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FREE ACTIONS BY ELEMENTARY ABELIAN 2-GROUPS ON STIEFEL MANIFOLDS

BY

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ABSTRACT. Let $V_{n,k}$ denote the Stiefel manifold of k-frames in $\mathbb{R}^n$. There is a free action on $V_{n,k}$ by the group $\mathbb{Z}_2^k$. We show that if $\mathbb{Z}_2^l$ acts freely on $V_{2^s m-1+k,k}$ and

$$2^s > \max\left\{\frac{3}{m}(k-1), k-1\right\}$$

then $l \leq k$.

§0. **Introduction.** Let $V_{n,k}$ denote the Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n$. Elements of $V_{n,k}$ may be written as k-tuples of orthonormal vectors $(v_1, \ldots, v_k)$ in $\mathbb{R}^n$. The elementary abelian 2-group of rank k, $\mathbb{Z}_2^k$, acts on $V_{n,k}$ as follows: if we write the generators of $\mathbb{Z}_2^k$ as $\tau_1, \ldots, \tau_k$, define $\tau_i(v_1, \ldots, v_k) = (v_1, \ldots, -v_i, \ldots, v_k)$. Clearly this is a free action, and one is tempted to

CONJECTURE. *If $\mathbb{Z}_2^l$ acts freely on $V_{n,k}$ then $l \leq k$.*

The main result of this paper is to prove the above conjecture for an infinite number of $V_{n,k}$'s for each k.

MAIN THEOREM. *Suppose that X is a finite CW-complex whose mod 2 cohomology is isomorphic (as algebras over the Steenrod algebra) to $H^*(V_{2^s m-1+k,k}; \mathbb{Z}_2)$ where*

$$2^s > \max\left\{\frac{3}{m}(k-1), k-1\right\},$$

and suppose X admits a free cellular action by the group $\mathbb{Z}_2^l$ then $l \leq k$.

The method of proof is to exploit a theorem due to G. Carlsson [1] that restricts varieties of Steenrod algebra invariant ideals of $H^*(\mathbb{Z}_2^l; \mathbb{Z}_2)$. The ideal we construct is generated by the transgressions of a set of multiplicative generators for $H^*(X; \mathbb{Z}_2)$ in the Serre spectral sequence associated to the fibration

$$X \to EG \underset{G}{\times} X \to BG.$$

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The generators will transgress if we require them to be cyclic over the Steenrod algebra $\mathscr{A}(2)$. Thus, the first section of this paper is devoted to classifying all $V_{n,k}$ for which the multiplicative generators form a cyclic module over $\mathscr{A}(2)$. This is where we need the restriction $2^s > k-1$.

In section 2 we attempt to show that the above ideal is $\mathscr{A}(2)$-invariant. We are able to do this only under the further restriction that

$$2^s > \frac{3}{m}(k-1).$$

Of course, if $m \geq 3$ then this is already implied by the first restriction.

As a final remark, we would like to point out that one can prove the above conjecture for any $V_{n,2}$ using some simple counting arguments in the spectral sequence.

§1. **Stiefel manifolds.** Let $V_{n,k}$ be the Stiefel manifold of orthonormal k-frames in n-space. Denote by P_l^N the $\mathscr{A}(2)$-module $H^*(\mathbb{R}P^N/\mathbb{R}P^{l-1}; \mathbb{Z}_2)$. So P_l^N has a $\mathbb{Z}_2$-basis $1, z_l, \ldots, z_n$ where the dimension of z_j is j and

$$Sq^j z_i = \begin{cases} \binom{i}{j} z_{i+j} & \text{if} \quad i+j \leq N \\ 0 & \text{otherwise} \end{cases}.$$

For the definition of a free $\mathscr{A}(2)$-algebra we refer the reader to [2].

PROPOSITION 1. [2] *$H^*(V_{n,k}; \mathbb{Z}_2)$ is isomorphic to the free $\mathscr{A}(2)$-algebra generated by P_{n-k}^{n-1}.*

DEFINITION. *An* $\mathscr{A}(2)$-module is called $\mathscr{A}(2)$-cyclic if it is generated over $\mathscr{A}(2)$ by a single element.

The next proposition classifies all P_l^N that are $\mathscr{A}(2)$-cyclic.

PROPOSITION 2.

(a) $P_{2^sm-1}^{2^sm-2+k}$ *is $\mathscr{A}(2)$-cyclic for any natural numbers m, s and k such that $k \leq 2^s$.*

(b) *Suppose that P_l^N is $\mathscr{A}(2)$-cyclic. If we write $l = 2^sm-1$, where m is odd, and $N = 2^sm-2+k$ then $k \leq 2^s$.*

Before proving proposition 2 we first need the following lemma on mod 2 binomial coefficients:

LEMMA. (a) *If $j \leq 2^s - 1$ then* $\binom{2^sm-1}{j} \equiv 1 \bmod 2$ *for any natural number m.*

(b) *If $0 < j \leq 2^s$ and m is odd then*

$$\binom{2^s(m+1)-1-j}{j} \equiv 0 \bmod 2$$

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Proof. (a) We may assume m is odd otherwise we can factor out the largest power of 2 dividing m which will only have the effect of increasing s. We write m in its dyadic expansion

$$m=\sum_{i=0}^{t} m_i 2^i,$$

where $m_0=1$ and each other $m_i=0$ or 1. The dyadic expansion of $2^s m-1$ is then

$$\begin{aligned} 2^s m-1 &= 2^s+\left(\sum_{i=1}^{t} m_i 2^{i+2}\right)-1 \\ &= 1+2+\cdots+2^{s-1}+\sum_{i=1}^{t} m_i 2^{i+s}. \end{aligned}$$

Since $j\le 2^s-1$ the dyadic expansion of j may be written as

$$j=\sum_{i=1}^{s-1} r_i 2^i.$$

Using the standard formula for computing binomial coefficients mod 2 from the dyadic expansions of its components [2] we have

$$\begin{aligned} \binom{2^s m-1}{j} &\equiv \prod_{i=1}^{s-1}\binom{1}{r_i}\cdot\prod_{i=1}^{t}\binom{m_i}{0} \\ &\equiv 1 \bmod 2. \end{aligned}$$

(b) If we assume $0<j\le 2^s$ then we may write the dyadic expansion of j as

$$j=\sum_{i=1}^{s} r_i 2^i.$$

From the proof of (a) we may write

$$\begin{aligned} 2^s(m+1)-1 &= 2^s m-1+2^s \\ &= 1+2+\cdots+2^{s-1}+2^s+\sum_{i=1}^{t} m_i 2^{i+s} \end{aligned}$$

Hence

$$\binom{2^s(m+1)-1-j}{j}\equiv\prod_{i=1}^{s}\binom{1-r_i}{r_i}\cdot\prod_{i=1}^{t}\binom{m_i}{0}\equiv\prod_{i=1}^{s}\binom{1-r_i}{r_i} \bmod 2.$$

We are assuming $j>0$ so at least one $r_i=1$, in which case

$$\binom{1-r_i}{r_i}=\binom{0}{1}=0.$$

This completes the proof of the lemma. ■

Proof of Proposition 2. (a) For $0 \leq j \leq k-1 \leq 2^s-1$ we have, by the lemma,

$$Sq^j z_{2^s m-1} = \binom{2^s m-1}{j} z_{2^s m-1+j}$$

$$= z_{2^s m-1+j}.$$

Thus every non-zero element is connected to $z_{2^s m-1}$ by a Steenrod square.

(b) Write $l = 2^s m - 1$ where m is odd, and $k = N - l + 1$. If $k > 2^s$ then $z_{2^s(m+1)-1}$ is non-zero in $P_{2^s m-1}^{2^s m-2+k}$.

We claim that $z_{2^s(m+1)-1}$ cannot be in the image of a non-zero class by any non-trivial Steenrod square. To see this suppose $0 < j \leq 2^s$ and compute, using the lemma

$$Sq^j z_{2^s(m+1)-1-j} = \binom{2^s(m+1)-1-j}{j} z_{2^s(m+1)-1}$$

$$= 0.$$

This completes the proof of the proposition. ∎

§2. **The main theorem.** The purpose of this section is to prove the main theorem. We assume that X is a finite CW-complex whose mod 2 cohomology is isomorphic (as algebras over the Steenrod algebra) to $H^*(V_{2^s m-1+k,k}; \mathbb{Z}_2)$, where $k \leq 2^s$. From the previous section we know that $H^*(X; \mathbb{Z}_2)$ is a free $\mathscr{A}(2)$-algebra on an $\mathscr{A}(2)$-cyclic module with $\mathbb{Z}_2$-basis $\{z_{2^s m-1}, \ldots, z_{2^s m-2+k}\}$.

Proposition 1. *If $2^s m - 1 \leq i \leq 2^s m - 2 + k$ then $H^i(X; \mathbb{Z}_2) \cong \mathbb{Z}_2$ with non-zero element z_i.*

Proof. The first possible non-zero product is $(z_{2^s m-1})^2$ which is in dimension $2^{s+1}m-2$. We are assuming $k \leq 2^s$, so $2^s m - 2 + k \leq 2^s m - 2 + 2^s = 2^s(m+1) - 2$. Now $2^s(m+1) - 2 \leq 2^{s+1}m - 2$ and is equal only when $m = 1$. But in this case

$$(z_{2^s-1})^2 = Sq^{2^s-1} z_{2^s-1}$$

$$= \binom{2^s-1}{2^s-1} z_{2^{s+1}-2}$$

$$= z_{2^{s+1}-2},$$

so the proposition holds. ∎

Assume that G is a finite group acting cellularly on X. Denote by X_G the space $EG \times_G X$. We let $E_r^{*,*}$ be the r-th term of the Serre spectral sequence associated to the fibration

$$X \to X_G \to BG.$$

It has E_2 term

$$E_2^{**} \cong H^*(BG; H^*(X; \mathbb{Z}_2)).$$

PROPOSITION 2. *If G acts freely on X then E^{**} is a finite dimensional $\mathbb{Z}_2$-vector space.*

Proof. Under the hypothesis we have a homotopy equivalence $EG\times_G X\simeq X/G$. Since X is a finite CW-complex, and G acts cellularly on X, X/G is a finite CW-complex. E^{**} is the associated graded groups to some filtration on $H^*(X;\mathbb{Z}_2)$ which is finite dimensionel. ■

PROPOSITION 3. *The induced G-action on $H^*(X;\mathbb{Z}_2)$ is trivial.*

Proof. It is enough to show that G acts trivially on the generators $\{z_i\}_i$. This follows immediately from propositon 1. ■

We now assume $G\cong\mathbb{Z}_2^l$. The mod 2 cohomology ring of $\mathbb{Z}_2^l$ is well known: $H^*(B\mathbb{Z}_2^l;\mathbb{Z}_2)\cong\mathbb{Z}_2[\gamma_1,\ldots,\gamma_l]$, where each γ_i is in dimension 1. An element $f\in H^n(BG;\mathbb{Z}_2)$ may be regarded, via the above ring isomorphism, as a homogeneous polynomial of degree n in l variables.

PROPOSITION 4. [1]. *For $G\cong\mathbb{Z}_2^l$ the polynomial $f\in H^n(BG;\mathbb{Z}_2)$ has a non-trivial zero point if, and only if there is a subgroup inclusion $i:\mathbb{Z}_2\hookrightarrow\mathbb{Z}_2^l$ such that $i^*f=0$.*

The first possible non-trivial differential in the above spectral sequence is d_{2^sm}. Let $f_{2^sm}=d_{2^sm}(z_{2^sm-1})$, an element of $H^{2^sm}(BG:\mathbb{Z}_2)$.

PROPOSITION 5. *If $\mathbb{Z}_2^l$ acts freely on X then f_{2^sm} cannot have a non-trivial zero point in $\mathbb{Z}_2^l$.*

Proof. Suppose f_{2^sm} had a non-trivial zero point in $\mathbb{Z}_2^l$. By proposition 4 there would exist a subgroup inclusion $i:\mathbb{Z}_2\hookrightarrow\mathbb{Z}_2^l$ such that $i^*f_{2^sm}=0$. The Serre spectral sequence associated to the fibration $X\to X_G\to BG$ is natural with respect to subgroup inclusions. It would follow that the sequence for $X\to X_{\mathbb{Z}_2}\to B\mathbb{Z}_2$ will collapse, $E_2=E_\infty$. This violates proposition 2. ■

We recall that the transgression operator, τ, in the Serre spectral sequence commutes with the action of the Steenrod algebra. Consequently, under our assumptions, each generator $z_{2^sm-1},\ldots,z_{2^sm-2+k}$ of $H^*(X;\mathbb{Z}_2)$ is transgressive and $\tau(z_i)$ is represented by $Sq^{i-2^sm+1}f_{2^sm}$. Let $f_{2^sm+j}=Sq^jf_{2^sm}$. We consider the ideal I, in $H^*(BG;\mathbb{Z}_2)$, generated by $f_{2^sm},\ldots,f_{2^sm+k-1}$.

LEMMA 1. *$E_\infty^{*,0}=H^*(BG;\mathbb{Z}_2)/I$ for $*<2^{s+1}m$.*

Proof. According to proposition 1 every element of $H^i(X;\mathbb{Z}_2)$ is transgressive for $2^sm-1\le i\le 2^sm-2+k$. Furthermore $E_2^{0,i}=0$ for $2^sm-2+k<i<2^{s+1}m-1$. The lemma now follows from dimension considerations. ■

LEMMA 2. *$Sq^if_{2^sm+j}\in I$ whenever $i+j<2^sm$.*

Proof. Since $E_\infty^{*,0}$ is the image of $H^*(BG;\mathbb{Z}_2)\to H^*(X_G;\mathbb{Z}_2)$ it is a module over the Steenrod algebra. $Sq^i f_{2^s m+j}$ represents zero in this module. The lemma now follows from lemma 1. ■

PROPOSITION 6. *If* $2^s > \max\left\{\frac{3}{m}(k-1), k-1\right\}$ *then* I *is* $\mathcal{A}(2)$*-invariant.*

Proof. Let $A(i, j)$ be the statement "$Sq^i Sq^j f_{2^s m} \in I$." We wish to show that $A(i, j)$ is true whenever $0 \le j \le k-1$ and $0 \le i \le 2^s m + j$. We proceed by induction on j. $A(i, 0)$ is true for $0 \le i < 2^s m$ by lemma 2 and $A(2^s m, 0)$ is true since $Sq^{2^s m} f_{2^s m} = (f_{2^s m})^2$, and $(f_{2^s m})^2$ is clearly in I.

Now for the inductive step, suppose that we have verified $A(i, j')$ whenever $0 \le j' \le j-1$ and $0 \le i \le 2^s m + j'$. If $i < 2^s m - j$ then $A(i, j)$ follows from lemma 2. Hence we may assume $i \ge 2^s m - j$. First observe that

$$\begin{aligned} i &\ge 2^s m - j \\ &> 3(k-1) - j \\ &\ge 3j - j \\ &= 2j. \end{aligned}$$

So by the Adem relations

$$Sq^i Sq^j f_{2^s m} = Sq^{2j} Sq^{i-j} f_{2^s m} + \sum_{a<j} C_a Sq^{i+j-a} Sq^a f_{2^s m}, \quad \text{where} \quad C_a \in \mathbb{Z}_2.$$

Inductively $Sq^{i+j-a} Sq^a f_{2^s m} \in I$ for $a < j$. We are left with showing $Sq^{2j} Sq^{i-j} f_{2^s m} \in I$. Because of the inequalities $2j \le 2(k-1) < 2^s m - (k-1)$ it follows from lemma 2 that $Sq^{2j} I \subset I$. We have already verified $A(i-j, 0)$, so $Sq^{2j} Sq^{i-j} f_{2^s m} \in I$. This completes the inductive step. ■

PROPOSITION 7. [1]. *Let* $I = (f_1, \ldots, f_s)$ *be an ideal generated by homogeneous polynomials* f_i *in* $\mathbb{Z}_2[\gamma_1, \ldots, \gamma_l]$. *Assume further that* I *is invariant under the action of the Steenrod algebra. Then if* $l > s$ *the polynomials* f_i *have a non-trivial common zero in* $\mathbb{Z}_2^l$.

Proof of the Main Theorem. We have seen, in proposition 6, that the ideal $I = (f_{2^s m}, \ldots, f_{2^s m+k-1})$ is $\mathcal{A}(2)$-invariant. By naturality of the Steenrod operations and proposition 4 there is a non-trivial common zero to $f_{2^s m}, \ldots, f_{2^s m+k-1}$ if, and only if $f_{2^s m}$ has a non-trivial zero. It follows from proposition 5 that $f_{2^s m}, \ldots, f_{2^s m+k-1}$ can have no non-trivial zero point. The result now follows from proposition 7. ■

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