta\colon A^{**} \otimes_{\mathrm{h}} B^{**} \to (A \otimes_{\mathrm{h}} B)^{**}$, respectively. For any $a^{**} \otimes b^{**} \in A^{**} \otimes B^{**}$ and $f \in (A \widehat{\otimes} B)^*$, we have

$$\mu J(a^{**} \otimes b^{**})(f) = (\chi f)(a^{**} \otimes b^{**}) = \tilde{f}(a^{**} \otimes b^{**}),$$

where $\tilde{f}\colon A^{**}\times B^{**}\to\mathbb{C}$ is the unique separately normal j.c.b. extension of $f\colon A\times B\to\mathbb{C}$. Also

$$j^{**}\zeta(a^{**}\otimes b^{**})(f)=\zeta(a^{**}\otimes b^{**})(j^*(f))=\widetilde{j^*(f)}(a^{**}\otimes b^{**}),$$

where $\widetilde{j^*(f)}\colon A^{**}\times B^{**}\to\mathbb{C}$ is the unique separately normal c.b. extension of $j^*(f)\colon A\times B\to\mathbb{C}$. Note that $\widetilde{j^*(f)}$ is also a j.c.b. extension of f, so, by uniqueness, $\widetilde{j^*(f)}=\tilde{f}$, which gives $\mu J=j^{**}\zeta$ on $A^{**}\otimes B^{**}$. Using the bi-continuity of μ we get $J=\mu^{-1}j^{**}\zeta$, with

$$\|J\|\leqslant\|\mu^{-1}\|\,\|j^{**}\|\,\|\zeta\|\leqslant 2c, \tag{2.3}$$

which proves our first claim.

Let A^{**} contain an isomorphic copy (not necessarily unital) of $\mathbb{M}_n$, for some $n\in\mathbb{N}$ and let B^{**} contain a copy of l_n^∞, both being infinite dimensional. Using the injectivity of the Haagerup norm and Lemma 2.10, $\mathbb{M}_n\otimes_{\mathrm{h}} l_n^\infty$ and $\mathbb{M}_n\mathbin{\widehat{\otimes}} l_n^\infty$ embed isometrically in $A^{**}\otimes_{\mathrm{h}} B^{**}$ and $A^{**}\mathbin{\widehat{\otimes}} B^{**}$, respectively. Let $\{e_{ij}\}$ denote the standard matrix units. Then, using [**18**, Lemma 3.1] and (2.3), we have

$$\begin{aligned} n^{1/2} &= \Big\|\sum_{j=1}^{n} e_{1j}\otimes e_{jj}\Big\|_{\mathrm{h}} \\ &\leqslant \Big\|\sum_{j} e_{1j}\otimes e_{jj}\Big\|_{\wedge} \\ &= \Big\|\sum_{j} e_{j1}\otimes e_{jj}\Big\|_{\wedge} \\ &\leqslant 2c\Big\|\sum_{j} e_{j1}\otimes e_{jj}\Big\|_{\mathrm{h}} \\ &= 2c. \end{aligned}$$

So A^{**} cannot contain an isomorphic copy of $\mathbb{M}_n$ for $n>4c^2$, which shows that A is $4c^2$-subhomogeneous. A similar argument gives that B is also $4c^2$-subhomogeneous.

The other implication is easy to prove. □

3. Ideal structure of $A\mathbin{\widehat{\otimes}}B$

The operator space projective tensor norm is symmetric, associative and projective but not injective [**10**]. For C^*-algebras A and B, $A\mathbin{\widehat{\otimes}}B$ is a Banach $*$-algebra with the natural isometric involution given by $*\colon a\otimes b\to a^*\otimes b^*$ [**16**]. This property is in contrast to the Haagerup norm, where the natural involution on $A\otimes_{\mathrm{h}}B$ is an isometry if and only if A and B are commutative [**17**]. This section is devoted to a systematic study of the ideal structure of this Banach $*$-algebra. If K and L are closed ideals of A and B, where A and B are C^*-algebras, then $K\mathbin{\widehat{\otimes}}L$ is a closed $*$-ideal of $A\mathbin{\widehat{\otimes}}B$ [**16**, Theorem 5], which is termed a *product ideal*. Allen, Sinclair and Smith [**1**] proved that sum of two product ideals in the Haagerup tensor product is again a closed ideal. In this section we discuss its analogue for the operator space projective tensor product, whose techniques also give a shorter proof of [**1**, Theorem 3.8]. Note that all the results in this section also hold true for closed $*$-ideals of $A \widehat{\otimes} B$. Throughout this section A and B denote the C^*-algebras, unless otherwise stated. We first state an elementary result, a proof of which for the Banach space projective norm can be found in [**15**].

Lemma 3.1. *If Banach algebras A and B both possess bounded approximate identities, then for any subcross norm α, $A \otimes^\alpha B$ possesses a bounded approximate identity, where $A \otimes^\alpha B$ is the completion of the algebraic tensor product $A \otimes B$ with respect to the α norm.*

Proposition 3.2. *Let I_1, I_2 and J_1, J_2 be closed ideals of A and B respectively. Then $I_1 \widehat{\otimes} J_1 + I_2 \widehat{\otimes} J_2$ is a closed $*$-ideal of $A \widehat{\otimes} B$.*

Proof. By Lemma 3.1 and [**16**, Theorem 5], it follows that $I_1 \widehat{\otimes} J_1$ and $I_2 \widehat{\otimes} J_2$ are closed $*$-ideals, both having bounded approximate identities. Using the fact that sum of two closed ideals is closed if at least one of them has a bounded approximate identity [**7**, Proposition 2.4], we obtain the required result. □

Remark 3.3. The above proposition is also true for the Haagerup norm and Banach space projective norm. In particular, this gives a shorter proof of [**1**, Theorem 3.8].

As a direct consequence of Proposition 3.2, we next show that the operator space projective tensor product is distributive over finite sums of closed ideals.

Corollary 3.4. *If M_i and N_i, $i = 1, 2, \dots, n$, are closed ideals in A and B, respectively, then*

(i) $A \widehat{\otimes} (\sum_{i=1}^n N_i) = \sum_{i=1}^n (A \widehat{\otimes} N_i)$,

(ii) $(\sum_{i=1}^n M_i) \widehat{\otimes} B = \sum_{i=1}^n (M_i \widehat{\otimes} B)$.

Proof. We shall only prove the first part, and the proof for (ii) follows the same lines. Using [**16**, Theorem 5], each $A \widehat{\otimes} N_i$ is a closed ideal of $A \widehat{\otimes} (\sum_i N_i)$, so it is easy to see that

$$A \widehat{\otimes} \left(\sum_i N_i \right) \supseteq \sum_i (A \widehat{\otimes} N_i).$$

For the other containment, note that $A \otimes (\sum_i N_i) \subseteq \sum_i (A \widehat{\otimes} N_i)$. By [**16**, Theorem 5] and Proposition 3.2, $A \widehat{\otimes} (\sum_i N_i)$ and $\sum_i (A \widehat{\otimes} N_i)$ are both closed in $A \widehat{\otimes} B$. So $A \widehat{\otimes} (\sum_i N_i) \subseteq \sum_i A \widehat{\otimes} N_i$, proving the result. □

We note that Allen *et al.* proved the analogue of the above result for Haagerup tensor product [**1**, Proposition 2.9]. However, their method was more technical. Using Remark 3.3 and the same argument as in the proof of the above result, a much shorter and simpler proof can be provided for the same result.

In the case of C^*-algebras, again using Proposition 3.2, we have the following modified version of [**10**, Proposition 7.1.7].

Proposition 3.5. *Let A, A_1, B and B_1 be C^*-algebras. Given the (complete) quotient mappings $\phi\colon A \to A_1$ and $\psi\colon B \to B_1$, the corresponding mapping $\phi \otimes \psi\colon A \otimes B \to A_1 \otimes B_1$ extends to a (complete) quotient mapping $\phi \widehat{\otimes} \psi\colon A \widehat{\otimes} B \to A_1 \widehat{\otimes} B_1$. Further,*

$$\ker(\phi \widehat{\otimes} \psi) = \ker \phi \widehat{\otimes} B + A \widehat{\otimes} \ker \psi.$$

Proof. From [**10**, Proposition 7.1.7], we know that

$$\ker(\phi \widehat{\otimes} \psi) = (\ker \phi \otimes B + A \otimes \ker \psi)^-,$$

so it is enough to check that

$$(\ker \phi \otimes B + A \otimes \ker \psi)^- = \ker \phi \widehat{\otimes} B + A \widehat{\otimes} \ker \psi.$$

Note that $\ker \phi \widehat{\otimes} B$ and $A \widehat{\otimes} \ker \psi$ are closed ideals of $A \widehat{\otimes} B$ [**16**, Theorem 5] and they can be realized as the closure of $\ker \phi \otimes B$ and $A \otimes \ker \psi$ in $A \widehat{\otimes} B$. The result now follows easily using the fact that $\ker \phi \widehat{\otimes} B + A \widehat{\otimes} \ker \psi$ is closed. □

In [**14**], we proved that the canonical map $i\colon A \widehat{\otimes} B \to A \otimes_{\min} B$ is injective. Making repeated use of this result along with some techniques of Allen *et al.* [**1**], we will now study the ideal structure of $A \widehat{\otimes} B$ in terms of the ideal structures of A and B.

Proposition 3.6. *Let I be a non-zero closed ideal of $A \widehat{\otimes} B$. Then I contains a non-zero elementary tensor and a non-zero product ideal.*

Proof. Let $I_{\min}$ denote the min-closure of I in $A \otimes_{\min} B$, i.e. $I_{\min}$ is the closure of $i(J)$ in $A \otimes_{\min} B$. Then $I_{\min}$ is a non-zero ideal of $A \otimes_{\min} B$ [**14**, Corollary 1], and thus contains a non-zero elementary tensor [**1**, Proposition 4.5], say $a \otimes b$, which also lies in I [**16**, Theorem 6]. Let K and L be the non-zero closed ideals in A and B generated by a and b. Then clearly I contains the product ideal $K \widehat{\otimes} L$. □

Theorem 3.7. *The Banach $*$-algebra $A \widehat{\otimes} B$ is simple if and only if A and B are simple.*

Proof. Let I be a non-zero closed ideal of $A \widehat{\otimes} B$. Then, by Proposition 3.6, I contains a non-zero product ideal $K \widehat{\otimes} L$, where K and L are non-zero ideals of A and B respectively. But A and B are simple so $K = A$ and $L = B$. Thus, $A \widehat{\otimes} B$ is simple.

For the reverse implication, assume that A is not simple. Then it contains a non-trivial closed ideal, say I, which gives rise to a non-zero closed ideal $I \widehat{\otimes} B$ of $A \widehat{\otimes} B$. Now Proposition 3.5 gives an isomorphism between the spaces $(A \widehat{\otimes} B)/(I \widehat{\otimes} B)$ and $(A/I) \widehat{\otimes} B$, which implies $I \widehat{\otimes} B$ is proper in $A \widehat{\otimes} B$. Thus, $I \widehat{\otimes} B$ is a non-trivial closed ideal of $A \widehat{\otimes} B$, which contradicts the fact that $A \widehat{\otimes} B$ is simple. Similarly, one can prove that B is simple. □

Theorem 3.8. *Let A and B be C^*-algebras with A simple. Then every closed ideal in $A \widehat{\otimes} B$ has the form $A \widehat{\otimes} L$ for some closed ideal L in B.*

Proof. Let K be a non-zero closed ideal in $A \widehat{\otimes} B$. By Proposition 3.6, since A is simple, K contains a non-zero product ideal of the form $A \widehat{\otimes} L_1$, L_1 being a non-zero closed ideal of B. Consider the non-empty family $\mathcal{F}$ of closed ideals L of B such that $A \widehat{\otimes} L \subseteq K$. Let $\mathcal{P} = \{L_i : i \in \Lambda\}$ be a chain in $\mathcal{F}$. Note that, by Corollary 3.4, $\mathcal{F}$ is closed under finite sums. So

$$J = \left\{ \sum_{\text{finite}} x_i : x_j \in L_j,\ j \in \Lambda \right\}$$

is an upper bound of $\mathcal{P}$ in $\mathcal{F}$. Thus, by Zorn's lemma, there is a largest closed ideal $L \subseteq B$ such that $A \widehat{\otimes} L \subseteq K$.

Consider the quotient map $1 \otimes \pi : A \widehat{\otimes} B \to A \widehat{\otimes} (B/L)$ with kernel $A \widehat{\otimes} L$. Then $\tilde{K} = (1 \otimes \pi)(K)$ is a closed ideal of $A \widehat{\otimes} (B/L)$. It is sufficient to show that $\tilde{K}$ is the zero ideal, as in that case $K \subseteq \ker(1 \otimes \pi) = A \widehat{\otimes} L$. If $\tilde{K}$ were non-zero, then it would contain a non-zero elementary tensor say $a \otimes (b + L) = (1 \otimes \pi)(a \otimes b)$, where $a \otimes b \in K$. Let N be the closed ideal in B generated by b. Since A is simple, K contains the closed ideal $A \widehat{\otimes} N$. But $A \widehat{\otimes} N$ is not contained in $A \widehat{\otimes} L$, which contradicts the maximality of L. Thus, $\tilde{K}$ is zero ideal and hence yields the result. □

Proposition 3.9. *A closed ideal J in $A \widehat{\otimes} B$ is minimal if and only if there exist minimal closed ideals $K \subseteq A$ and $L \subseteq B$ such that $J = K \widehat{\otimes} L$.*

Proof. Let J be minimal in $A \widehat{\otimes} B$. By Proposition 3.6, there is a non-zero product ideal $K \widehat{\otimes} L$ contained in J. Since J is minimal, $J = K \widehat{\otimes} L$, and it is clear that K and L must be minimal in A and B, respectively.

Conversely, let K and L be minimal closed ideals. Then they are both simple C^*-algebras. By Theorem 3.7, $K \widehat{\otimes} L$ is simple and thus contains no proper non-zero closed ideal of $A \widehat{\otimes} B$. Hence, it is minimal. □

Theorem 3.10. *A closed ideal J is maximal in $A \widehat{\otimes} B$ if and only if there exist maximal closed ideals M in A and N in B such that*

$$J = A \widehat{\otimes} N + M \widehat{\otimes} B.$$

Proof. Let M and N be maximal ideals of A and B respectively. Note that, by Proposition 3.2, $J = A \widehat{\otimes} N + M \widehat{\otimes} B$ is a closed ideal of $A \widehat{\otimes} B$. Also if $\pi_1 : A \to A/M$ and $\pi_2 : B \to B/N$ are quotient maps, then by Proposition 3.5 J is equal to $\ker(\pi_1 \otimes \pi_2)$, and there is an isomorphism between $(A \widehat{\otimes} B)/J$ and $(A/M) \widehat{\otimes} (B/N)$. By Theorem 3.7, $(A \widehat{\otimes} B)/J$ is a simple Banach*-algebra. Thus, J is maximal in $A \widehat{\otimes} B$.

Conversely, let J be a maximal ideal of $A \widehat{\otimes} B$. Let $J_{\min}$ be the min-closure of J in $A \otimes_{\min} B$. Then $J_{\min}$ is a non-zero closed ideal of $A \otimes_{\min} B$ [**14**] and it is proper since $J_{\min} = A \otimes_{\min} B$ would imply $J = A \widehat{\otimes} B$ [**16**]. Let $\pi : A \otimes_{\min} B \to B(H)$ be an irreducible representation annihilating $J_{\min}$. Since the canonical map $i : A \widehat{\otimes} B \to A \otimes_{\min} B$ is a bounded $*$-homomorphism, we get a $*$-representation $\tilde{\pi} = \pi \circ i$ of $A \widehat{\otimes} B$ on H such that

https://doi.org/10.1017/S001309151300045X Published online by Cambridge University Press

$\tilde{\pi}(J) = \{0\}$. By [**23**, Lemma IV.4.1], there exist commuting representations π_1 and π_2 of A and B on H, respectively, such that $\tilde{\pi}(a \otimes b) = \pi_1(a)\pi_2(b)$ for all $a \in A$, $b \in B$. Let $M = \ker \pi_1$, $N = \ker \pi_2$ and $I = A \widehat{\otimes} N + M \widehat{\otimes} B$. Clearly, $\tilde{\pi}(M \widehat{\otimes} B) = \{0\} = \tilde{\pi}(A \widehat{\otimes} N)$, which gives $\tilde{\pi}(J + I) = \{0\}$. So $J + I$ is a proper ideal of $A \widehat{\otimes} B$, which by maximality of J gives $I \subseteq J$. For the reverse inclusion, using Proposition 3.5, there is a quotient map $q\colon A \widehat{\otimes} B \to (A/M) \widehat{\otimes} (B/N)$ with kernel I. It is sufficient to show that $q(J) = \{0\}$. Now, the representations π_1 and π_2 induce faithful commuting representations $\tilde{\pi_1}$ of A/M and $\tilde{\pi_2}$ of B/N on H. Using [**23**, Proposition IV.4.7] and the fact that the canonical map $i\colon A \widehat{\otimes} B \to A \otimes_{\max} B$ is a bounded $*$-homomorphism, there exists a representation π_0 of $(A/M) \widehat{\otimes} (B/N)$ on H such that $\pi_0(x \otimes y) = \tilde{\pi_1}(x)\tilde{\pi_2}(y)$ for all $x \in A/M$, $y \in B/N$. It is easy to verify that $\tilde{\pi}$ and $\pi_0 \circ q$ agree on $A \otimes B$, which by continuity gives $\pi_0(q(J)) = 0$. Now, π is an irreducible representation, so $\tilde{\pi_1}$ and $\tilde{\pi_2}$ are both faithful factor representations with commuting ranges. Using [**23**, Proposition IV.4.20], π_0 is faithful on $(A/M) \otimes (B/N)$, so it is faithful on $(A/M) \widehat{\otimes} (B/N)$ [**14**, Theorem 2]. Thus, $q(J) = 0$. Finally, since $(A \widehat{\otimes} B)/J$ is isomorphic to $(A/M) \widehat{\otimes} (B/N)$, using Theorem 3.7, it is easy to see that M and N are maximal in A and B, respectively. □

Finally, we obtain a complete picture of the lattice of closed ideals of $\mathcal{B}(H) \widehat{\otimes} \mathcal{B}(H)$.

Theorem 3.11. *The only non-trivial closed ideals of $\mathcal{B}(H) \widehat{\otimes} \mathcal{B}(H)$ are*

$$\mathcal{K}(H) \widehat{\otimes} \mathcal{K}(H), \quad \mathcal{B}(H) \widehat{\otimes} \mathcal{K}(H), \quad \mathcal{K}(H) \widehat{\otimes} \mathcal{B}(H) \quad \text{and} \quad \mathcal{B}(H) \widehat{\otimes} \mathcal{K}(H) + \mathcal{K}(H) \widehat{\otimes} \mathcal{B}(H),$$

H being an infinite-dimensional separable Hilbert space.

Proof. It is known that $\mathcal{K}(H)$ is the only non-trivial closed ideal of $\mathcal{B}(H)$, so, using Proposition 3.9 and Theorem 3.10, we have

$$\mathcal{K}(H) \widehat{\otimes} \mathcal{K}(H) \quad \text{and} \quad \mathcal{B}(H) \widehat{\otimes} \mathcal{K}(H) + \mathcal{K}(H) \widehat{\otimes} \mathcal{B}(H)$$

as the unique minimal and maximal closed ideals of $\mathcal{B}(H) \widehat{\otimes} \mathcal{B}(H)$, respectively. Now, consider any non-trivial closed ideal K of $\mathcal{B}(H) \widehat{\otimes} \mathcal{B}(H)$. Using Proposition 3.6 and the fact that any proper closed ideal in a ring with unity must be contained in some maximal ideal, we note that

$$\mathcal{K}(H) \widehat{\otimes} \mathcal{K}(H) \subseteq K \subseteq \mathcal{B}(H) \widehat{\otimes} \mathcal{K}(H) + \mathcal{K}(H) \widehat{\otimes} \mathcal{B}(H). \tag{3.1}$$

Let us write $I = \mathcal{B}(H) \widehat{\otimes} \mathcal{K}(H)$ and $J = \mathcal{K}(H) \widehat{\otimes} \mathcal{B}(H)$. We first claim that

$$K \cap (I + J) = K \cap I + K \cap J. \tag{3.2}$$

Consider any $x \in K \cap (I + J)$. By Lemma 3.1, [**15**, Lemma 1.4.9] and Proposition 3.2, $I + J$ has a bounded approximate identity. So, using Cohen's factorization theorem [**6**], there exist $y, z \in (I + J)$ such that $x = yz$, and z belongs to the closed left ideal generated by x. Thus, $z \in K$, which further gives $x \in K \cap I + K \cap J$. The other inclusion is easy to prove.

https://doi.org/10.1017/S001309151300045X Published online by Cambridge University Press

Now $K \cap I$ and $K \cap J$ are (non-zero) closed ideals of I and J respectively, so, using Theorem 3.8, we can write

$$K \cap I = L \mathbin{\widehat{\otimes}} \mathcal{K}(H) \quad \text{and} \quad K \cap J = \mathcal{K}(H) \mathbin{\widehat{\otimes}} M, \tag{3.3}$$

where L and M are either $\mathcal{B}(H)$ or $\mathcal{K}(H)$. Using (3.1)–(3.3), we have

$$K = L \mathbin{\widehat{\otimes}} \mathcal{K}(H) + \mathcal{K}(H) \mathbin{\widehat{\otimes}} M,$$

which proves the result. □

Acknowledgements. The research of R.J. was supported in part by the Research Foundation Flanders (FWO), under the Research Programme G.0231.07.

References

1. S. D. Allen, A. M. Sinclair and R. R. Smith, The ideal structure of the Haagerup tensor product of C^*-algebras, *J. Reine Angew. Math.* **442** (1993), 111–148.
2. R. J. Archbold, E. Kaniuth, G. Schlichting and D. W. B. Somerset, Ideal spaces of the Haagerup tensor product of C^*-algebras, *Int. J. Math.* **8** (1997), 1–29.
3. B. Blackadar, *Operator algebras: theory of C^*-algebras and von-Neumann algebras* (Springer, 2006).
4. D. P. Blecher and C. LeMerdy, *Operator algebras and their modules: an operator space approach*, London Mathematical Soceity Monographs, New Series, Volume 30 (Clarendon Press, Oxford University Press, Oxford, 2004).
5. D. P. Blecher and V. I. Paulsen, Tensor product of operator spaces, *J. Funct. Analysis* **99** (1991), 262–292.
6. P. J. Cohen, Factorization in group algebras, *Duke Math. J.* **26** (1959), 199–205.
7. P. G. Dixon, Non-closed sums of closed ideals in Banach algebras, *Proc. Am. Math. Soc.* **128** (2000), 3647–3654.
8. E. G. Effros and Z.-J. Ruan, On approximation properties for operator spaces, *Int. J. Math.* **19** (1990), 163–187.
9. E. G. Effros and Z.-J. Ruan, A new approach to operator spaces, *Can. Math. Bull.* **34** (1991), 329–337.
10. E. G. Effros and Z.-J. Ruan, *Operator spaces*, London Mathematical Society Monographs, New Series, Volume 23 (Clarendon Press, Oxford University Press, New York, 2000).
11. C. C. Graham and O. C. McGehee, *Essays in commutative harmonic analysis* (Springer, 1979).
12. U. Haagerup, The Grothendieck inequality for bilinear forms on C^*-algebras, *Adv. Math.* **56** (1985), 93–116.
13. U. Haagerup and M. Musat, The Effros–Ruan conjecture for bilinear forms on C^*-algebras, *Invent. Math.* **174** (2008), 139–163.
14. R. Jain and A. Kumar, Operator space tensor products of C^*-algebras, *Math. Z.* **260** (2008), 805–811.
15. E. Kaniuth, *A course in commutative Banach algebras* (Springer, 2009).
16. A. Kumar, Operator space projective tensor product of C^*-algebras, *Math. Z.* **237** (2001), 211–217.
17. A. Kumar, Involution and the Haagerup tensor product, *Proc. Edinb. Math. Soc.* **44** (2001), 317–322.

18. A. Kumar and A. M. Sinclair, Equivalence of norms on operator space tensor products of C^*-algebras, *Trans. Am. Math. Soc.* **350** (1998), 2033–2048.
19. G. Pisier and D. Shlyakhtenko, Grothendieck's theorem for operator spaces, *Invent. Math.* **150** (2002), 185–217.
20. W. Rudin, *Functional analysis*, 2nd edn (McGraw-Hill, 1991).
21. F. C. Sanchez and R. Garcia, The bidual of a tensor product of Banach spaces, *Rev. Mat. Ibero.* **21** (2005), 843–861.
22. R. R. Smith, Completely bounded module maps and the Haagerup tensor product, *J. Funct. Analysis* **102** (1991), 156–175.
23. M. Takesaki, *Theory of operator algebras, I* (Springer, 1979).
24. S. Wassermann, A pathology in the ideal space of $L(H) \otimes L(H)$, *Indiana Univ. Math. J.* **27** (1978), 1011–1020.