Bull. Aust. Math. Soc. **108** (2023), 11–18
doi:10.1017/S0004972722001137

AFFINE CONVOLUTIONS, RAMANUJAN–FOURIER EXPANSIONS AND SOPHIE GERMAIN PRIMES

ELIEZER FUENTES

(Received 28 August 2022; accepted 6 September 2022; first published online 25 October 2022)

Abstract

For a fixed integer h, the standard orthogonality relations for Ramanujan sums $c_r(n)$ give an asymptotic formula for the shifted convolution $\sum_{n\le N} c_q(n)c_r(n+h)$. We prove a generalised formula for affine convolutions $\sum_{n\le N} c_q(n)c_r(kn+h)$. This allows us to study affine convolutions $\sum_{n\le N} f(n)g(kn+h)$ of arithmetical functions f, g admitting a suitable Ramanujan–Fourier expansion. As an application, we give a heuristic justification of the Hardy–Littlewood conjectural asymptotic formula for counting Sophie Germain primes.

2020 *Mathematics subject classification*: primary 11A25; secondary 11N05.

Keywords and phrases: Ramanujan–Fourier expansion, convolution, Sophie Germain primes.

1. Introduction

Let us write $e(x) = \exp(2\pi i x)$ and let $\mathbb{N}$ be the set of positive integers. In [11], Ramanujan introduced the following sums for $r \in \mathbb{N}$ and $n \in \mathbb{Z}$, now known as *Ramanujan sums*:

$$c_r(n) = \sum_{\substack{a=1\\(a,r)=1}}^{r} e(an/r),$$

where (a, r) denotes the greatest common divisor (gcd) of a and r. An important feature of Ramanujan sums is that they satisfy certain *orthogonality relations*, first noticed by Carmichael [1].

THEOREM 1.1 (Shifted orthogonality relations). *Let $h \in \mathbb{Z}$ and let $q, r \in \mathbb{N}$. Then,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_q(n)c_r(n+h) = \begin{cases} c_r(h) & \text{if } r = q,\\ 0 & \text{otherwise.}\end{cases}$$

That is, the orthogonality relations give an asymptotic formula for the shifted convolution of Ramanujan sums. Our first result is a generalisation for affine convolutions.

THEOREM 1.2 (Affine orthogonality relations). *Let $h \in \mathbb{Z}$ and let $k, q, r \in \mathbb{N}$. Then,*

$$\sum_{n=1}^{N} c_q(n)c_r(kn+h) = \begin{cases} c_r(h)N + O(q^2r^2) & \text{if there is } d \mid k \text{ with } r = dq \text{ and } (k/d, q) = 1, \\ O(q^2r^2) & \text{otherwise.} \end{cases}$$

Here and in the rest of this article, the term *affine* refers to the fact that the arguments used in the convolutions are n and $kn + h$; thus, they are related by an affine transformation.

A *Ramanujan–Fourier expansion* for an arithmetical function $f : \mathbb{N} \to \mathbb{C}$ is a series representation of the form

$$f(n) = \sum_{r=1}^{\infty} \hat{f}(r)c_r(n),$$

where the series converges for each positive integer n. The coefficients $\hat{f}(r)$ are called Ramanujan–Fourier coefficients for f. In [11], Ramanujan obtained Ramanujan–Fourier expansions for several classical arithmetical functions. The literature on Ramanujan–Fourier expansions is vast, and we refer the reader to [9] for a survey.

For example, we have the following result due to Ramanujan [11]:

$$\frac{\sigma_s(n)}{n^s} = \zeta(s+1) \sum_{q=1}^{\infty} \frac{c_q(n)}{q^{s+1}} \tag{1.1}$$

for $s > 0$, where $\sigma_s(n) = \sum_{d|n} d^s$ and $\zeta(z)$ is the Riemann-zeta function. In this case, the series is absolutely convergent.

A more delicate example due to Hardy [7] is a Ramanujan–Fourier expansion for (essentially) the von Mangoldt function $\Lambda(n)$:

$$\frac{\varphi(n)}{n}\Lambda(n) = \sum_{r=1}^{\infty} \frac{\mu(r)}{\varphi(r)} c_r(n), \tag{1.2}$$

where $\varphi(n)$ is the Euler totient function and $\mu(n)$ is the Möbius function. In this case, the series converges for each n but not absolutely.

A topic that has captured considerable attention is that of shifted convolutions of arithmetical functions admitting a Ramanujan–Fourier expansion (see, for instance, [2–5, 10, 12]). Namely, given $f(n) = \sum_r \hat{f}(r)c_r(n)$ and $g(n) = \sum_r \hat{g}(r)c_r(n)$, two arithmetical functions admitting Ramanujan–Fourier expansions, the orthogonality relations for Ramanujan sums heuristically suggest

$$\lim_{n \le N} \frac{1}{N} \sum_{n \le N} f(n)g(n+h) = \sum_{r=1}^{\infty} \hat{f}(r)\hat{g}(r)c_r(h), \tag{1.3}$$

https://doi.org/10.1017/S0004972722001137 Published online by Cambridge University Press

in analogy with the Wiener–Khintchine formula from the theory of Fourier series. In fact, such a formula has been proved in several cases, as in the examples just cited.

Under suitable convergence conditions, we use Theorem 1.2 to prove an analogous formula for affine convolutions.

THEOREM 1.3. *Let $f(n) = \sum_r \hat{f}(r)c_r(n)$ and $g(n) = \sum_r \hat{g}(r)c_r(n)$ be arithmetical functions admitting Ramanujan–Fourier expansions. Suppose that*

$$\sum_{r=1}^{\infty} |\hat{f}(r)|r^2 \quad \text{and} \quad \sum_{r=1}^{\infty} |\hat{g}(r)|r^2$$

converge. Let $k \in \mathbb{N}$ and $h \in \mathbb{Z}$. Then,

$$\sum_{n \le N} f(n)g(kn+h) = \kappa(f, g, k, h) \cdot N + O(1),$$

where

$$\kappa(f, g, k, h) = \sum_{q=1}^{\infty} \sum_{\substack{d \mid k \\ (k/d, q)=1}} \hat{f}(q)\hat{g}(dq)c_{dq}(h).$$

This result is applicable, for instance, to sums of powers of divisors.

COROLLARY 1.4. *Let $s \in \mathbb{R}$ with $s > 2$, let $h \in \mathbb{Z}$ and let $k \in \mathbb{N}$. Then,*

$$\sum_{n \le N} \frac{\sigma_s(n)}{n^s} \cdot \frac{\sigma_s(kn+h)}{(kn+h)^s} = C(s, k, h) \cdot N + O(1),$$

where $C(s, k, h) > 0$ is a constant depending only on s, k and h.

Corollary 1.4 gives a variation of a result of Ingham [5, 8]. Theorem 1.3 motivates the following question.

QUESTION 1.5. Let $f(n) = \sum_r \hat{f}(r)c_r(n)$ and $g(n) = \sum_r \hat{g}(r)c_r(n)$ be arithmetical functions admitting Ramanujan–Fourier expansions and let $k \ge 1$ and h be integers. Does the asymptotic formula

$$\sum_{n \le N} f(n)g(kn+h) \sim \kappa(f, g, k, h) \cdot N$$

hold with

$$\kappa(f, g, k, h) = \sum_{q=1}^{\infty} \sum_{\substack{d \mid k \\ (k/d, q)=1}} \hat{f}(q)\hat{g}(dq)c_{dq}(h)?$$

A motivation to study shifted convolutions via Ramanujan–Fourier expansions comes from [6], where it is shown that if (1.3) holds for $f(n) = g(n) = \varphi(n)\Lambda(n)/n$ with the Ramanujan–Fourier expansion (1.2), then the twin prime conjecture would hold in the following strong form conjectured by Hardy and Littlewood.

https://doi.org/10.1017/S0004972722001137 Published online by Cambridge University Press

CONJECTURE 1.6 (Twin prime conjecture). Let $\pi_2(x) = \#\{p < x : p, p+2 \text{ are primes}\}$ and let

$$C_2 = 2\prod_{p>2}\Big(1 - \frac{1}{(p-1)^2}\Big).$$

Then we have the asymptotic formula

$$\pi_2(x) \sim C_2 \cdot \frac{x}{(\log x)^2}.$$

Arguing as in [6] but using affine convolutions instead of shifted convolutions, one can study prime pairs of the form p, $kp+h$. In particular, let

$$\pi_G(x) = \#\{p < x : p \text{ and } 2p+1 \text{ are primes}\}$$

be the counting function of Sophie Germain primes.

THEOREM 1.7 (Heuristic for Sophie Germain primes). *If Question 1.5 has a positive answer for $f(n) = g(n) = \varphi(n)\Lambda(n)/n$ with the Ramanujan–Fourier expansion (1.2) and $k = 2$, $h = 1$, then there are infinitely many Sophie Germain primes and, moreover,*

$$\pi_G(x) \sim C_2 \cdot \frac{x}{(\log x)^2},$$

where C_2 is the same constant appearing in the twin prime conjecture.

It will be clear from the proof that our argument works, more generally, for prime pairs of the form $p, kp+h$. We restrict our attention to the case of Sophie Germain primes for the sake of exposition.

2. Orthogonality relations

We need the following simple estimate.

LEMMA 2.1 (Trigonometric sums). *Let α be a nonintegral rational number and let $D > 0$ be an integer with $\alpha D \in \mathbb{Z}$. Let $P < Q$ be positive integers. Then,*

$$\Big|\sum_{n=P}^{Q} e(n\alpha)\Big| \le D/2.$$

PROOF. Using the elementary formula for a geometric sum, one finds

$$\Big|\sum_{n=P}^{Q} e(n\alpha)\Big| \le \frac{2}{e(\alpha)-1} = \frac{1}{\sin(\pi\alpha)}$$

and the result follows. □

https://doi.org/10.1017/S0004972722001137 Published online by Cambridge University Press

PROOF OF THEOREM 1.2. Let $S = \sum_{n=1}^{N} c_q(n)c_r(kn+h)$ and note that

$$S = \sum_{\substack{b=1\\(b,r)=1}}^{r} e(bh/r) \sum_{\substack{a=1\\(a,q)=1}}^{q} E(a,b),$$

where

$$E(a,b) = \sum_{n=1}^{N} e\Big(n\Big(\frac{a}{q} + \frac{kb}{r}\Big)\Big).$$

We consider three cases.

Case 1: $dq \neq r$ for every $d \mid k$. We claim that for a, b as in S, the number $a/q + kb/r$ is not an integer. For otherwise, let m be this integer and note that $ar + kbq = mqr$. Since $(a,q) = 1$, this implies $q \mid r$, say $r = dq$, which in turn gives $ad + kb = mdq$. As $(b,dq) = (b,r) = 1$, we have $(b,d) = 1$, and hence $d \mid k$, which is a contradiction.

By Lemma 2.1, $|E(a,b)| \leq qr/2$, and hence $|S| \leq \varphi(r)\varphi(q)qr/2 < (qr)^2$.

Case 2: $dq = r$ for some $d \mid k$ with $(k/d, q) > 1$. We claim that for a, b as in S, the number $a/q + kb/r$ is not an integer. Indeed, let $s = k/d$ and note that $a/q + kb/r = (a + sb)/q$, which is not an integer because $(a,q) = 1$, while $(sb,q) \geq (s,q) > 1$.

As in the previous case, we apply Lemma 2.1 to deduce $|S| < (qr)^2$.

Case 3: $dq = r$ for some $d \mid k$ with $(k/d, q) = 1$. Let $s = k/d$ so that $(s,q) = 1$. Let a and b be as in S. Note that $(b,q) = 1$ because $(b,dq) = (b,r) = 1$. Thus, for each b, there is a unique $a \in (\mathbb{Z}/q\mathbb{Z})^\times$ satisfying $a \equiv -sb \bmod q$, which is equivalent to $a/q + kb/r = (a + sb)/q \in \mathbb{Z}$.

Consequently, $E(a,b) = N$ if $a \equiv -sb \bmod q$ (which, for a given b, occurs exactly for one a), while $|E(a,b)| \leq qr/2$ otherwise (by Lemma 2.1). It follows that

$$\begin{aligned} S &= \sum_{\substack{b=1\\(b,r)=1}}^{r} e(bh/r) \sum_{\substack{a=1\\(a,q)=1}}^{q} E(a,b) \\ &= N \sum_{\substack{b=1\\(b,r)=1}}^{r} e(bh/r) + O(\varphi(r)(\varphi(q)-1)qr) \\ &= N \cdot c_r((h) + O((qr)^2). \end{aligned}$$

□

3. Convolutions

PROOF OF THEOREM 1.3. Since $|c_r(n)| \leq \phi(r) \leq r$, the convergence hypothesis implies that the Ramanujan–Fourier expansions $f(n) = \sum_r \hat{f}(r)c_r(n)$ and $g(n) = \sum_r \hat{g}(r)c_r(n)$ are absolutely convergent, which justifies the following computation:

https://doi.org/10.1017/S0004972722001137 Published online by Cambridge University Press

$$\sum_{n\le N} f(n)g(kn+h) = \sum_{n\le N}\Big(\sum_q \hat{f}(q)c_q(n)\Big)\Big(\sum_r \hat{g}(r)c_r(kn+h)\Big)$$
$$= \sum_{q,r\ge 1} \hat{f}(q)\hat{g}(r)\sum_{n\le N} c_q(n)c_r(kn+h).$$

By Theorem 1.2, this last expression is

$$N\sum_{q\ge 1}\sum_{\substack{d\mid k\\(k/d,q)=1}} \hat{f}(q)\hat{g}(dq)c_{dq}(h) + O\Big(\sum_{q,r}|\hat{f}(q)\hat{g}(r)|q^2r^2\Big).$$

Finally, by the convergence hypothesis, the error term is $O(1)$. □

PROOF OF COROLLARY 1.4. Invoke (1.1), the estimate $|c_q(n)|\le q$ and Theorem 1.3. □

4. Sophie Germain primes

Before proving Theorem 1.7, we need the following lemma.

LEMMA 4.1. *We have*

$$\frac{1}{(\log N)^2}\sum_{n\le N}\Lambda(n)\Lambda(2n+1) = (1+O((\log N)^{-1/2}))\cdot \pi_G(N) + O((\log N)^{-5/2}).$$

PROOF. Let

$$L(N) = \sum_{\substack{p\le N\\ p,2p+1 \text{ are primes}}} \log(p)\log(2p+1).$$

First we note that

$$\sum_{n\le N}\Lambda(n)\Lambda(2n+1) = L(N) + O\Big(\log N\sum_{\alpha=2}^{2\log N}\sum_{p\le 2N^{1/\alpha}}\log p\Big).$$

By Chebyshev's bounds, the sum inside the error term is $O(N^{1/2}\log N)$ from which

$$\sum_{n\le N}\Lambda(n)\Lambda(2n+1) = L(N) + O(N^{1/2}\log^2 N). \tag{4.1}$$

We observe that

$$L(N) = \sum_{\substack{p\le N\\ p,2p+1 \text{ are primes}}} \log^2(p) + O(\pi_G(N)\log N). \tag{4.2}$$

By partial summation,

$$\sum_{\substack{p\le N\\ p,2p+1 \text{ are primes}}} \log^2(p) = \pi_G(N)\log^2(N) + O\Big(\int_2^N \frac{\pi_G(t)\log(t)}{t}\,dt\Big). \tag{4.3}$$

https://doi.org/10.1017/S0004972722001137 Published online by Cambridge University Press

At this point, we could use an unconditional upper bound for $\pi_G(t)$ coming from sieve theory, but let us proceed in a more elementary way. We split the integral at $T = N/\sqrt{\log N}$ and use Chebyshev's bound $\pi_G(t) \le \pi(t) = O(t/\log t)$ to find

$$\begin{aligned}\int_2^N \frac{\pi_G(t)\log(t)}{t}\,dt &= \int_2^T \frac{\pi_G(t)\log(t)}{t}\,dt + \int_T^N \frac{\pi_G(t)\log(t)}{t}\,dt \\ &\le \int_2^T dt + \pi_G(N)\int_T^N \frac{\log(t)}{t}\,dt \\ &= O(T) + O(\pi_G(N)\cdot N\cdot(\log T)/T) \\ &= O(N/(\log N)^{1/2} + \pi_G(N)(\log N)^{3/2}).\end{aligned}$$

The result now follows from (4.1), (4.2) and (4.3). □

PROOF OF THEOREM 1.7. Under the given assumptions,

$$\sum_{n\le N} \frac{\varphi(n)}{n}\Lambda(n)\cdot\frac{\varphi(2n+1)}{2n+1}\Lambda(2n+1) \sim \kappa\cdot N,$$

where

$$\kappa = \sum_{q=1}^{\infty}\sum_{\substack{d|2\\(2/d,q)=1}} \frac{\mu(q)\mu(dq)}{\varphi(q)\varphi(dq)}c_{dq}(1).$$

By a computation similar to (14) in [6], we deduce

$$\sum_{n\le N}\Lambda(n)\Lambda(2n+1) \sim \kappa\cdot N.$$

By Lemma 4.1, we obtain

$$\pi_G(N) \sim \kappa\cdot\frac{N}{(\log N)^2}.$$

It only remains to show $\kappa = C_2$. Since $c_r(1) = \mu(r)$ and the only divisors of 2 are $d = 1$ and $d = 2$, we find

$$\begin{aligned}\kappa &= \sum_{q=1}^{\infty}\sum_{\substack{d|2\\(2/d,q)=1}} \frac{\mu(q)\mu(dq)^2}{\varphi(q)\varphi(dq)} \\ &= \sum_{q\ge 1\text{ odd}} \frac{\mu(q)^3}{\varphi(q)^2} + \sum_{q\ge 1}\frac{\mu(q)\mu(2q)^2}{\varphi(q)\varphi(2q)} \\ &= \sum_{q\ge 1\text{ odd}} \frac{\mu(q)}{\varphi(q)^2} + \sum_{q\ge 1\text{ odd}}\frac{\mu(q)\mu(2q)^2}{\varphi(q)\varphi(2q)}\end{aligned}$$

https://doi.org/10.1017/S0004972722001137 Published online by Cambridge University Press

$$= \sum_{q\geq 1\text{ odd}} \frac{\mu(q)}{\varphi(q)^2} + \sum_{q\geq 1\text{ odd}} \frac{\mu(q)\mu(2)^2}{\varphi(q)^2\varphi(2)}$$

$$= 2\sum_{q\geq 1\text{ odd}} \frac{\mu(q)}{\varphi(q)^2} = C_2. \qquad \square$$

Acknowledgement

This research was conducted while the author was an MSc student at Pontificia Universidad Católica de Chile under the supervision of Hector Pasten. The author wishes to thank Professor Pasten for suggesting the topic and for his help in writing up the results.

References

[1] R. D. Carmichael, 'Expansions of arithmetical functions in infinite series', *Proc. Lond. Math. Soc. (2)* **34** (1932), 1–26.

[2] G. Coppola and M. R. Murty, 'Finite Ramanujan expansions and shifted convolution sums of arithmetical functions, II', *J. Number Theory* **185** (2018), 16–47.

[3] G. Coppola, M. R. Murty and B. Saha, 'On the error term in a Parseval type formula in the theory of Ramanujan expansions II', *J. Number Theory* **160** (2016), 700–715.

[4] G. Coppola, M. R. Murty and B. Saha, 'Finite Ramanujan expansions and shifted convolution sums of arithmetical functions', *J. Number Theory* **174** (2017), 78–92.

[5] H. G. Gadiyar, M. R. Murty and R. Padma, 'Ramanujan–Fourier series and a theorem of Ingham', *Indian J. Pure Appl. Math.* **45**(5) (2014), 691–706.

[6] H. G. Gadiyar and R. Padma, 'Ramanujan–Fourier series, the Wiener–Khintchine formula and the distribution of prime pairs', *Phys. A* **269**(2–4) (1999), 503–510.

[7] G. H. Hardy, 'Note on Ramanujan's trigonometrical function $c_q(n)$ and certain series of arithmetical functions', *Math. Proc. Cambridge Philos. Soc.* **20** (1921), 263–271.

[8] A. E. Ingham, 'Some asymptotic formulae in the theory of numbers', *J. Lond. Math. Soc. (2)* **2** (1927), 202–208.

[9] M. R. Murty, 'Ramanujan series for arithmetical functions', *Hardy–Ramanujan J.* **36** (2013), 21–33.

[10] M. R. Murty and B. Saha, 'On the error term in a Parseval type formula in the theory of Ramanujan expansions', *J. Number Theory* **156** (2015), 125–134.

[11] S. Ramanujan, 'On certain trigonometric sums and their applications in the theory of numbers', *Trans. Cambridge Philos. Soc.* **22** (1918), 179–199.

[12] B. Saha, 'A note on arithmetical functions with absolutely convergent Ramanujan expansions', *Int. J. Number Theory* **12**(6) (2016), 1595–1611.

ELIEZER FUENTES, Mathematics Department,
Pontificia Universidad Católica de Chile, Santiago, Chile
e-mail: eifuentesq@gmail.com