PERSISTENCE OF THE NORMALIZED EIGENVECTORS OF A PERTURBED OPERATOR IN THE VARIATIONAL CASE

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Abstract. Let H be a real Hilbert space and denote by S its unit sphere. Consider the nonlinear eigenvalue problem $Ax + \varepsilon B(x) = \delta x$, where $A : H \to H$ is a bounded self-adjoint (linear) operator with nontrivial kernel Ker A, and $B : H \to H$ is a (possibly) nonlinear perturbation term. A unit eigenvector $x_0 \in S \cap \mathrm{Ker}\, A$ of A (thus corresponding to the eigenvalue $\delta = 0$, which we assume to be isolated) is said to be *persistent*, or a *bifurcation point* (from the sphere $S \cap \mathrm{Ker}\, A$), if it is close to solutions $x \in S$ of the above equation for small values of the parameters $\delta \in \mathbb{R}$ and $\varepsilon \neq 0$. In this paper, we prove that if B is a C^1 gradient mapping and the eigenvalue $\delta = 0$ has finite multiplicity, then the sphere $S \cap \mathrm{Ker}\, A$ contains at least one bifurcation point, and at least two provided that a supplementary condition on the potential of B is satisfied. These results add to those already proved in the non-variational case, where the multiplicity of the eigenvalue is required to be odd.

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1. Introduction and statement of the results. Let S be the unit sphere in a real Hilbert space H and consider the equation

$$Tx + \varepsilon B(x) = \lambda x \tag{1.1}$$

in the unknown $(\lambda, \varepsilon, x) \in \mathbb{R} \times \mathbb{R} \times S$, where T is a bounded self-adjoint (linear) operator in H and B is a (possibly nonlinear) continuous operator in H. We consider the following bifurcation problem for (1.1): suppose that λ_0 is an eigenvalue of T, let $\mathrm{Ker}(T - \lambda_0 I)$ be the corresponding eigenspace, and put

$$S_0 := S \cap \mathrm{Ker}(T - \lambda_0 I).$$

Then, (1.1) possesses the family

$$Z_0 := \{(\lambda_0, 0, v) | v \in S_0\}$$

of *trivial* solutions. A trivial solution $(\lambda_0, 0, v_0) \in Z_0$ – essentially, a normalized λ_0-eigenvector v_0 of T – is said to be a *bifurcation point from Z_0 for the solutions of* (1.1), or briefly a *bifurcation point from Z_0 for* (1.1), if any neighbourhood of $(\lambda_0, 0, v_0)$

in $\mathbb{R} \times \mathbb{R} \times S$ contains *nontrivial* solutions – that is, solutions not in Z_0 – of (1.1). If we assume that λ_0 is an isolated eigenvalue, then any nontrivial solution $(\lambda, \varepsilon, x)$ of (1.1) with λ close to λ_0 must have $\varepsilon \neq 0$. Therefore, $v_0 \in S_0$ is a bifurcation point in the above sense if and only if there exists a sequence $\{(\lambda_n, \varepsilon_n, x_n)\}$ in $\mathbb{R} \times \mathbb{R} \setminus \{0\} \times S$ which converges to $(\lambda_0, 0, v_0)$ and such that $Tx_n + \varepsilon_n B(x_n) = \lambda_n x_n, \forall n \in n$. In this case, we say equivalently that v_0 is a *persistent* normalized λ_0-eigenvector of T.

In the papers [2–4], we have discussed both necessary conditions and sufficient conditions for this particular kind of bifurcation to occur when λ_0 is an isolated eigenvalue of T *of finite multiplicity* (algebraic and geometric multiplicity coincide for self-adjoint operators). Particularly in [3], using as a main tool the Lefschetz fixed-point theorem (see [9] for example) we have shown that bifurcation from Z_0 does indeed take place at some point of Z_0 provided that B is of class C^1 and the multiplicity of λ_0 is *odd*. It is well known that a similar result holds for bifurcation in the classical sense, that is for the one-parameter equation

$$Tx + B(x) = \lambda x, \tag{1.2}$$

where B is C^1 near $x = 0$ and satisfies the conditions $B(0) = B'(0) = 0$. Here, the trivial solutions are $\{(\lambda, 0)|\lambda \in \mathbb{R}\}$, and so one is interested in bifurcation from this line at a given point $(\lambda_0, 0)$. It is equally well known that if in addition B is a *gradient* operator – so that (1.2) has a variational character – then the requirement of oddness about the multiplicity of λ_0 can be removed. For the precise statement of these basic results about (1.2), we refer for instance to [1, 5, 6, 10, 12, 13].

In this paper, we give more evidence of the similarity existing between (1.1) and (1.2), as we show that when B is a gradient, there is bifurcation from Z_0 irrespective of the (finite) multiplicity of λ_0. The formal statement is given in Theorem 1.1 below. Before doing this, we wish to remark that 'nonclassical' bifurcation problems of the kind considered here have been proposed by several authors: see in particular Ambrosetti and Prodi [11], who called them *atypical* bifurcation problems and Furi and Pera [8], who introduced the related concept of *cobifurcation*; see in addition the paper [7], where a unifying framework for both classical and 'atypical' bifurcation is proposed. Also, we believe that the analysis of (1.1) is far from complete and in fact many open problems await attention, for instance the possible extension of the results in [3] to Banach space operators.

It is now time to state formally the main result of the present paper, and before this we make precise our assumptions about the nonlinear term B. These are as follows:

(B1) $B : U \to H$ is a C^1 operator defined on a neighbourhood U of S;
(BG) B is the gradient of a functional: that is, there exists a differentiable functional b defined on U such that

$$\langle B(x), y \rangle = b'(x)y, \qquad \forall x \in U, \ \forall y \in H. \tag{1.3}$$

In (1.3), $\langle \cdot, \cdot \rangle$ denotes the scalar product in H and $b'(x)$ denotes the (Fréchet) derivative of b at the point $x \in U$. We then write for short $B = \nabla b$.

THEOREM 1.1. *Let T be a bounded self-adjoint operator in a real Hilbert space H, and let λ_0 be an isolated eigenvalue of T of finite multiplicity. If B satisfies (B1) and (BG), then there exists at least one bifurcation point from Z_0 for (1.1).*

For the proof of Theorem 1.1, we take advantage of a technical lemma proved in [3] and of an idea of Stuart [13]. The former is needed for an appropriate global

reduction of (1.1) to a finite-dimensional equation in $\mathrm{Ker}(T - \lambda_0 I)$, while the latter permits to identify the solutions of this equation – for ε fixed – as the critical points of a C^1 functional α_ε (related to the quadratic form of T) on a compact C^1 manifold M_ε which is essentially a deformation of S_0. Then the minimum and the maximum points of α_ε on M_ε give approximate solutions v_ε, and letting $\varepsilon \to 0$ in a suitable manner yields the desired bifurcation point in S_0.

This passage to the limit can be better controlled, and permits a considerable improvement of Theorem 1.1, when B is *bounded* in U (or in a smaller neighbourhood of S). Indeed in this case we show – supposing in addition that b is not constant on S_0 – that there are (at least) *two* distinct bifurcation points for (1.1), which moreover have the property of minimizing (resp. maximizing) b over S_0. This is formally stated as Theorem 2.1 in the next section. It is not clear to us whether the condition that b is not constant on S_0 is essential (so that this adds to the open problems about (1.1)). For further remarks about (1.1) we refer to the review [4], where the results of the present paper were announced.

2. Proof of the results.
Set

$$A = T - \lambda_0 I, \qquad \delta = \lambda - \lambda_0$$

and write (1.1) as

$$Ax + \varepsilon B(x) = \delta x. \tag{2.1}$$

Also put

$$N = \mathrm{Ker}\, A = \{x \in H : Ax = 0\}, \qquad W = \mathrm{Im}\, A = \{Ax : x \in H\}.$$

Then, by our assumptions on T and λ_0, H is the orthogonal sum

$$H = N \oplus W. \tag{2.2}$$

Let P and $Q = I - P$ be the orthogonal projections onto N and W respectively; then writing $x = Px + Qx =: v + w$ according to (2.2) and applying in turn P, Q to both members of (2.1), we see that this equation is equivalent to the system

$$\varepsilon PB(v + w) = \delta v, \tag{2.3}$$

$$Aw + \varepsilon QB(v + w) = \delta w. \tag{2.4}$$

In the first part of our analysis we proceed essentially as in [3]. Let V' and W' be neighbourhoods of S_0 in N and of 0 in W respectively such that $V' + W' \subset U$. Identify H with $N \times W$ and consider the map f of $\mathbb{R} \times \mathbb{R} \times V' \times W'$ into W defined putting

$$f(\delta, \varepsilon, v, w) := Aw + \varepsilon QB(v + w) - \delta w. \tag{2.5}$$

We have

$$\frac{\partial f}{\partial w}(0, 0, v_0, 0) = A|_W$$

for any $v_0 \in V'$, and $A|_W : W \to W$ is an isomorphism. Then (B1) and a straightforward application of the implicit function theorem guarantee that, given any $v_0 \in S_0 \subset V'$, equation (2.4) – the so-called *complementary equation* – can be solved uniquely w.r.t. w for each given (δ, ε, v) in a neighbourhood $Y_0 \subset \mathbb{R} \times \mathbb{R} \times N$ of $(0, 0, v_0)$. Moreover if $w(\delta, \varepsilon, v)$ denotes the solution corresponding to $(\delta, \varepsilon, v) \in Y_0$, then $w(0, 0, v) = 0$ for any v and the mapping $(\delta, \varepsilon, v) \to w(\delta, \varepsilon, v)$ of Y_0 into W is of class C^1. Therefore, by definition

$$Aw(\delta, \varepsilon, v) + \varepsilon QB(v + w(\delta, \varepsilon, v)) = \delta w(\delta, \varepsilon, v) \tag{2.6}$$

for any $(\delta, \varepsilon, v) \in Y_0$; and – replacing $w(\delta, \varepsilon, v)$ in (2.3) – we see that in order to solve (2.1) it is enough to find $(\delta, \varepsilon, v) \in Y_0$ satisfying the finite-dimensional equation (the *bifurcation equation*)

$$\varepsilon PB(v + w(\delta, \varepsilon, v)) = \delta v \tag{2.7}$$

and the additional normalization constraint

$$v + w(\delta, \varepsilon, v) \in S. \tag{2.8}$$

However, a strengthened version of the implicit function theorem is needed here in order to solve the complementary equation *globally* with respect to S_0. Thus, for $\eta > 0$, consider the compact neighbourhood of S_0

$$M = \{v \in N : \big|\|v\| - 1\big| \leq \eta\} \tag{2.9}$$

and take $\eta > 0$ so small that $M \subset V'$. Then applying Lemma 2.2 of [**3**] with f as in (2.5), $\Omega = \mathbb{R} \times \mathbb{R} \times V' \times W'$ and $G = \{0\} \times \{0\} \times M \times \{0\}$, it follows that the function $w = w(\delta, \varepsilon, v)$ can be taken to be defined and of class C^1 on an open neighbourhood $Y_1 = I_1 \times J_1 \times V_1 \subset \mathbb{R} \times \mathbb{R} \times N$ of $\{0\} \times \{0\} \times M$.

A further reduction can be made on 'eliminating δ' from our equations. Indeed, take the scalar product in (2.7) to get

$$\langle \varepsilon PB(v + w(\delta, \varepsilon, v)), v \rangle = \delta \|v\|^2. \tag{2.10}$$

Dividing both members of (2.10) by $\|v\|^2$ and applying again Lemma 2.2 of [**3**] to the resulting equation, we see that δ can be written as a C^1 function $\delta(\varepsilon, v)$ of (ε, v), defined on the closure of an open subset $Z = J \times V$ of $\mathbb{R} \times N$ containing $\{0\} \times M$ and such that $\delta(0, v) = 0$ for any $v \in V$, and $(\delta(\varepsilon, v), \varepsilon, v) \in Y_1$ for $(\varepsilon, v) \in Z$. We shall also assume that the closure $\overline{V}$ of V is compact and does not contain the origin of N. For convenience put

$$\phi(\varepsilon, v) := w(\delta(\varepsilon, v), \varepsilon, v), \quad (\varepsilon, v) \in Z. \tag{2.11}$$

Then, by definition

$$\langle \varepsilon PB(v + \phi(\varepsilon, v)), v \rangle = \delta(\varepsilon, v)\|v\|^2 \tag{2.12}$$

for $(\varepsilon, v) \in Z$, and we also have in particular from (2.6):

$$A\phi(\varepsilon, v) + \varepsilon QB(v + \phi(\varepsilon, v)) = \delta(\varepsilon, v)\phi(\varepsilon, v). \tag{2.13}$$

Moreover, $\phi(0, v) = 0$ for $v \in V$ and we see – from (2.7) and (2.8) – that in order to have a nontrivial solution, we need to find $(\varepsilon, v) \in Z$, $\varepsilon \neq 0$, such that

$$\varepsilon PB(v + \phi(\varepsilon, v)) = \delta(\varepsilon, v)v \tag{2.14}$$

and

$$\|v + \phi(\varepsilon, v)\|^2 = \|v\|^2 + \|\phi(\varepsilon, v)\|^2 = 1. \tag{2.15}$$

A normalized solution (δ, ε, x) of the original equation (2.1) will then be given by $(\delta(\varepsilon, v), \varepsilon, v + \phi(\varepsilon, v))$.

To proceed to the main step of the proof, we also put for convenience

$$F_\varepsilon(x) := Ax + \varepsilon B(x), \quad \delta_\varepsilon(v) := \delta(\varepsilon, v), \quad \phi_\varepsilon(v) := \phi(\varepsilon, v)$$

for $\varepsilon \in J$ and $v \in V$. Then, the system (2.14)–(2.15) in the unknowns ε and v can be written

$$PF_\varepsilon(v + \phi_\varepsilon(v)) = \delta_\varepsilon(v)v, \qquad \|v + \phi_\varepsilon(v)\|^2 = 1 \tag{2.16}$$

while the identities (2.12) and (2.13) become

$$\langle PF_\varepsilon(v + \phi_\varepsilon(v)), v \rangle = \delta_\varepsilon(v)\|v\|^2, \tag{2.17}$$

$$QF_\varepsilon(v + \phi_\varepsilon(v)) = \delta_\varepsilon(v)\phi_\varepsilon(v). \tag{2.18}$$

We now show, using (BG), that for any ε there exist (at least) *two* distinct solutions $v = v_\varepsilon$, $z = z_\varepsilon$ of (2.16). For this purpose, let $B = \nabla b$. Then $F_\varepsilon = \nabla f_\varepsilon$ with

$$f_\varepsilon(x) = \frac{1}{2}\langle Ax, x \rangle + \varepsilon b(x).$$

Our claim – suggested by [13] – is that for any fixed $\varepsilon \in J$ sufficiently small the solutions $v \in V$ of (2.16) are precisely the critical points of the C^1 functional α_ε defined by

$$\alpha_\varepsilon(v) := f_\varepsilon(v + \phi_\varepsilon(v)) = \frac{1}{2}\langle A\phi_\varepsilon(v), \phi_\varepsilon(v) \rangle + \varepsilon b(v + \phi_\varepsilon(v)), \quad v \in V, \tag{2.19}$$

over

$$M_\varepsilon = \{v \in V : \|v + \phi_\varepsilon(v)\|^2 = 1\}, \tag{2.20}$$

which, for ε small, is a C^1 manifold. To see this, for any $\varepsilon \in J$ put

$$\beta_\varepsilon(v) = \|v + \phi_\varepsilon(v)\|^2, \quad v \in \overline{V},$$

and observe that the gradient of the functional $v \mapsto \beta_0(v) = \|v\|^2$ does not vanish on $\overline{V}$, since this set does not contain the origin. Therefore, because of the compactness of $\overline{V}$, the gradient of β_ε remains different from zero on V for ε sufficiently small. This shows that for these values of ε the set M_ε is actually a C^1 manifold. We shall assume that J is such that any $\varepsilon \in J$ has this property.

At a critical point v_0 of α_ε on M_ε, we have

$$\alpha'_\varepsilon(v_0)h = \frac{\mu}{2}\beta'_\varepsilon(v_0)h \tag{2.21}$$

for all $h \in N$ and for some Lagrange multiplier μ. Now compute the first and second member of (2.21) by the chain rule; we have

$$\begin{aligned}
\alpha'_\varepsilon(v_0)h &= f'_\varepsilon(v_0 + \phi_\varepsilon(v_0))[h + \phi'_\varepsilon(v_0)h] \\
&= \langle F_\varepsilon(v_0 + \phi_\varepsilon(v_0)), h + \phi'_\varepsilon(v_0)h \rangle \\
&= \langle PF_\varepsilon(v_0 + \phi_\varepsilon(v_0)) + QF_\varepsilon(v_0 + \phi_\varepsilon(v_0)), h + \phi'_\varepsilon(v_0)h \rangle \\
&= \langle PF_\varepsilon(v_0 + \phi_\varepsilon(v_0)), h \rangle + \langle QF_\varepsilon(v_0 + \phi_\varepsilon(v_0)), \phi'_\varepsilon(v_0)h \rangle.
\end{aligned}$$

Using (2.18) it follows that

$$\alpha'_\varepsilon(v_0)h = \langle PF_\varepsilon(v_0 + \phi_\varepsilon(v_0)), h \rangle + \delta_\varepsilon(v_0)\langle \phi_\varepsilon(v_0), \phi'_\varepsilon(v_0)h \rangle.$$

On the other hand,

$$\begin{aligned}
\beta'_\varepsilon(v_0)h &= 2\langle v_0 + \phi_\varepsilon(v_0), h + \phi'_\varepsilon(v_0)h \rangle \\
&= 2[\langle v_0, h \rangle + \langle \phi_\varepsilon(v_0), \phi'_\varepsilon(v_0)h \rangle].
\end{aligned}$$

Replacing in (2.21) we then have

$$\begin{aligned}
\langle PF_\varepsilon(v_0 + \phi_\varepsilon(v_0)), h \rangle &+ \delta_\varepsilon(v_0)\langle \phi_\varepsilon(v_0), \phi'_\varepsilon(v_0)h \rangle \\
&= \mu\big(\langle v_0, h \rangle + \langle \phi_\varepsilon(v_0), \phi'_\varepsilon(v_0)h \rangle\big), \quad h \in N. \tag{2.22}
\end{aligned}$$

Putting $h = v_0$ and using (2.17), it follows in particular that

$$\delta_\varepsilon(v_0)\|v_0\|^2 + \delta_\varepsilon(v_0)\langle \phi_\varepsilon(v_0), \phi'_\varepsilon(v_0)v_0 \rangle = \mu\big(\|v_0\|^2 + \langle \phi_\varepsilon(v_0), \phi'_\varepsilon(v_0)v_0 \rangle\big).$$

This shows that $\mu = \delta_\varepsilon(v_0)$, and replacing this in (2.22) yields

$$\langle PF_\varepsilon(v_0 + \phi_\varepsilon(v_0)), h \rangle = \delta_\varepsilon(v_0)\langle v_0, h \rangle$$

for all $h \in N$, showing that v_0 satisfies (2.16), as claimed.

Now observe that for $v \in V$, $\phi_\varepsilon(v) = \phi(\varepsilon, v) \to \phi(0, v) = 0$ as $\varepsilon \to 0$. The definitions (2.9) and (2.20) then show that $M_\varepsilon \subset M$ for ε sufficiently small. It follows in particular that M_ε is compact; and thus for each ε there exist $v_\varepsilon, z_\varepsilon \in M_\varepsilon$ such that

$$\alpha_\varepsilon(v_\varepsilon) = \min_{v \in M_\varepsilon} \alpha_\varepsilon(v), \qquad \alpha_\varepsilon(z_\varepsilon) = \max_{v \in M_\varepsilon} \alpha_\varepsilon(v). \tag{2.23}$$

Therefore, for each ε, v_ε and z_ε solve (2.16).

It is now quite easy to conclude that S_0 has at least one bifurcation point: we can use for this an argument already employed in [3]. Take a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \neq 0$ for each n and $\varepsilon_n \to 0$ as $n \to \infty$. Let $v_n := v_{\varepsilon_n}$, $\delta_n := \delta_{\varepsilon_n}(v_n) = \delta(\varepsilon_n, v_n)$ and $\phi_n := \phi_{\varepsilon_n}(v_n) = w(\delta_n, \varepsilon_n, v_n)$; then putting $x_n := v_n + \phi_n$, we have a sequence $\{(\delta_n, \varepsilon_n, x_n)\}$ with $x_n \in S$ for each $n \in n$ and

$$Ax_n + \varepsilon_n B(x_n) = \delta_n x_n \qquad \forall n \in n. \tag{2.24}$$

By the compactness of M, we can assume – passing if necessary to a subsequence – that $v_n \to v_0 \in M \subset V$. Consequently,

$$\delta_n \to \delta(0, v_0) = 0, \qquad \phi_n \to w(0, 0, v_0) = 0.$$

It follows that $x_n \to v_0$, which in turn implies that $v_0 \in S$ and – by (2.24) – that $Av_0 = 0$, showing that $v_0 \in S_0$ and is a bifurcation point of (1.1). This completes the proof of Theorem 1.1.

We continue our discussion of (1.1) assuming, in addition to the previous hypotheses, that

(BB) B is **bounded** in a neighbourhood of S.

Then we claim that as $\varepsilon \to 0$, the family (v_ε) of minimizers of α_ε on M_ε – (see (2.23)) – converges (through a sequence) to a minimizer v_0 of b on S_0, and similarly for (z_ε). Before stating this formally, we need to list some facts about the functional

$$h_\varepsilon := \frac{\alpha_\varepsilon}{\varepsilon}, \qquad \varepsilon \neq 0.$$

(i) Since v_ε minimizes α_ε on M_ε, then it also minimizes h_ε for $\varepsilon > 0$: that is

$$h_\varepsilon(v_\varepsilon) \leq h_\varepsilon(v), \qquad \forall \varepsilon > 0, \ \forall v \in M_\varepsilon.$$

(ii) The assumption (BB) implies that $\phi_\varepsilon(v) = O(|\varepsilon|)$ – that is, $\|\phi_\varepsilon(v)\| \leq K|\varepsilon|$ for some $K > 0$ – as $\varepsilon \to 0$, uniformly w.r.t. v in a bounded neighbourhood of S_0. To see this, recall the definition of $\phi_\varepsilon(v) = \phi(\varepsilon, v)$ via (2.11) and recall that $w(\delta, \varepsilon, v)$ solves (2.4), that is $Aw - \delta w = -\varepsilon QB(v + w)$ or, putting $H_\delta = -((A - \delta I)|_W)^{-1}$,

$$w = \varepsilon H_\delta QB(v + w).$$

As H_δ and Q are bounded linear operators and B is bounded in a neighbourhood of S, it follows that $w(\delta, \varepsilon, v) = O(|\varepsilon|)$ uniformly with respect to (δ, v) in a bounded neighbourhood of $\{0\} \times S_0 \subset \mathbb{R} \times N$. This implies in particular the desired bound for $\|\phi_\varepsilon(v)\|$.

(iii) For each $\varepsilon > 0$ small and each $y \in S_0$, there exists a number $t = t(\varepsilon, y)$, with $0 < t \leq 1$, such that $t(\varepsilon, y)y \in M_\varepsilon$; moreover, $t(\varepsilon, y) \to 1$ as $\varepsilon \to 0$ for each $y \in S_0$. Indeed pick an $y \in S_0$ and consider the point ty running on the ray from 0 through y; by (2.20) we see that it meets M_ε if and only if

$$t^2 + \|\phi_\varepsilon(ty)\|^2 = 1.$$

The remark made above about ϕ_ε implies that this equation has (for each ε small) a solution in the interval $(0, 1]$.

As a consequence of (i) and (iii), we obtain in particular the inequality

$$h_\varepsilon(v_\varepsilon) \leq h_\varepsilon(t(\varepsilon, y)y), \qquad \varepsilon > 0, \ y \in S_0. \qquad (2.25)$$

(iv) Our next claim is that as $\varepsilon \to 0$, $h_\varepsilon \to b$ uniformly in a neighbourhood of S_0. To see this, consider the expression (2.19) of α_ε, which gives

$$h_\varepsilon(v) = \frac{1}{2\varepsilon}\langle A\phi_\varepsilon(v), \phi_\varepsilon(v)\rangle + b(v + \phi_\varepsilon(v)). \qquad (2.26)$$

We know from (ii) that $\phi_\varepsilon(v) = O(|\varepsilon|)$ as $\varepsilon \to 0$, uniformly w.r.t. v in a bounded neighbourhood of S_0. Since A is a bounded linear operator, we have $|\langle Ax, x\rangle| \le C\|x\|^2$ for some $C > 0$ and all $x \in H$, and therefore $\langle A\phi_\varepsilon(v), \phi_\varepsilon(v)\rangle = O(\varepsilon^2)$ as $\varepsilon \to 0$, uniformly w.r.t. v in a bounded neighbourhood of S_0. Looking at (2.26), this proves our claim.

We can now establish formally the desired property of (v_ε).

PROPOSITION 2.1. *Let $v_\varepsilon \in M_\varepsilon$ be such that $\alpha_\varepsilon(v_\varepsilon) = \min_{v \in M_\varepsilon} \alpha_\varepsilon(v)$. Then, there exists a sequence $\{v_n\} := \{v_{\varepsilon_n}\}$ such that $v_n \to v_0$ as $n \to \infty$, where $v_0 \in S_0$ and satisfies*

$$b(v_0) = \min_{y \in S_0} b(y).$$

Proof. Take any sequence $\{\varepsilon_n\}$ with $\varepsilon_n > 0$ for each n and $\varepsilon_n \to 0$ as $n \to \infty$. Let $v_n := v_{\varepsilon_n}$ and $h_n := h_{\varepsilon_n}$; then we have from (2.25)

$$h_n(v_n) \le h_n(t(\varepsilon_n, y)y), \qquad \forall n \in n, \ \forall y \in S_0. \tag{2.27}$$

As before we can assume – passing if necessary to a subsequence – that $v_n \to v_0 \in S_0$. Moreover if D is a neighbourhood of S_0,

$$|h_n(v_n) - b(v_0)| \le |h_n(v_n) - b(v_n)| + |b(v_n) - b(v_0)| \le \sup_{v \in D} |h_n(v) - b(v)|$$
$$+ |b(v_n) - b(v_0)|$$

and since – by (iv) above – $h_n \to b$ uniformly in a neighbourhood of S_0, it follows that $h_n(v_n) \to b(v_0)$ as $n \to \infty$.

The same argument shows that the right-hand side of (2.27) converges to $b(y)$ as $n \to \infty$, for each fixed $y \in S_0$, because $t(\varepsilon_n, y) \to 1$ for each $y \in S_0$ as proven in (iii). Therefore, letting $n \to \infty$ in (2.27), we obtain in the limit

$$b(v_0) \le b(y), \qquad \forall y \in S_0,$$

which is the assertion. $\square$

Proposition 2.1 leads (as announced in Section 1) to the following improvement of Theorem 1.1.

THEOREM 2.1. *Let T, B and λ_0 be as in Theorem 1.1, and assume, in addition, that B satisfies the assumption (BB). Then, (1.1) admits two bifurcation points from Z_0, v_0 and v_1, such that*

$$b(v_0) = \min_{y \in S_0} b(y), \qquad b(v_1) = \max_{y \in S_0} b(y).$$

Consequently, if b is not constant on S_0, there exist at least two distinct bifurcation points from Z_0 for (1.1).

Now a natural question is: can we drop the assumption 'b is not constant on S_0' in the statement of Theorem 2.1? Unable until to give an answer, we present a simple example in which b is constant on S_0, and still there are infinitely many bifurcation points.

EXAMPLE 2.2. Consider (2.1) with $A, B : \mathbb{R}^3 \to \mathbb{R}^3$ defined as follows. Write $x = (x_1, x_2, z)$ for $x \in \mathbb{R}^3$ and put

$$A(x_1, x_2, z) = (0, 0, z).$$

In the notations used before, we thus have $v = (x_1, x_2, 0)$ – identified with $(x_1, x_2) \in \mathbb{R}^2$ – and $w = (0, 0, z)$ – identified with $z \in \mathbb{R}$. As for B, put

$$B(x_1, x_2, z) = \left(2x_1 z, 2x_2 z, x_1^2 + x_2^2\right)$$

and observe that $B = \nabla b$ with $b(x_1, x_2, z) = (x_1^2 + x_2^2)z$. Equations (2.3) and (2.4) are here respectively

$$2\varepsilon z(x_1, x_2) = \delta(x_1, x_2) \tag{2.28}$$

and

$$z + \varepsilon \left(x_1^2 + x_2^2\right) = \delta z.$$

Solve the latter uniquely with respect to z:

$$z = \frac{\varepsilon}{\delta - 1} \left(x_1^2 + x_2^2\right)$$

so that – putting $x_1^2 + x_2^2 = R^2$ – we have

$$z = w(\delta, \varepsilon, v) = w(\delta, \varepsilon, \|v\|) = \frac{\varepsilon}{\delta - 1} R^2 := w(\delta, \varepsilon, R). \tag{2.29}$$

Using this and (2.28) yield the equation in δ :

$$\delta = 2\varepsilon z = \frac{2\varepsilon^2}{\delta - 1} R^2$$

that can be written

$$\delta^2 - \delta - 2\varepsilon^2 R^2 = 0$$

and can thus be explicitly solved to give

$$\delta = \delta(\varepsilon, R) = \frac{1}{2}(1 - \sqrt{1 + 8\varepsilon^2 R^2}). \tag{2.30}$$

Replacing this in (2.29) permits us to compute the function ϕ appearing in the formulae (2.16)–(2.18). Here we have $\phi(\varepsilon, v) = \phi(\varepsilon, \|v\|) = w(\delta(\varepsilon, R), \varepsilon, R) := \phi(\varepsilon, R)$ with

$$\phi(\varepsilon, R) = \frac{\varepsilon R^2}{\frac{1}{2}(1 - \sqrt{1 + 8\varepsilon^2 R^2}) - 1} = -\frac{2\varepsilon R^2}{1 + \sqrt{1 + 8\varepsilon^2 R^2}}. \tag{2.31}$$

In turn, this allows us to determine M_ε by (2.20), which becomes here

$$R^2 + \frac{4\varepsilon^2 R^4}{(1 + \sqrt{1 + 8\varepsilon^2 R^2})^2} = 1.$$

This equation has for each $\varepsilon \neq 0$ one solution $R(\varepsilon)$ with $0 < R(\varepsilon) < 1$, and moreover $1 - R(\varepsilon)^2 = O(\varepsilon^2)$ as $\varepsilon \to 0$. We see in particular that M_ε is a circle in the plane (x_1, x_2) approaching S_0 (the circle of radius 1) as $\varepsilon \to 0$. Replacing this value $R(\varepsilon)$ in the above expressions (2.30) and (2.31) produces the functions

$$\delta(\varepsilon) := \delta(\varepsilon, R(\varepsilon)), \qquad \phi(\varepsilon) := \phi(\varepsilon, R(\varepsilon)),$$

which finally allow us to express the solutions (δ, ε, x) in terms of the sole parameter ε. Indeed, these are (for any ε)

$$(\delta(\varepsilon), \varepsilon, x(\varepsilon)) \quad \text{with} \quad x(\varepsilon) = (v, z), \qquad \|v\| = R(\varepsilon), \quad z = \phi(\varepsilon).$$

It follows in particular that each $v_0 \in S_0$ is a bifurcation point: just take

$$x(\varepsilon) = (R(\varepsilon)v_0, \phi(\varepsilon))$$

to have a family of nontrivial solutions that converges to $(v_0, 0)$ as $\varepsilon \to 0$.

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