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CONJUGACY CLASSES AND NILPOTENT VARIETY OF A REDUCTIVE MONOID

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ABSTRACT. We continue in this paper our study of conjugacy classes of a reductive monoid M. The main theorems establish a strong connection with the Bruhat-Renner decomposition of M. We use our results to decompose the variety M_{nil} of nilpotent elements of M into irreducible components. We also identify a class of nilpotent elements that we call standard and prove that the number of conjugacy classes of standard nilpotent elements is always finite.

Introduction. In the study of a reductive group G, the variety G_{uni} of unipotent elements plays an important role, *cf.* [1]. In particular, this variety is irreducible and has only finitely many conjugacy classes. We will study in this paper the variety M_{nil} of nilpotent elements in a reductive monoid M with zero. While the two varieties are isomorphic when M is the multiplicative monoid of a finite dimensional algebra, this is in general not true. In fact M_{nil} is usually a reducible variety. We will obtain in this paper a description of the irreducible components of M_{nil}. We accomplish this by first refining our earlier results on conjugacy classes of M. The relevant affine subsets $M(ey)$ are shown to generate the same conjugacy classes as the double $B \times B$ orbit $BeyB$. Next the order on these conjugacy classes is determined within the Renner monoid R. This yields a description of the irreducible components of M_{nil}.

The number of conjugacy classes of M_{nil} is usually infinite. In an earlier paper we showed that the number of conjugacy classes of rank 1 nilpotent elements is always finite. We generalize this result to standard (exponent $=1+$ rank) nilpotent elements.

1. **Preliminaries.** Let M be a reductive monoid over an algebraically closed field k, *cf.* [6], [14]. We will assume that M has a zero 0. Let G denote the reductive unit group of M. The $G \times G$ orbits ($=\mathcal{J}$-classes) of M form a finite lattice $\mathcal{U}$ with order defined by:

$$J_1 \leq J_2 \quad \text{if } J_1 \subseteq \bar{J}_2.$$

There is a cross-section Λ of idempotents e_J, $J \in \mathcal{U}$ so that

$$e_{J_1}e_{J_2} = e_{J_2}e_{J_1} = e_{J_1 \wedge J_2} \quad \text{for all } J_1, J_2 \in \mathcal{U}.$$

Then $\Lambda(\cong \mathcal{U})$ is called the *cross-section lattice* of M. It turns out that

$$T = C_G(\Lambda) = \{g \in G \mid ge = eg \text{ for all } e \in \Lambda\}$$

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is a maximal torus of G. If $\Gamma \subseteq \Lambda$, then

$$P = P(\Gamma) = \{g \in G \mid ge = ege \text{ for all } e \in \Gamma\},$$
$$P^- = P^-(\Gamma) = \{g \in G \mid eg = ege \text{ for all } e \in \Gamma\}$$

are opposite parabolic subgroups of G relative to T. In particular $B = P(\Lambda)$ and $B^- = P^-(\Lambda)$ are opposite Borel subgroups of G relative to T. If $X \subseteq M$, then

$$E(X) = \{e \in X \mid e^2 = e\}$$

is the idempotent set of X. Clearly $E = E(M)$ is a partially ordered set if we define:

$$e \leq f \quad \text{if } ef = fe = e.$$

Moreover,

$$E(M) = \{x^{-1}ex \mid x \in G, e \in \Lambda\}.$$

If $\dim T = n$, then all maximal chains in Λ, $E(M)$, $E(\bar{T})$ have the same length, n. In particular this yields a *rank* function

$$\text{rk}\colon M \longrightarrow \{0, \ldots, n\}$$

such that

$$\text{rk}(0) = 0, \quad \text{rk}(1) = n.$$

Let

$$\Lambda_{\text{min}} = \{e \in \Lambda \mid \text{rk}(e) = 1\}$$
$$\Lambda_{\text{max}} = \{e \in \Lambda \mid \text{rk}(e) = n - 1\}$$

Let $a \in M$, $\text{rk}(a) = m$. Then there is a smallest positive integer t such that $\text{rk}(a^t) = \text{rk}(a^{t+1})$. Then a^t lies in a subgroup of M and

(1) $$\text{rk}(a) > \text{rk}(a^2) > \cdots > \text{rk}(a^t) = \text{rk}(a^{t-1}).$$

So $\text{rk}(a^t) \leq \text{rk}(a) - t + 1$. We will call *a standard* if,

(2) $$\text{rk}(a^t) = \text{rk}(a) - t + 1$$

We note that an element of rank ≤ 1 is necessarily standard. An element a is *nilpotent* if $a^t = 0$ where t is as in (1). Then t is the *exponent* of a. Clearly then a is standard if $\text{rk}(a) = t - 1$. We note that in the case of the full matrix monoid $M_n(k)$, a nilpotent element a is standard if and only it has atmost one non-zero Jordan block.

As usual let $W = N_G(T)/T$ denote the *Weyl group* of G with generating set S of simple reflections, length function l and Bruhat-Chevalley order $\leq$, *cf.* [1], [2]. Then by the Bruhat decomposition,

$$G = \bigsqcup_{x \in W} BxB$$

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and for $x, y \in W$,

$$x \le y \Longleftrightarrow BxB \subseteq \overline{ByB}.$$

If $x \in W$, then we denote by $\dot{x}$, a coset representative of $N_G(T)$. If $I \subseteq S$, then $W_I = \langle I \rangle$ is a (standard) parabolic subgroup of W and

$$P_I = BW_IB, \quad P_I^- = B^- W_I B^-$$

are (standard) opposite parabolic subgroups of G with *Levi decomposition*

$$P_I = L_I U_I, P_I^- = L_I U_I^-, L_I = P_I \cap P_I^-.$$

In particular

$$B = TU, \quad B^- = TU^-, \quad T = B \cap B^-.$$

If $K \subseteq I$, then define $K \lhd I$ if K is a union of some components of I with respect to the Coxeter graph structure of S. Clearly $\lhd$ is a transitive relation and for all $J \subseteq S$,

$$K \lhd I \Rightarrow J \cap K \lhd J \cap I.$$

For $I \subseteq S$, let

$$D_I = \{y \in W \mid \ell(yw) = \ell(y) + \ell(w) \text{ for all } w \in W_I\}$$
$$D_I^{-1} = \{y \in W \mid \ell(wy) = \ell(y) + \ell(w) \text{ for all } w \in W_I\}.$$

Then by [1; Chapter 2], for all $y \in D_I^{-1}$,

(3) $$y^{-1}(B \cap L_I)y \subseteq B \quad \text{and} \quad yBy^{-1} \subseteq U_I^- B.$$

Now $W = D_I W_I = W_I D_I^{-1}$. Hence associated with $x \in W$ is a unique element of D_I^{-1} (and also of D_I). We will need to associate an element of D_I^{-1} in a different way. For $x, y \in W$, define:

$$x \equiv_I y \quad \text{if } \bigcap_{i \ge 0} x^i W_I y^{-i} \ne \emptyset.$$

Clearly $\equiv_I$ is an equivalence relation on W. We also note that

(4) $$x \equiv_I wxw^{-1} \quad \text{if } w \in W_I$$
$$x \equiv_I ux \quad \text{if } u \in \bigcap_{i \ge 0} x^i W_I x^{-i}$$

PROPOSITION 1.1. *Let $x \in W$. Then $x \equiv_I y$ for a unique $y \in D_I^{-1}$. Moreover $\ell(y) \le \ell(x)$.*

PROOF. Let

(5) $$x_1 = x = w_1 y_1, \quad w_1 \in W_I, \quad y_1 \in D_I^{-1}$$
$$x_2 = y_1 w_1 = w_2 y_2, \quad w_2 \in W_I, \quad y_2 \in D_I^{-1}$$
$$x_3 = y_2 w_2 = w_3 y_3, \quad w_3 \in W_I, \quad y_3 \in D_I^{-1}$$
$$\vdots$$

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Now

$$x_{j+1} = y_j w_j = w_j^{-1}(w_j y_j) w_j = w_j^{-1} x_j w_j.$$

Hence by (4),

$$x_1 \equiv_I x_2 \equiv_I x_3 \equiv \cdots. \tag{6}$$

Also

$$\ell(x_{j+1}) = \ell(y_j w_j) \leq \ell(y_j) + \ell(w_j) = \ell(w_j y_j) = \ell(x_j).$$

Hence

$$\ell(x_1) \geq \ell(x_2) \geq \cdots.$$

So for some N,

$$\ell(x_N) = \ell(x_{N+1}) = \cdots.$$

Hence for $j \geq N$

$$y_j w_j = w_{j+1} y_{j+1}, \quad \ell(y_j w_j) = \ell(y_j) + \ell(w_j).$$

Since $y_j \in D_I^{-1}$, we see by the exchange condition [2; Theorem 5.8] that for $j \geq N$,

$$y_{j+1} = y_j u_j, \quad u_j \in W_I, \ \ell(y_{j+1}) = \ell(y_j) + \ell(u_j).$$

In particular

$$\ell(y_N) \leq \ell(y_{N+1}) \leq \cdots.$$

Hence there exists $K \geq N$ such that

$$y_K = y_{K+1} = \cdots.$$

So for $j \geq K$,

$$y_K w_j = w_{j+1} y_K.$$

So

$$y_K w_j y_K^{-1} = w_{j+1} \in W_I.$$

Hence

$$w_K \in \bigcap_{i \geq 0} y_K^{-i} W_I y_K^i.$$

So by (4), (6)

$$x = x_1 \equiv_I x_K = w_K y_K \equiv_I y_K \in D_I^{-1}.$$

Clearly $\ell(y_K) \leq \ell(x_K) \leq \ell(x)$.

Next we prove uniqueness. Let $y, z \in D_I^{-1}$ such that $y \equiv_I z$. Then there exists

$$w \in \bigcap_{i \geq 0} y^i W_I z^{-i}.$$

Let $w_0 = w$ and for $i \geq 1$,

$$w_i = y^{-1} w_{i-1} z = y^{-i} w z^i \in W_I. \tag{7}$$

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Now $w_1 = y^{-1}wz$ and hence

$$wz = yw_1. \tag{8}$$

Since $z \in D_I^{-1}$,

$$\ell(w) + \ell(z) = \ell(yw_1) \le \ell(y) + \ell(w_1). \tag{9}$$

By (8), $w^{-1}y = w_1z^{-1}$. Since $y \in D_I^{-1}$,

$$\ell(w) + \ell(y) = \ell(w_1z^{-1}) \le \ell(z) + \ell(w_1). \tag{10}$$

Adding (9), (10), we see that $\ell(w) \le \ell(w_1)$. Thus by (7),

$$\ell(w) \le \ell(w_1) \le \ell(w_2) \le \cdots. \tag{11}$$

If $N = |W|$, then by (7), $w_N = w$. Hence by (11), $\ell(w) = \ell(w_1)$. By (9), (10), $\ell(y) = \ell(z)$. Since $y, z \in D_I^{-1}$, we see by (8) and the exchange condition that $y = z$. ∎

Let $I \subseteq S$. Then for all $J \subseteq S$,

$$D_I^{-1} \subseteq (D_I^{-1} \cap D_J)W_J.$$

Hence for all $y \in D_I^{-1}$, we see by [1; Theorem 2.7.4] that

$$W_I \cap yW_Jy^{-1} \quad \text{is a standard parabolic subgroup.} \tag{12}$$

Hence

$$\begin{aligned} W_I \cap yW_Iy^{-1} &= W_{I_1}, \quad I_1 \subseteq I \\ W_I \cap yW_{I_1}y^{-1} &= W_{I_2}, \quad I_2 \subseteq I_1 \\ W_I \cap yW_{I_2}y^{-1} &= W_{I_3}, \quad I_3 \subseteq I_2 \\ &\vdots \end{aligned}$$

Let $K = K_0 \lhd I$. Then by (12),

$$\begin{aligned} W_I \cap yW_{K_0}y^{-1} &= W_{K_1}, \quad K_1 \lhd I_1 \\ W_I \cap yW_{K_1}y^{-1} &= W_{K_2}, \quad K_2 \lhd I_2 \\ &\vdots \end{aligned}$$

Let

$$D_I^*(K) = \{y \in D_I^{-1} \mid y \in D_{K_j} \text{ for all } j \ge 0\}. \tag{13}$$

We note that

$$D_I^*(\emptyset) = D_I^{-1}, \quad D_I^*(I) = D_I \cap D_I^{-1}. \tag{14}$$

COROLLARY 1.2. *Let* $y \in D_I^*(K)$, $z \in W_K$, $yz \equiv {}_I y' \in D_I^{-1}$. *Then* $\ell(y') \geq \ell(y)$. *If* $\ell(y) = \ell(y')$, *then* $y = y'$.

PROOF. Let $X_0 = K$ and for $i \geq 0$

$$X_{i+1} = K_{i+1} \setminus \bigcup_{j=0}^{k} X_j \triangleleft K_{i+1}.$$

Then

$$X_i \triangleleft X_i \sqcup X_j,\ X_j \triangleleft X_i \sqcup X_j \quad \text{for } i \neq j. \tag{15}$$

Let

$$X = X_0 \sqcup X_1 \sqcup \cdots = K_0 \cup K_1 \cup K_2 \cup \cdots.$$

Now for $i \geq 0$,

$$W_I \cap yW_{X_i}y^{-1} \subseteq W_I \cap yW_{K_i}y^{-1} \subseteq W_{K_{i+1}} \subseteq W_X.$$

Hence by (15),

$$W_I \cap yW_X y^{-1} \subseteq W_X. \tag{16}$$

Since $y \in D_X$,

$$\ell(yv) = \ell(y) + \ell(v) \quad \text{for all } v \in W_X. \tag{17}$$

Now we apply the algorithm (5) in Proposition 1.1 to yz, along with the exchange condition and (17) to obtain:

$$yz = v_1 y z_1, \quad v_1 \in W_I,\ z_1 \in W_K,\ yz_1 \in D_I^{-1}.$$

Then $v_1 = y(zz_1^{-1})y^{-1} \in W_X$ by (16). So

$$yz_1 v = v_2 y z_2, \quad v_2 \in W_I,\ z_2 \in W_X,\ yz_2 \in D_I^{-1}.$$

Then $v_2 = y(z_1 v_1 z_2^{-1})y^{-1} \in W_X$ by (17). Continuing,

$$yz_i v_i = v_{i+1} y z_{i+1}, \quad v_{i+1} \in W_I,\ z_{i+1} \in W_X,\ yz_{i+1} \in D_I^{-1}.$$

Then as in Proposition 1.1, for some j, $yz \equiv {}_I yz_j \in D_I^{-1}$, $z_j \in W_X$. By (17), $l(yz_j) = \ell(y) + \ell(z_j)$. This completes the proof. ∎

THE RENNER MONOID $R = \overline{N_G(T)}/T$. This is a finite inverse monoid with unit group W and idempotent set $E(\bar{T})$. Moreover

$$R = W\Lambda W, \quad E(\bar{T}) = \bigcup_{x \in W} x^{-1}\Lambda x.$$

By [12], the Bruhat decomposition for G can be extended to M as:

$$M = \bigsqcup_{r \in R} \mathrm{B}r\,B \tag{18}$$

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If $\Gamma \subseteq E(\bar{T})$, let

$$W(\Gamma) = \{x \in W \mid xe = ex \text{ for all } e \in \Gamma\}.$$

Let $e \in \Lambda$. Then

(19) $$W(e) = W_I \quad \text{for some } I = \lambda(e) \subseteq S$$
$$W_e = \{x \in W \mid xe = ex = e\} = W_K \quad \text{for some } K \lhd I.$$

Also let

(20) $$D(e) = D_I, \quad D^*(e) = D_I^*(K), \quad D_e = D_K.$$

Then by (14),

$$D^*(e) = D(e) \cap D(e)^{-1} \quad \text{if } e \in \Lambda_{\min}$$
$$D^*(e) = D(e)^{-1} \quad \text{if } e \in \Lambda_{\max}.$$

We note that $W(e)$ is the Weyl group of $L(e) = C_G(e)$ and W_e is Weyl group of G_e where

$$G_e = \{g \in G \mid ge = e = eg\}^c.$$

If $r \in WeW$, then

$$r = xey, \quad x \in D_e,\ y \in D(e)^{-1}.$$

This is the *standard form* of r. Let $r_1 = xey$, $r_2 = sft$ in standard form. Define

(21) $$r_1 \leq r_2 \quad \text{if } e \leq f,\ x \leq sw,\ w^{-1}t \leq y \quad \text{for some } w \in W(f)W_e.$$

Then by [4],

(22) $$r_1 \leq r_2 \Longleftrightarrow \mathrm{B}r_1 B \subseteq \overline{\mathrm{B}r_2 B}.$$

Let $e, f \in \Lambda$, $y \in D(e)^{-1}$. Then

$$\begin{aligned} Beyfy^{-1} &= eC_B(e)yfy^{-1} \\ &= ey \cdot y^{-1}C_B(e)y \cdot fy^{-1} \\ &\subseteq eyBfy^{-1}, \quad \text{by (3)} \\ &\subseteq eyfBy^{-1} \\ &= eyfy^{-1} \cdot yBy^{-1}. \end{aligned}$$

It follows that if $h = e \cdot yfy^{-1}$, then $Bh = hBh$. So $h \in \Lambda$. Thus we have the following analogue of (12):

(23) $$e \cdot yfy^{-1} \in \Lambda \quad \text{for all } e, f \in \Lambda,\ y \in D(e)^{-1}.$$

The monoid analogue of the Coxeter-Dynkin diagram is the *type map* $\lambda \colon \Lambda \longrightarrow 2^S$ where λ is as in (19). λ along with the Tits building determines the (biordered set) $E(M)$,

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cf. [9]. The determination of all possible type maps remains an important open problem. However the problem has been solved in [9] when $|\Lambda_{\min}| = 1$. These are called *$\mathcal{J}$-irreducible monoids* of type I where $\lambda(e) = I$, $\Lambda_{\min} = \{e\}$. Such monoids arise as the lined closures of irreducible representations of a semisimple groups. For $\mathcal{J}$-irreducible monoids of type I, λ has the following description. Let

$$\Lambda_I = \{e_X \mid X \subseteq S, \text{ no component of } X \text{ is contained in } I\} \cup \{0\}. \tag{24}$$

Define $e_X \leq e_Y$ if $X \subseteq Y$ and let

$$\lambda_I(e_X) = X \cup \{\alpha \in I \mid \alpha\beta = \beta\alpha \text{ for all } \beta \in X\}.$$

Then $\Lambda_I \cong \Lambda$.

2. **Conjugacy classes.** Let $a, b \in M$. Then a is *conjugate* to b ($a \sim b$) if $b = a^x = x^{-1}ax$ for some $x \in G$. If $X, Y \subseteq M$, then we write $X \sim Y$ to mean that every element of X is conjugate to an element of Y and every element of Y is conjugate to an element of X. We will further refine here our earlier results on conjugacy classes [7], [8], while at the same time finding some surprising connections with the Bruhat-Renner decomposition (18).

LEMMA 2.1. *Let $y \in D(e)^{-1}$, $H = C_G(e^z \mid z \in \langle y \rangle)$. Then for all $b \in C_B(e)$, $h \in H$, $ebh\dot{y} \sim eh'y$ for some $h' \in H$.*

PROOF. Let $L = C_G(e)$. Then by (3),

$$\dot{y}^{-1}(U \cap L)\dot{y} \subseteq U = (U \cap L)U_I. \tag{25}$$

Let $V_0 = 1$ and for $i \geq 1$,

$$V_i = \bigcap_{j=0}^{i-1} \dot{y}^j(U \cap L)\dot{y}^{-j} \cap \dot{y}^i U_I \dot{y}^{-i} \subseteq U \cap L$$

$$V = \bigcap_{i \geq 0} \dot{y}^i(U \cap L)\dot{y}^{-i} \subseteq H.$$

If $N = |W|$, then clearly $V_i = 1$ for $i \geq N$. Let

$$U_j = V_j \cdots V_0, \quad j \geq 0.$$

Then since U is a product of root subgroups in any order, we see by (25) that $U \cap L = U_N V$. Also

$$\dot{y}^{-i} V_1 \dot{y} \subseteq U_I, \; \dot{y}^{-1} V_{i+1} \dot{y} \subseteq V_i \quad \text{for } i \geq 1.$$

So

$$\dot{y}^{-1} U_{i+1} \dot{y} \subseteq U_I U_i, \quad i \geq 0. \tag{26}$$

Now

$$bh \in (U \cap L)TH = U_N VTH = U_N H.$$

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So

$$bh = u_1 h_1 \quad \text{for some } u_1 \in U_N,\ h_1 \in H.$$

Suppose $u_1 \in U_{i+1}$, $i \geq 0$. Then

$$\begin{aligned} eu_1h_1\dot{y} &= e\dot{y} \cdot \dot{y}^{-1}u_1h_1\dot{y} \sim \dot{y}^{-1}u_1h_1\dot{y} \cdot e\dot{y} \\ &= \dot{y}^{-1}u_1\dot{y} \cdot \dot{y}^{-1}h_1\dot{y}e\dot{y}. \end{aligned}$$

By (26), $\dot{y}^{-1}u_1\dot{y} = vu_2$ for some $v \in U_I$, $u_2 \in U_i$. Also $h_2 = \dot{y}^{-1}h_1\dot{y} \in H$. So

$$ebh\dot{y} \sim vu_2h_2e\dot{y} = veu_2h_2\dot{y} = eu_2h_2\dot{y}.$$

Hence by induction $ebh\dot{y} \sim eh'\dot{y}$ for some $h' \in H$. ∎

Let $e \in \Lambda$ and let $D(e), D^*(e)$ be as in (20). Let $y \in D(e)^{-1}$, $H = C_G(e^z \mid z \in \langle y \rangle)$. Define

$$M(ey) = eHy$$
$$G(ey) = H \Big/ \prod_{z \in \langle y \rangle} H'_{e^z}$$

where

$$H'_{e^z} = \{h \in H \mid he^z = e^z h = e^z\}.$$

Clearly $\dot{y}$ yields a natural map and automorphism,

(27) $$\xi \colon M(ey) \longrightarrow G(ey), \qquad \sigma \in \operatorname{Aut} G(ey)$$

where $\xi(ehy)$ is the coset of h and $\sigma(h) = \dot{y}h\dot{y}^{-1}$. If $a, b \in G(ey)$, then a is *σ-conjugate* to b if $ga\sigma(g)^{-1} = b$ for some $g \in G(ey)$. Let

$$\tilde{M}(ey) = \bigcup_{g \in G} g \cdot M(ey) \cdot g^{-1}.$$

THEOREM 2.2. *Let $e \in \Lambda$. Then*
(i) If $y \in D(e)^{-1}$, then

$$\tilde{M}(ey) = \bigcup_{g \in G} g \cdot BeyB \cdot g^{-1}$$

(ii) GeG is the disjoint union:

$$GeG = \bigsqcup_{y \in D^*(e)} \tilde{M}(ey).$$

(iii) If $y \in D(e)^{-1}$, $a, b \in M(ey)$, then $a \sim b$ in M if and only if $\xi(a)$, $\xi(b)$ are σ-conjugate in $G(ey)$, where ξ, σ are as in (27).

PROOF. (i) Let $H = C_G(e^z \mid z \in \langle y \rangle)$. Then $\dot{y}$ yields an automorphism σ of H given by: $\sigma(h) = \dot{y}h\dot{y}^{-1}$. Then by (3), $\sigma(B \cap H) = B \cap H$. So if $h \in H$, then by [15; Lemma 7.3], there exists $g \in H$ such that $gh\sigma(g)^{-1} \in B \cap H$. Hence

$$eh\dot{y} \sim g \cdot eh\dot{y} \cdot g^{-1} = egh\sigma(g)^{-1}\dot{y} \in e(B \cap H)y \subseteq BeyB.$$

Also $BeyB \sim Bey = eC_B(e)y$. Combined with Lemma 2.1, we see that $M(ey) \sim BeyB$.

(ii) Let $I = \lambda(e)$. If $x \in W$, then by Proposition 1.1, $x \equiv_I y$ for some $y \in D(e)^{-1}$, $l(y) \leq l(x)$. Hence by [8; Theorem 2.1, 2.6], every element of GeG is conjugate to an element of $M(ey)$ for some $y \in D(e)^{-1}$. Moreover if $y_1, y_2 \in D(e)^{-1}$, then $M(ey_1) \sim M(ey_2)$ if and only if for some $x \in W$, $ey_1 \sim ex$ in R and $x \equiv_I y_2$. In such a case, write $y_1 \approx y_2$. If $y_1 \not\approx y_2$, then by [8], no element of $M(ey_1)$ is conjugate to an element of $M(ey_2)$. We can assume that $\ell(y)$ is minimum in the $\approx$-class of y. Then if $ey \sim ey'$, $y' \in D(e)^{-1}$, then $\ell(y') \geq \ell(y)$. We claim that $y \in D^*(e)$. Suppose $y = y'z^{-1}$, $\ell(y) = \ell(y') + \ell(z) > \ell(y)$,

$$z \in W(e, \ldots, e^{y^{-i}}) \cap W_{e^{y^{-i}}}.$$

Then in R,

$$ey' = eyz \sim zey = ezy = ey \cdot z^y \sim z^y \cdot ey = \cdots \sim z^{y^j} \cdot ey = ey. \tag{28}$$

This contradiction shows that $y \in D^*(e)$. Next let $y_1, y_2 \in D^*(e)$ such that $y_1 \approx y_2$. Let $\ell(y_1) \geq \ell(y_2)$. Then by [8; Theorem 2.6] and (4), there exists $z \in W_e$ such that $zy_1 \equiv_I y_2$. So $y_1 z \equiv_I y_2$. By Proposition 1.1 and Corollary 1.2, $y_1 = y_2$. This proves (ii).

(iii) This is proved in [7; Theorem 2.4]. ∎

Let $\preceq$ denote the transitive relation on R generated by:

1. If $r_1 \leq r_2$, then $r_1 \preceq r_2$.
2. If $y \in D(e)^{-1}$, $x \in W$, then $eyx \preceq xey$. Let

$$R^* = \{ey \mid e \in \Lambda, y \in D^*(e)\}.$$

THEOREM 2.3. *(i) $\preceq$ is a partial order on R^*.*

(ii) M is the disjoint union:

$$M = \bigsqcup_{r \in R^*} \tilde{M}(r).$$

(iii) If $r_1, r_2 \in R^*$, then

$$\tilde{M}(r_1) \subseteq \overline{\tilde{M}(r_2)} \iff r_1 \preceq r_2.$$

(iv) If $r \in R^*$, then $\overline{\tilde{M}(r)} = \bigsqcup_{\substack{r' \in R^* \\ r' \preceq r}} \tilde{M}(r')$.

PROOF. (ii) This follows from Theorem 2.2.

(iii) For $r \in R$, let

$$X(r) = \bigcup_{g \in G} g \cdot \mathrm{Br}B \cdot g^{-1}.$$

If $r \in R^*$, then by Theorem 2.2, $X(r) = \tilde{M}(r)$. For $r \in R$, G acts on $\overline{X(r)}$ by conjugation and B stabilizes $\overline{\mathrm{Br}B}$ under this action. Since G/B is a projective variety, it follows that

$$\overline{X(r)} = \bigcup_{g \in G} g \cdot \overline{\mathrm{Br}B} \cdot g^{-1} = \bigcup_{r' \leq r} X(r'). \tag{29}$$

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Let $e \in \Lambda, y \in D(e)^{-1}, x \in W$. Let $L = C_G(e)$. Then

$$\begin{aligned} BeyxB \sim Beyx = e(B \cap L)yx &\sim xe(B \cap L)y \\ &= xey \cdot y^{-1}(B \cap L)y \\ &\subseteq xeyB, \quad \text{by (3)} \\ &\subseteq BxeyB. \end{aligned}$$

Hence

$$X(eyx) \subseteq X(xey) \quad \text{for } y \in D(e)^{-1},\ x \in W. \tag{30}$$

Also

$$\begin{aligned} BxeyB &\sim eyBx \\ &\subseteq \bigcup_{x' \leq x} Beyx'B, \quad \text{by [13; Theorem 1.4].} \end{aligned}$$

Hence

$$X(xey) \subseteq \bigcup_{x' \leq x} X(eyx') \quad \text{for } y \in D(e)^{-1},\ x \in W. \tag{31}$$

Note also that

$$eyx' \preceq x'ey \leq xey \quad \text{for } y \in D(e)^{-1},\ x' \leq x.$$

By (20), (30), for all $r, r' \in R$,

$$r' \preceq r \Longrightarrow X(r') \subseteq \overline{X(r)}.$$

Now let $e \in \Lambda$, $x \in D^*(e)$, $r \in R^*$ such that $X(ex) \subseteq \overline{X(r)}$. Then by (29), (30), (31), there exists $x_1 \in W$ such that $ex_1 \preceq r$ and $e\dot{x} \in X(ex_1)$. Choose x_1 such that $\ell(x_1)$ is minimum. Then applying the algorithm (5) in Proposition 1.1 and using (30), (31) and the minimality of $\ell(x_1)$, we see that $e\dot{x} \in X(evy)$ for some $y \in D(e)^{-1}$, $v \in W(e^{\alpha} | \alpha \in \langle y \rangle)$ such that $evy \preceq ex_1$ and $\ell(vy) = \ell(x_1)$. Now

$$BevyB \sim Bevy = eC_B(e)vy.$$

By Lemma 2.1, it follows that $e\dot{x}$ is conjugate to an element of $M(ey) \subseteq X(ey)$. Also $ey \leq vey = evy \preceq r$. Hence $\ell(y) = \ell(x_1)$. We claim that $y \in D^*(e)$. Otherwise $y = y_1 z^{-1}$, $\ell(y) = \ell(y_1) + \ell(z) > \ell(y_1)$,

$$z \in W(e, \ldots, e^{y^{-i}}) \cap W_{e^{y^{-i}}}.$$

Then $y_1 \in D(e)^{-1}$ and

$$ey_1 = ey \cdot z \preceq zey = ezy = ey \cdot z^y \preceq z^y \cdot ey = \cdots \preceq z^{y_i} \cdot ey = ey.$$

Hence $ey_1 \preceq ey$. Also by (28), $ey \sim ey_1$ in R. By [8; Theorem 2.6], $M(ey) \sim M(ey_1) \subseteq X(ey_1)$. Hence $e\dot{x} \in X(ey_1)$, $ey_1 \preceq r$, $\ell(y_1) < \ell(x_1)$. This contradiction shows that $y \in D^*(e)$. By Theorem 2.2 (ii), $x = y$. Hence $ex \preceq r$, proving (ii).

(iv) This follows from the proof of (iii).

(i) Let $r_1, r_2 \in R^*$ such that $r_1 \preceq r_2 \preceq r_1$. Then $\overline{\tilde{M}(r_1)} = \overline{\tilde{M}(r_2)}$. Since this is an irreducible variety, there exist non-empty open subsets O_1, O_2 such that $O_1 \subseteq \tilde{M}(r_1)$ and $O_2 \subseteq \tilde{M}(r_2)$. In particular $\tilde{M}(r_1) \cap \tilde{M}(r_2) \neq \emptyset$. By Theorem 2.2, $r_1 = r_2$. This completes the proof. ∎

EXAMPLE 2.4. $\preceq$ is not a partial order on R. If $M = M_3(k)$, then

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \leq \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $e \in \Lambda$, $y \in D(e)^{-1}$. Then by (23), $e \cdot yey^{-1} \in \Lambda$. So again by (23),

$$e \cdot yey^{-1} \cdot y^2ey^{-1} = e \cdot y(e \cdot yey^{-1})y^{-1} \in \Lambda.$$

Continuing, we see that

$$f = e \cdot yey^{-1} \cdot y^2ey^{-2} \cdots y^{N-1}ey^{1-N} \in \Lambda$$

where $N = |W|$. Hence $(ey)^N = fy^N = f$. So

$$(32) \qquad (ey)^N \in \Lambda \quad \text{for all } y \in D(e)^{-1}.$$

In particular

$$y \in W_f, \quad ey \in M_f = \{a \in M \mid af = fa = f\}^c.$$

So ey is a nilpotent element of $R(M_f)$ and the study of conjugacy within $M(ey)$ reduces to studying conjugacy within $C_G(G_f)$ and conjugacy within $M_f(eY)$. See [5; Theorem 4.1]. We note that every element of $M_f(ey)$ is nilpotent in M_f. We are therefore naturally led to studying nilpotent elements in reductive monoids.

3. Nilpotent variety.

While the variety G_{uni} of unipotent elements in a reductive group G is always irreducible, the variety M_{nil} of nilpotent elements in a reductive monoid M is usually not irreducible. We will use the results of the previous section to decompose M_{nil} into irreducible components. Let R^*_{nil} denote the set of nilpotent elements in R^*.

THEOREM 3.1. *(i) Let $e \in \Lambda$, $e \neq 0$, $y \in D^*(e)$. Then $ey \in R^*_{\text{nil}}$ if and only if $y \notin W(f)$ for all $f \in \Lambda_{\min}$ with $f \leq e$*

(ii) M_{nil} is the disjoint union:

$$M_{\text{nil}} = \bigsqcup_{r \in R^*_{\text{nil}}} \tilde{M}(r).$$

(iii) The irreducible components of M_{nil} are $\overline{\tilde{M}(r)}$ where r is a maximal element of R^*_{nil} with respect to the partial order $\preceq$.

PROOF. (i) Suppose ey is not nilpotent. Then by (32), there exists $f \in \Lambda_{\min}$ such that $eyf = fey = f$. This implies that $f \leq e$ and $y \in W(f)$. Conversely suppose $f \in \Lambda_{\min}$,

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$f \leq e$ such that $y \in W(f)$. Then $eyf = efy = fy$. So $(ey)^i f = fy^i$ for all i. So ey is not nilpotent.

(ii) It is easy to see that $\tilde{M}(r)$ has a nilpotent element if and only if r is nilpotent in R. In this case every element of $\tilde{M}(r)$ is nilpotent. Hence (ii) follows from Theorem 2.3.

(iii) This follows from Theorem 2.3 since each $\tilde{M}(r)$ is irreducible. ∎

We will now apply Theorem 3.1 to two special cases. By a *canonical monoid* on G, we mean a $\mathcal{J}$-irreducible monoid of type $\emptyset$. Such monoids are obtained by taking the lined closure of an irreducible representation of a semisimple group with the highest weight being in the interior of the Weyl chamber. They are also related to the canonical compactification of a reductive group. We refer to [10] for details. In the case of $\mathrm{SL}_n(k)$ such a monoid is obtained by taking the lined closure of the representation:

$$A \longrightarrow \otimes_i \wedge^i A.$$

We will also consider the *dual canonical monoid* (see [11]) where the cross section lattice of the canonical monoid is turned upside down. For $\mathrm{SL}_n(k)$ such a monoid is obtained by taking the lined closure of the representation:

$$A \longrightarrow \oplus_i \wedge^i A.$$

THEOREM 3.2. *(i) Let M be a canonical monoid with $\Lambda_{\max} = \{f_\alpha \mid \alpha \in S\}$, where $\lambda(f_\alpha) = S \setminus \{\alpha\}$. Then M_{nil} has $|S|$ irreducible components: $\overline{\tilde{M}(f_\alpha \alpha)}$, $\alpha \in S$.*

(ii) Let M be a dual canonical monoid with $\Lambda_{\max} = \{e\}$. If S has t components, then M_{nil} has $2^{|S|-t}$ irreducible components: $\overline{\tilde{M}(ey)}$ where y is a Coxeter element of W of length $|S|$.

PROOF. (i) Now

$$\Lambda = \{e_X \mid X \subseteq S\} \cup \{0\}$$

with $\lambda(e_X) = X$. For $\alpha \in S$, let $f_\alpha = e_X$ where $X = S \setminus \{\alpha\}$. Let $X \subseteq S$, $e_X y \in R^*_{\mathrm{nil}}$. Then y starts with $\alpha \notin X$. So $e_X y \leq f_\alpha \alpha$. By Theorem 3.1 (ii), $f_\alpha \alpha \in R^*_{\mathrm{nil}}$. The result now follows from Theorem 3.1 (iii).

(ii) Now

$$\Lambda = \{1\} \cup \{e_X \mid X \subseteq S\}$$

with $0 = e_S$ and $\Lambda_{\max} = \{e_\emptyset\}$. Let $e = e_\emptyset$. Let $X \subseteq S, X \neq S$. Then $W(e_X) = W_{e_X} = W_X$. Let $e_X y \in R^*_{\mathrm{nil}}$. By Theorem 3.1(ii), $y \notin W_Y$ for any proper subset Y of S containing X. Thus y involves each $\alpha \in S \backslash X$. Thus $y \geq z$ for some Coxeter elements of $W_{S \backslash X}$ of length $|S \setminus X|$. Let v be a Coxeter element of W_X of length $|X|$. Then vz is a Coxeter element of W of length $|S|$. Since $v \in W_{e_X}$, $e_X y \leq e_X z \leq evz$. Since $W(e) = 1$, $\preceq = \leq$ on eW. Also if $x \in W$ is a Coxeter element of length $|S|$, then by Theorem 3.1(i), $ex \in R^*_{\mathrm{nil}}$. It is a consequence of induction and the exchange condition that the number of Coxeter element of length $|S|$ is $2^{|S|-t}$. This completes the proof. ∎

EXAMPLE 3.3. Let

$$M = \{A \otimes B \mid A, B \in M_4(K), A^t B = BA^t \text{ is a scalar matrix}\}.$$

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Then $S = \{\alpha - \beta - \gamma\}$ and M is a $\mathcal{J}$-irreducible monoid of type $\{\beta\}$. Hence

$$\Lambda_{\max} = \{e_1, e_2, e_3\}, \quad \lambda(e_1) = \{\alpha, \beta\}, \quad \lambda(e_2) = \{\alpha, \gamma\}, \quad \lambda(e_3) = \{\beta, \gamma\}.$$

The maximal elements with respect to $\leq$ as well as $\preceq$ of R^*_{nil} are:

$$\{e_1\gamma, e_2\beta\alpha, e_2\beta\gamma, e_3\alpha\}.$$

Correspondingly the irreducible components of M_{nil} are:

$$\overline{\tilde{M}(e_1\gamma)}, \quad \overline{\tilde{M}(e_2\beta\alpha)}, \quad \overline{\tilde{M}(e_2\beta\gamma)}, \quad \overline{\tilde{M}(e_3\alpha)}.$$

CONJECTURE 3.4. The maximal element of R^*_{nil} with respect to $\leq$ are also the maximal element of R^*_{nil} with respect to $\preceq$.

4. **Finiteness.** We study in this section the problem of when the number of conjugacy classes within $\tilde{M}(ey)$ is finite.

THEOREM 4.1. *Let $e \in \Lambda$, $y \in D(e)^{-1}$. Then the following conditions are equivalent:*
(i) $\tilde{M}(ey)$ has finitely many conjugacy classes.
(ii) $\tilde{M}(ey)$ is a single conjugacy class.
(iii) $G(ey)$ is a torus and for all $f \in E(\bar{T})$ with $f^y = f$, $f \in \overline{\prod_{z \in \langle y \rangle} T_{e^z}}$.
In this case ey is nilpotent.

PROOF. (i) $\Rightarrow$ (iii). Let σ denote the automorphism of $G(ey)$ associated with $\dot{y}$. By Theorem 2.2, $G(ey)$ has finitely many σ-conjugacy classes. So for some $x \in G(ey)$, the σ-conjugacy class of x is dense in $G(ey)$. Let θ denote the automorphism of $G(ey)$ given by: $\theta(g) = x\sigma(g)x^{-1}$. So the map:

$$g \longrightarrow g\theta(g)^{-1} = gx\sigma(g)^{-1} \cdot x^{-1}$$

from $G(ey)$ to $G(ey)$ is dominant. By [15; 10.2],

$$G(ey)_\theta = \{g \in G(ey) \mid \theta(g) = g\}$$

is finite. By [15; Corollary 10.12], $G(ey)$ is solvable. Since $G(ey)$ is reductive it follows that $G(ey) = T'$ is a torus. So

$$T' = T/T_1, \quad T_1^c = \overline{\prod_{z \in \langle y \rangle} T_{e^z}}.$$

Hence $\sigma = \theta$ and T'_σ is finite. Now let $f \in (E(\bar{T})$ such that $f^y = f$. Let $y^{n+1} = 1$,

$$T_2 = \{t \cdot t^y \cdots t^{y^n} \mid t \in T_f\}.$$

Then T_2 is a torus and $f \in \bar{T}_2$. Clearly the image of T_2 in T' is contained in T'_σ. Since T'_σ is finite, $T_2 \subseteq T_1$. So $f \in \bar{T}_1$. In particular $0 \in \bar{T}_1$. This implies that $\prod_{z \in \langle y \rangle} e^z = 0$ and hence ey is nilpotent.

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(iii) $\Rightarrow$ (ii). Let $y^{n+1} = 1$,

$$T_1 = \{t \cdot t^y \cdots t^{y^n} \mid t \in T\}.$$

Then T_1 is a torus, $0 \in \bar{T}_1$. Let

$$T_2 = \Big\{t \cdot t^y \cdots t^{y^n} \mid t \in \prod_{z \in \langle y \rangle} T_{e^z}\Big\}.$$

Then $E(\bar{T}_1) = E(\bar{T}_2)$, $T_2 \subseteq T_1$. Hence $T_1 = T_2$. So for all $t \in G(ey)_\sigma$, $t^{n+1} = 1$. Hence $G(ey)_\sigma$ is finite. By [15; Theorem 10.1], the σ-conjugacy class of 1 is $G(e\sigma)$. By Theorem 2.2, $\tilde{M}(ey)$ is a single conjugacy class.

(ii) $\Rightarrow$ (i). This is obvious. ∎

REMARK 4.2. Since $G(ey)$ is a reductive group, we see that $G(ey) = 1$ if and only if $T = \prod_{z \in \langle y \rangle} T_{e^z}$.

Finally we generalize our earlier result [5; Theorem 4.8] on rank 1 nilpotent elements.

THEOREM 4.3. *The number of conjugacy classes of standard nilpotent elements in M is finite and is equal to the number of standard nilpotent elements in R^*.*

PROOF. Let $e \in \Lambda, y \in D(e)^*$. Then clearly an element of $\tilde{M}(ey)$ is standard nilpotent if and only if ey is standard nilpotent in R. Let ey be standard nilpotent of rank p. Then $\mathrm{rk}\big((ey)^i\big) = p - i + 1$. Let

$$e_i = e \cdot yey^{-1} \cdots y^i e y^{-i}, \quad i = 0, \ldots, p.$$

Then $(ey)^{i+1} = e_i y^i$, $i = 0, \ldots, p$. Hence $\mathrm{rk}(e_i) = p - i$ and

$$e = e_0 > e_1 > \cdots > e_p = 0.$$

Let $T_1 = \prod_{z \in \langle y \rangle} T_{e^z}$. Then $T_e \subseteq T_1$ and $e_0, \ldots, e_p \in \bar{T}$. Hence we have a maximal chain of $E(\bar{T})$ contained in $\bar{T}_1$. Hence $\dim T_1 = \dim T$. So $T = T_1$. By Remark 4.2, $G(ey) = 1$. We are now done by Theorem 2.3. ∎

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