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SPLIT SUBDIRECT PRODUCTS AND PIECEWISE DOMAINS

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1. Preliminaries. Throughout this paper all rings will have unity and all modules will be unital.

If $X \subset R$, then $r(X)$ (respectively, $l(X)$) denotes the right (left) annihilator of x.

An element d of R is called *right (left) regular* if $r(d) = 0$ ($l(d) = 0$). An element which is both right and left regular is called *regular*. A ring Q is called the *right classical quotient ring* of R if (i) $R \subset Q$, (ii) every regular element of R is invertible in Q, and (iii) every element of Q is of the form rd^{-1} where r, $d \in R$ and d is regular in R. In this case R is called a *right order* in Q.

A ring R is called a *piecewise domain* (PWD) [**11**] if it possesses a complete set of orthogonal idempotents $\{e_e, \ldots, e_n\}$ such that $xy = 0$ implies that $x = 0$ or $y = 0$ whenever $x \in e_iRe_k$ and $y \in e_kRe_j$.

2. Split subdirect products. In this section we introduce a special subdirect product of rings which we call a (right) split subdirect product. We show that split sub-direct products preserve many (homological) properties of the factor rings. One unusual feature is that if R is a split subdirect product of rings R_1 and R_2, then every left R-module canonically decomposes as a subdirect product of an R_1-module with an R_2-module.

Now let R be the subdirect product of rings R_1 and R_2. The following ideas are motivated by Goodearl [**5**]. Define $E_1 \subset R_1$, $E_2 \subset R_2$ by the conditions $E_1 \times 0 = R \cap (R_1 \times 0)$ and $0 \times E_2 = R \cap (0 \times R_2)$. Then E_1 and E_2 are two-sided ideals of R_1 and R_2 respectively and we have the ring isomorphisms: $R/(0 \times E_2) \simeq R_1$, $R/(E_1 \times 0) \simeq R_2$ and $R/(E_1 \times E_2) \simeq R_1/E_1 \simeq R_2/E_2$. Let $\pi_i : R \rightarrow R_i$ be the restriction to R of the natural projection map for $i = 1, 2$.

We say that R is a (right) *split subdirect product* of R_1 and R_2 provided that E_1 and E_2 are direct summands as right ideals of R_1 and R_2 respectively. In this case $E_1 \times E_2$ is a direct summand of R as a right ideal.

The following proposition will be our main tool in showing that split subdirect products preserve homological properties.

PROPOSITION 2.1. *Let $I = eR$ be a two-sided ideal of a ring R where $e^2 = e$ and let $\bar{R} = R/I$. Then*

(i) $PD(M_R) = PD(M_{\bar{R}})$ *for all right $\bar{R}$-modules M.*

(ii) inj. dim $({}_RM)$ = inj. dim $({}_{\bar{R}}M)$ *for all left $\bar{R}$-modules M.*

(iii) $WD(M_R) = WD(M_{\bar{R}})$ *for all right $\bar{R}$-modules M.*

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Proof. (i) Let M be an arbitrary right $\bar{R}$-module. By [**1**, Problem 10, p. 123], $PD(M_R) \leqq PD(M_{\bar{R}})$. Since $\bar{R}_R$ is projective, $\mathrm{Ext}_R{}^n\,(M, N) \simeq \mathrm{Ext}_{\bar{R}}{}^n\,(M, N)$ for all $\bar{R}$-modules N by [**1**, Proposition 4.1.4]. Thus $PD(M_{\bar{R}}) \leqq PD(M_R)$.

The proofs of (ii) and (iii) follow by similar arguments using Problem 10 on page 123 along with Propositions 4.1.3 and 4.1.2 of [**1**].

We are now in a position to state our main results which relate the homological properties of R with those of R_1 and R_2 when R is a split subdirect product of R_1 and R_2.

THEOREM 2.2. *Let R be a split subdirect product of R_1 and R_2. Then* r.gl.dim $R = \sup\,\{\text{r.gl.dim } R_i\}$, l.gl.dim $R = \sup\,\{\text{l.gl.dim } R_i\}$ *and* WGD(R) $= \sup\,\{WGD(R_i)\}$. *Furthermore R is right semihereditary (right p.p.) if and only if each R_i is right semihereditary (right p.p.).*

Proof. We will show that r.gl.dim $R = \sup\,\{\text{r.gl.dim } R_i\}$. By Proposition 2.1 r.gl.dim $(R_i) \leqq$ r.gl.dim R for $i = 1, 2$. Thus $\sup\,\{\text{r.gl.dim } R_i\} \leqq$ r.gl.dim R. To show the reverse equality let M be an arbitrary right R-module and consider the exact sequence

$$0 \rightarrow MK \rightarrow M \rightarrow M/MK \rightarrow 0$$

where $K = E_1 \times 0$. For any right R-module X we have the exact sequence

$$\mathrm{Ext}_R{}^k\,(M/MK, X) \rightarrow \mathrm{Ext}_R{}^k\,(M, X) \rightarrow \mathrm{Ext}_R{}^k\,(MK, X).$$

Since MK is a R_1-module and M/MK is a R_2-module, by Proposition 2.1 $PD_{R_1}(MK) = PD_R(MK)$ while $PD_{R_2}(M/MK) = PD_R(M/MK)$. Thus if $k > \sup\,\{\text{r.gl.dim } R_i\}$, $\mathrm{Ext}_R{}^k\,(MK, X) = \mathrm{Ext}_R{}^k\,(M/MK, X) = 0$; hence $\mathrm{Ext}_R{}^k\,(M, X) = 0$. Thus r.gl.dim $R \leqq \sup\,\{\text{r.gl.dim } R_i\}$.

The proofs for the left global dimension and weak dimension are similar.

Now let R be right semihereditary. Then since $E_1 \times 0$ and $0 \times E_2$ are direct summands of R as right ideals, both R_1 and R_2 are right semihereditary. Assume that R_1 and R_2 are right semihereditary and let I be a finitely generated right ideal of R. Then we have the exact sequence

$$0 \rightarrow I \cap K \rightarrow I \rightarrow I/(I \cap K) \rightarrow 0$$

where $K = E_1 \times 0$. Now $I/(I \cap K) \simeq (I + K)/K$ is a finitely generated R_2-ideal; hence $(I + K)/K$ is a finitely generated projective R-module by Proposition 2.1. Thus the sequence splits and $I \cap K$ is a finitely generated R_1-ideal. Thus again by Proposition 2.1, $I \cap K$ is R-projective. Hence I is R-projective and R is right semihereditary.

The proof for right p.p. is identical to the proof for right semihereditary.

Let R be a subdirect product of rings R_1 and R_2 with E_1 and E_2 defined as above. R is called an *essential* (*right*) *subdirect product* if E_1 and E_2 are essential as right ideals of R_1 and R_2 respectively. Essential products were introduced by Goodearl [**5**]. Among many other results he showed that the essential product of nonsingular rings is again nonsingular and that is maximal right quotient rings was the direct product of those of the factor rings. The following easy lemma shows that a converse is true. This lemma will be used in the next section.

LEMMA 2.3. *Let R be a right nonsingular ring with maximal right quotient ring Q. Suppose that $Q = Q_1 \times Q_2$, and let R_i be the projection of R in Q_i for $i = 1, 2$. Then R is the essential subdirect product of R_1 and R_2.*

Proof. Write elements of Q as ordered pairs.

Clearly R is the subdirect product of R_1 and R_2 with $E_i = R \cap R_i$ for each i. It remains to be shown that E_i is essential in R_i. Let $0 \neq (x, 0) \in R_1$. Since R is right essential in Q, there exists $(r, s) \in R$ for which $(x, 0)(r, s) = (xr, 0)$ is a nonzero element of R. Then $(r, 0) \in R_1$ and $(xr, 0)$ is a nonzero element of $R \cap R_1 = E_1$. Hence E_1 is essential in R_1. Similarly E_2 is essential in R_2.

For right split subdirect products we have the following proposition which shows that any left R-module is a subdirect product of R_1 and R_2 modules. For the remainder of this section R will be as in the following proposition.

PROPOSITION 2.4. *Let R be a right split subdirect product of rings R_1 and R_2. Then any left R-module is isomorphic to a subdirect product of R_1 and R_2 modules.*

Proof. Let E_1 and E_2 be defined as above. If M is any left R-module, then M/I_1M and M/I_2M are R_1 and R_2 modules respectively where $I_1 = 0 \times E_2$ and $I_2 = E_1 \times 0$. Furthermore there is an R-homomorphism h of M into $M/I_1M \oplus M/I_2M$ defined by $h(m) = ([m + I_1M], [m + I_2M])$.

We first show that h is an R-monomorphism. Since I_1 and I_2 are direct summands of R as right ideals, $I_1 = e_1R$ and $I_2 = e_2R$ for orthogonal idempotents e_1 and e_2. Suppose $m \in \ker (h)$; that is, $m \in I_1M \cap I_2M$. Since $I_1 = 0 \times E_2$ and $I_2 = E_1 \times 0$, $e_1m = e_2m = m$. But $0 = (e_1e_2)m = e_1(e_2m) = e_1m = m$ and $m = 0$. Thus $\ker (h) = 0$ and h is a monomorphism. Hence M can be considered as a submodule of $M/I_1M \oplus M/I_2M$. Since the projection maps restricted to M are clearly onto, M is a subdirect product of the R_1-module M/I_1M with the R_2-module M/I_2M.

By the above proposition every R-module is a canonical subdirect product of R_1 and R_2 modules. Denote this representation of a left module by (M_1, M_2) and represent the elements of M as ordered pairs.

The following proposition shows that homomorphisms also decompose.

PROPOSITION 2.5. *Let $f : M \to N$ and $g : N \to P$ be R-homomorphisms for left R-modules M, N, and P. Let e_i and I_i be as in Proposition* 2.4.

(i) *f decomposes as (f_1, f_2) where $f_i : M_i \to N_i$ is an R_i-homomorphism for $i = 1, 2$. If $m \in M$ and $m = (m_1, m_2)$, then $f(m) = (f_1(m_1), f_2(m_2))$.*

(ii) *gf decomposes as (g_1f_1, g_2f_2).*

(iii) *If f_1 and f_2 are monomorphisms, then f is a monomorphism.*

(iv) *If f is an epimorphism, then f_1 and f_2 are epimorphisms.*

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Proof: (i) Suppose that $f : M \to N$ is an R-homomorphism. For each i $f(I_iM) \subset I_iN$; thus f induces a map $f_i : M/I_iM \to N/I_iN$ such that the following diagram commutes.

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ M/I_iM & \xrightarrow{f_i} & N/I_iN \end{array}$$

Since the diagram commutes and since the vertical maps are the coordinate maps of the subdirect product representations for M and N, it follows that $f(m)_i = f_i(m_i)$ for $i = 1, 2$. Hence $f(m) = (f_1(m_1), f_2(m_2))$.

(ii) Let $m = (m_1, m_2)$. Then $g(f(m)) = g[(f_1(m_1), f_2(m_2))] = (g_1f_1(m_1), g_2f_2(m_2))$. Hence $gf = (g_1f_1, g_2f_2)$.

(iii) Let $f = (f_1, f_2)$ be a monomorphism and suppose that $0 = f(m) = (f_1(m_1), f_2(m_2))$. Then $f_1(m_1) = f_2(m_2) = 0$ and hence $m_1 = m_2 = 0$ since f_1 and f_2 are monomorphisms. Therefore $m = 0$ and f is a monomorphism.

(iv) Let $f = (f_1, f_2)$ be an epimorphism. Since $f(M) = N$ and $N = (N_1, N_2)$, then $f_1(M_1) = N_1$ and $f_2(M_2) = N_2$. Therefore f_1 and f_2 are epimorphisms.

The following proposition shows that the injective modules over R are completely determined by those over R_1 and R_2.

PROPOSITION 2.6. *An R-module is R-injective if and only if it is the direct sum of an injective R_1-module and an injective R_2-module.*

Proof. Necessity: This direction follows from Lemma 2.1 (ii).

Sufficiency: Let $E = (E_1, E_2)$ be an injective left R-module, and let H_i be the R_i-injective hull of E_i. Then $H_1 \oplus H_2$ is an injective R-module by the above. Consider the exact sequence $0 \to E \xrightarrow{f} H_1 \oplus H_2$ under the natural embedding f. Since E is injective, there is a map $g : H_1 \oplus H_2 \to E$ such that $gf = id_E$. Then g_1f_1 is the identity function on E_1. Hence E_1 is injective since it's isomorphic to a direct summand of the injective module H_1. The map g_1 is the restriction of the map g to H_1, thus $E_1 = \text{Im}\,(g_1)$ is a submodule of E. Since E_1 is injective, we have $E = E_1 \oplus A$. $A \simeq E/E_1 \subset (E_1 \oplus E_2)/E_1$ and is hence isomorphic to an R-submodule of E_2. Every R-submodule of an R_2-module is an R_2-module; therefore A is an R_2-module. A is injective as an R-module and hence as an R_2-module by Lemma 2.1 (ii).

It should be noted that the decomposition in Proposition 2.6 is not in general unique, since there are injective R-modules which are simultaneously R_1 and R_2 modules.

3. Split products of piecewise domains. In this section the results of Section 2 are applied to yield results concerning the structure of a PWD and its quotient rings. It is first shown that if a PWD can be represented as a split subdirect product of rings, then most of its properties are determined and shared by the factor rings. It is shown that split products of PWD's occur fairly naturally; in particular, if a PWD is an essential subdirect product of rings, then the product is split (but not necessarily direct). Finally these results are applied to obtain that the maximal quotient ring of a PWD is a direct product of finitely many rings each of which is a prime self-injective regular ring whose ideals form a well ordered chain.

Let R be a PWD relative to the complete set of orthogonal idempotents $\{e_1, \ldots, e_n\}$. By the proof of the Main Theorem of [**10**], the relation $i \sim j$ if $e_iRe_j \neq 0$ and $e_jRe_i \neq 0$ is an equivalence relation on $\{1, \ldots, n\}$. Let $C_1, \ldots, C_r$ denote the equivalence classes and set $f_i = \sum_{j \in C_i} e_j$. Then the $f_1, \ldots, f_r$ can be ordered so that R has the triangular structure:

$$R \simeq \begin{bmatrix} P_1 & & & & & \\ P_{21} & P_2 & & & & \\ P_{31} & P_{32} & P_3 & & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot \; \cdot & \cdot & \\ P_{r1} & P_{r2} & P_{r3} & \cdot \; \cdot & \cdot & P_r \end{bmatrix}$$

where $P_{ij} = f_iRf_j$ and $P_i = f_iRf_i$. This notation will be used throughout this section.

For the next three technical lemmas, eR will be a two-sided ideal of R where $e^2 = e$.

LEMMA 3.1. *There exists* $\{e_{i(1)}, \ldots, e_{i(k)}\} \subset \{e_1, \ldots, e_n\}$ *such that* $eR = \oplus_{j=1}^k e_{i(j)}R$.

Proof. For each i consider e_ie. Since eR is a two-sided ideal of R, $e_ie \in eR$; thus $ee_ie = e_ie$. Hence $(e_ie)^2 = e_i(ee_ie) = e_i(e_ie) = e_ie$ and e_ie is an idempotent element of e_iR. Since e_iR is indecomposable, $e_ieR = 0$ or $e_ieR = e_iR$. Thus if $e_ie \neq 0$, then $e_iR \subseteq eR$. We have $e = 1 \cdot e = (e_1 + \ldots + e_n)e = e_1e + \ldots + e_ne$. Hence $e = \sum\{e_ie : e_ie \neq 0\} = \sum_{j=1}^k e_{i(j)}e$ and it follows that $eR = \oplus \{e_iR: e_ie \neq 0\} = \oplus_{j=1}^k e_{i(j)}R$.

Since $e' = e_{i(1)} + \ldots + e_{i(k)}$ is an idempotent generator of eR, there is no loss of generality in assuming that $e = e_{i(1)} + \ldots + e_{i(k)}$.

LEMMA 3.2. *If* $k, l \in C_j$, *then* $e_ke \neq 0$ *if and only if* $e_le \neq 0$. *Hence there exists* $\{f_{i(1)}, \ldots, f_{i(m)}\} \subset \{f_1, \ldots, f_r\}$ *such that* $e = \sum_{j=1}^m f_{i(j)}$ *and* $eR = \oplus_{j=1}^m f_{i(j)}R$.

Proof. Suppose that $k, l \in C_j$ and $e_ke \neq 0$. By the definition of C_j, there exists $t, s \in R$ such that $e_kte_l \neq 0$ and $e_lse_k \neq 0$. Let $r = e_kte_l + e_lse_k$. Then $e_kre_l = e_kte_l \neq 0$ and $e_lre_k = e_lse_k \neq 0$. By the above comments $e_ke = e_k$, therefore $e_lre_kee_k = e_lre_k \neq 0$. Now $re_kee_k \in eR$ as eR is a two-sided ideal; thus $re_kee_k = es$ for some $s \in R$. Therefore $e_lre_kee_k = e_les \neq 0$ so $e_le \neq 0$. Thus it follows that $e = \sum\{f_i: f_ie = f_i\} = \sum\{f_i: f_ie \neq 0\}$ and the last statement of the lemma is immediate.

LEMMA 3.3. *The f_j's can be reordered, still preserving the triangular structure of R, in such a manner that $eR = \sum_{j>k} f_jR = \sum_{i>l} e_iR$ for some k, l.*

Proof. Since $e = \sum\{f_i: f_ie \neq 0\} = \sum\{f_i: f_ie = f_i\}$ by the proof of Lemma 3.2, let i be the first index for which $f_ie \neq 0$ and $f_{i+1}e = 0$. If $r < i$, $f_rRf_i = f_rRf_{i+1} = 0$ by the triangular structure of R. Again by the triangular structure of R, $f_iRf_{i+1} = 0$. Now $f_{i+1}Rf_i = f_{i+1}Rf_ie \subset f_{i+1}(eR) = 0$. Thus f_i and f_{i+1} can be interchanged while still preserving the triangular structure of R. Continuing this process yields the desired ordering.

It is an open question whether a *PWD* is a *PWD* relative to every complete set of orthogonal idempotents. Hence the assumption in the following proposition that each R_i is a *PWD* with respect to every complete set of orthogonal idempotents.

PROPOSITION 3.4. *Let R be a split subdirect product of R_1 and R_2. If R is a PWD, then each R_i is a PWD. Conversely if each R_i is a PWD relative to every complete set of orthogonal idempotents of R_i, then R is a PWD relative to every complete set of orthogonal idempotents of R.*

Proof. Let R be a *PWD* relative to a complete set of orthogonal idempotents $\{e_1, \ldots, e_n\}$. Since $E_1 \times 0$ and $0 \times E_2$ are direct summands of R as right ideals, it follows directly that $R_1 \simeq R/(0 \times E_2)$ and $R_2 \simeq R/(E_1 \times 0)$ are *PWD*'s relative to the complete sets of orthogonal idempotents $\{\pi_1(e_1), \ldots, \pi_1(e_n)\}$ and $\{\pi_2(e_1), \ldots, \pi_2(e_n)\}$ respectively.

For the converse, first note that R has a complete set of orthogonal idempotents. Since $E_1 \times 0$ is a direct summand of R as a right ideal, $E_1 \times 0 = eR$ where $e^2 = e$. Now $R/(E_1 \times 0) \simeq R_2$ has a complete set of orthogonal idempotents so there exists a complete set of orthogonal idempotents of $(1 - e)R$. But E_1 has a complete set of orthogonal idempotents by Lemma 3.1, so $E_1 \times 0 = eR$ has a complete set of orthogonal idempotents. Hence R has a complete set of orthogonal idempotents.

Now let R have a complete set of orthogonal idempotents $\{e_1, \ldots, e_n\}$. Since $\{\pi_1(e_1), \ldots, \pi_1(e_n)\}$ is a complete set of orthogonal idempotents of R_1, R_1 is a *PWD* relative to this set of idempotents. E_1 is a direct summand of R_1 as a right ideal of R_1 so by Lemma 3.3 there is a rearrangement of $\{e_1, \ldots, e_n\}$ such that $E_1 = \sum_{j>k_1} \pi_1(e_j)R_1$ and $\pi_1(e_k)R_1\, \pi_1(e_l) = 0$ for $k \leqq k_1 < l$. Then $\{\pi_2(e_1), \ldots, \pi_2(e_{k_1})\}$ is a complete set of orthogonal idempotents for R_2 so there exists a rearrangement of $\{e_1, \ldots, e_{k_1}\}$ such that $E_2 = \sum_{k_1<j\leqq k_1} \pi_2(e_j)R_2$ and $\pi_2(e_k)R_2\pi_2(e_l) = 0$ for $k \leqq k_2 < l$ by Lemma 3.3

Thus there are integers $1 \leqq k_2 \leqq k_1 \leqq n$ such that $E_1 \times 0 = \sum_{j>k_1} e_jR$ and $0 \times E_2 = \sum_{k_2<j\leqq k_1} e_jR$. For $k \leqq k_2 < l$, $e_kRe_l = 0$ and if $k_2 < k \leqq k_1 < l \leqq n$, $e_kRe_l \subset E_1 \cap E_2 = 0$ and $e_lRe_k \subset E_1 \cap E_2 = 0$.

Now let $0 \neq x \in e_iRe_j$ and $0 \neq y \in e_jRe_k$. In view of the remarks in the preceding paragraph, we need only consider two cases:

i) If $i, j, k \leqq k_1$, then $xy \neq 0$ since $R/(E_1 \times 0) \simeq R_2$ is a PWD relative to $\{\pi_2(e_1), \ldots, \pi_2(e_{k_1})\}$.

(ii) If $i, j, k \in \{1, \ldots, k_2, k_1 + 1, \ldots, n\}$, then $xy \neq 0$ since $R/(0 \times E_2) \simeq R_1$ is a PWD relative to $\{\pi_1(e_1), \ldots, \pi_1(e_{k_2}), \pi_1(e_{k_1+1}), \ldots, \pi_1(e_n)\}$.

Remark. The class of rings which are PWD's relative to every complete set of orthogonal idempotents includes all p.p. rings with no infinite sets of orthogonal idempotents. Also included are those rings which have a complete set of orthogonal idempotents and which satisfy a Krull-Schmidt type theorem such as the semiperfect rings.

We now establish that split subdirect products arise fairly naturally. Let R be a PWD relative to the complete set of orthogonal idempotents $\{e_1, \ldots, e_n\}$ and suppose that R has a right flat overring $Q = Q_1 \times Q_2$ (ring direct product). Let $\pi_i : Q \rightarrow Q_i$ be the natural projection map and $\pi_i(R) = R_i$ for $i = 1, 2$. Since any PWD is a nonsingular ring [**9**], the maximal left quotient ring of R will serve as the overring Q in many cases.

We state the following well known lemma which appears in Cartan and Eilenberg [**1**, Problem 5, p. 122] for completeness.

LEMMA 3.5. *Let R be a ring and A be a right R-module. If I is a left ideal of R, then $A \otimes_R I \rightarrow A \otimes_R R$ is a monomorphism if and only if whenever $\sum_i a_i\mu_i = 0$ for $\{a_i\} \subset A$ and $\{\mu_i\} \subset I$, there exists $\{b_j\} \subset A$ and $\{\lambda_{ij}\} \subset R$ such that $\sum_j b_j\lambda_{ij} = a_i$ for all i and $\sum_i \lambda_{ij}\mu_i = 0$ for all j.*

Then we have the following technical lemma.

LEMMA 3.6. *Let R be a PWD as above. If $\pi_i(e_k) \neq 0$ and $0 \neq x \in e_kRe_l$, then $\pi_i(x) \neq 0$.*

Proof. Without loss of generality, let us assume that $\pi_1(e_k) \neq 0$ and $0 \neq x \in e_kRe_l$. If $\pi_1(x) = 0$, then $x = (0, r_2)$ where $r_2 \in R_2$. Let $q = (q_1, 0) \in Q$ be such that $qe_k = q$. Then $qx = 0$, so by the flatness of Q_R, there exists $\{b_j\} \subset Q$ and $\{\lambda_j\} \subset R$ such that $q = \sum_j b_j\lambda_j$ and $\lambda_jx = 0$ for all j. Since $qe_k = q$ and $x \in e_kRe_l$, we may assume that $\lambda_j \in Re_k$ for all j. Right multiplication by x induces a monomorphism [**10**] of Re_k into Re_l. Since $\lambda_jx = 0$, $\lambda_j = 0$ for all j, a contradiction.

COROLLARY 3.7. *Let R be a PWD as above. If $\pi_i(e_k) \neq 0$ and $\pi_i(e_l) = 0$, then $e_kRe_l = 0$.*

Proof. Assume that $0 \neq x \in e_kRe_l$. Since $\pi_i(e_l) = 0$ and x is a left multiple of e_l, $\pi_i(x) = 0$. But this contradicts Lemma 3.6.

We now show that R is a split subdirect product of R_1 and R_2.

PROPOSITION 3.8. *Let R be a PWD and suppose R has a right flat overring $Q = Q_1 \times Q_2$. Then R is a split subdirect product of PWD's R_1 and R_2.*

Proof. Let R be a PWD relative to the complete set of orthogonal idempotents $\{e_1, \ldots, e_n\}$ and let $\{f_1, \ldots, f_r\}$ be defined as in the beginning of this section. Then $f_i = \sum_{j \in C_i} e_j$ where $e_l R e_k \neq 0$ and $e_k R e_l \neq 0$ for all $k, l \in C_i$. Then by Corollary 3.7, $\pi_i(e_k) \neq 0$ if and only if $\pi_i(e_l) \neq 0$ for all $k, l \in C_i$. Let j be the first index for which $\pi_i(f_j) = 0$ and $\pi_i(f_{j+1}) \neq 0$. Then for $e_k \in C_{j+1}$ and $e_l \in C_j$, $\pi_i(e_k) \neq 0$ and $\pi_i(e_l) = 0$ so $e_k R e_l = 0$ by Corollary 3.7. Therefore $f_{j+1} R f_j = 0$ so that the ordering of the idempotents $\{f_1, \ldots, f_{j+1}, f_j, \ldots, f_r\}$ still preserves the triangular structure of R. Then we can find integers $1 \leqq k_2 \leqq k_1 \leqq n$ such that $f_i \in E_1$ if and only if $i > k_1$ and $f_i \in E_2$ if and only if $k_2 < i \leqq k_1$. By Lemma 3.6, it follows that $E_1 = \sum_{j > k_1} f_j R$ and $E_2 = \sum_{k_2 < j \leqq k_1} f_j R$. Thus R is a split subdirect product of R_1 and R_2. By Proposition 3.4 R_1 and R_2 are PWD's.

The following corollary follows directly from Propositions 3.4 and 3.8.

COROLLARY 3.9. *Let R have a right flat overring $Q = Q_1 \times Q_2$. Then R is a PWD relative to every complete set of orthogonal idempotents of R if and only if R is a split subdirect product of R_1 and R_2 which are PWD's relative to every complete set of orthogonal idempotents of R_1 and R_2 respectively.*

We now let R be a PWD and let R have a right flat overring $Q = Q_1 \times \ldots \times Q_t$. Let R_i be the image of R under the projection map $Q \to Q_i$. We then have the following theorem.

THEOREM 3.10. *Let R be a PWD possessing a right flat overring $Q = Q_1 \times \ldots \times Q_t$ where $t \geqq 2$. Then* r.gl.dim $R = \sup \{$r.gl.dim $R_i\}$, l.gl.dim $R = \sup \{$l.gl.dim $R_i\}$, *and* $WGD(R) = \sup \{WGD(R_i)\}$. *Furthermore R is right semihereditary (right p.p.) if and only if each R_i is right semihereditary (right p.p.).*

Proof. We will prove the theorem by induction on t. If $t = 2$, then R is a split subdirect product of R_1 and R_2. Hence the result is just that of Theorem 2.2. Now assume the result is true for $t = r$ and let $Q = Q_1 \times \ldots \times Q_{r+1}$. Set $A = Q_2 \times \ldots \times Q_{r+1}$ and let $\pi_1 : Q \to Q_1$, $\pi_2 : Q \to A$ be the natural projection maps. Then by Proposition 3.8, R is a split subdirect product of PWD's $\pi_1(R) = R_1$ and $\pi_2(R) = S$. Q_1 and A are right R-flat modules as they are direct summands of Q_R. Hence by Proposition 2.1 Q_1 and A are respectively right R_1 and S-flat. Hence by induction, the theorem is valid for S. Thus, in particular, r.gl.dim $S = \sup \{$r.gl.dim $R_i : i \geqq 2\}$. Again by induction, r.gl.dim $R = \sup \{$r.gl.dim R_1, r.gl.dim $S\} = \sup \{$r.gl.dim $R_i : i \geqq 1\}$. The other conclusions follow in a similar manner.

Remark. For the theorem to be of value, the ring R must have a right flat overring Q. Since any PWD is nonsingular [**9**], an obvious candidate for Q is the maximal left quotient ring. Cateforis [**2**] has given necessary and sufficient conditions for the maximal left quotient ring to be right flat. Left semi-hereditary rings are included in this class of rings. In fact the maximal quotient ring need not be flat for a similar theorem to hold as is shown by the next proposition.

PROPOSITION 3.11. *Let R be an essential right subdirect product of rings R_1 and R_2. If R is a PWD, then so are R_1 and R_2 and the product is split (left).*

Proof. Let E_1 and E_2 be as above. Since R is an essential product of R_1 and R_2, E_i is an essential right ideal of R_i. Identify E_i with $E_i \times 0$. Without loss of generality it is sufficient to show that E_1 is a direct summand of $_RR$. We will do this by showing that $e_i \in E_1$ whenever $E_1e_i \neq 0$, where R is a PWD with respect to the complete set of idempotents $\{e_1, \ldots, e_n\}$. Suppose that $E_1e_i \neq 0$ and $e_i \notin E_1$. Then there exists e_j such that $e_jEe_i \neq 0$. Since $e_i \notin E_1$, the projection of e_i in R_2 is not zero. E_2 is an essential right ideal of R_2, thus $e_iE_2 \neq 0$ since R_2 is right nonsingular. Hence there exists $e_k \in R$ such that $e_iE_2e_k \neq 0$. As a result $(e_jE_1e_i)(e_iE_2e_k) \neq 0$ since R is a PWD. This is a contradiction, for $(e_jE_1e_i)(e_iE_2e_k) \subset E_1E_2 \subset E_1 \cap E_2 = 0$. Therefore $e_i \in E_1$. Hence $E_1 = \oplus \{Re_i : E_1e_i \neq 0\}$ and is a direct summand of $_RR$.

R_1 and R_2 are PWD's by Proposition 3.4.

COROLLARY 3.12. *Let R be a PWD with maximal right quotient ring Q. If $Q = Q_1 \times Q_2$, then R is a split subdirect product of R_1 and R_2 where R_i is the projection of R in Q_i.*

Proof. By Lemma 2.3, R is the right essential subdirect product of R_1 and R_2. Then by Proposition 3.11 the subdirect product is split.

The following corollary shows that information about Q can sometimes be used to represent R as a ring direct product.

PROPOSITION 3.13. *Let R, Q, Q_1, Q_2, R_1, and R_2 be as in Corollary 3.12. If Q is also the maximal left quotient ring of R, or if Q_R is flat, then $R = R_1 \times R_2$.*

Proof. Each E_i is right essential by Lemma 2.3. If Q is the maximal left quotient ring of R (or Q_R is flat), then E_i is a right direct summand of R by Corollary 3.12 (Proposition 3.8). This can only happen if $E_i = R_i$; it follows that $R = R_1 \times R_2$.

The following theorem characterizes the maximal right quotient ring of a PWD.

THEOREM 3.14. *Let R be a PWD with maximal right quotient ring Q. Then $Q = Q_1 \times \ldots \times Q_t$ where each Q_i is an indecomposable prime self-injective regular ring whose ideals form a well ordered chain.*

Proof. We will first show that Q is a ring direct product of finitely many indecomposable rings. Let $n(R)$ be the smallest positive integer for which R

has a complete set of $n(R)$ idempotents with respect to which R is a PWD. The proof is by induction on $n(R)$.

If $n(R) = 1$, then R is a domain since R is a PWD. If $Q = Q_1 \times Q_2$, then by Corollary 3.12 R is the split subdirect product of rings R_1 and R_2. This cannot happen since R is a domain.

Inductively assume the result for all PWD's S for which $n(S) < n(R)$. If the maximal right quotient ring of R is indecomposable, then the result trivially holds. Otherwise $Q = Q_1 \times Q_2$, and by Corollary 3.12 R is a split subdirect product of rings R_1 and R_2. It is easy to see using reasoning similar to that of Lemma 2.3 that Q_i is the maximal right quotient ring of R_i for each i. Also, it is shown in the proof of Proposition 3.4 that $n(R_i) < n(R)$. Hence the induction hypothesis implies that Q_1 and Q_2 are each ring direct products of finitely many indecomposable rings. It follows that Q is a direct product of finitely many rings, say $Q = Q_1 \times \ldots \times Q_t$ where each Q_i is indecomposable.

Since a PWD is nonsingular [**9**], Q is a right self-injective regular ring [**3**]. Hence each Q_i is a self-injective regular ring. By Goodearl [**6**, Proposition 3] each Q_i is a prime ring since it is indecomposable. By [**6**, Theorem 6] the ideals of Q_i are linearly ordered. The zero ideal is a closed prime ideal in Q_i, hence by [**6**, Theorem 8] the ideals of Q_i are well ordered.

The following definitions are those of Handelman and Lawrence [**11**]. A *(right) insulator* of an element a of R is a finite subset $S \subset R$ such that $r(aS) = 0$. The ring R is called *(right) strongly prime* (SP) if every nonzero element of R has a right insulator.

The maximal right quotient ring of a domain is simple. The next proposition shows that this property is shared by a prime piecewise domain.

PROPOSITION 3.15. *If R is a prime PWD with maximal right quotient ring Q, then Q is a simple right self-injective regular ring.*

Proof. As in Theorem 3.14 Q is a right self-injective regular ring.

Let R be a prime PWD with respect to the complete set of idempotents $\{e_1, \ldots, e_n\}$. Since R is prime, $e_iRe_j \neq 0$ for all i and j. Let $a \in R$. There exists an e_i such that $ae_i \neq 0$. For all k there exists $r_k \in R$ such that $e_ir_ke_k \neq 0$. Since R is a PWD, $ae_ir_ke_k \neq 0$ for each k. Let $S = \{e_ir_ke_k : k = 1, \ldots, n\}$. Since R is a PWD, $r(ae_ir_ke_k) = \oplus_{j \neq k} e_jR = A_k$. Hence $r(aS) = \bigcap_{k=1}^n A_k = 0$. Thus S is a right insulator for the arbitrary element a, and the ring R is strongly prime. The ring Q is then simple by Proposition 1.1 of Goodearl and Handelman [**7**].

We now turn our attention to the classical right quotient ring of a PWD. Gordon [**8**] has given necessary and sufficient conditions for a PWD to be a right order in a semiprimary ring. We will make use of the next theorem which is due to Gordon [**8**, Theorem A].

THEOREM 3.16. *Suppose e and f are nonzero idempotents in a ring R which is* *a right order in a semiprimary ring Q. Then $eQf = eRffQf$ and the semiprimary ring fQf is the classical right quotient ring of fRf.*

The next theorem shows that the classical quotient ring of a PWD which is a split product can be represented as a split product of the quotient rings of the factor rings.

THEOREM 3.17. *Let a PWD R be a split subdirect product of PWD's R_1 and R_2. Then R has a semiprimary classical right quotient ring Q if and only if each R_i has a semiprimary classical right quotient ring Q_i. Each Q_i is a PWD, and Q is a split subdirect product of Q_1 and Q_2.*

Proof. Since R is a split product of R_1 and R_2, $E_i = e_iR$ for $i = 1, 2$ where each e_i is an idempotent. Let $e_3 = 1 - e_1 - e_2$, $h_1 = e_1 + e_3$, and $h_2 = e_2 + e_3$. Then R has the following triangular structure:

$$R \simeq \begin{bmatrix} e_1Re_1 & & \\ 0 & e_2Re_2 & \\ e_3Re_1 & e_3Re_2 & e_3Re_3 \end{bmatrix}.$$

Furthermore R_i can be identified with h_iRh_i.

All of the Q's are PWD's since by Gordon [**8**, page 40] a semiprimary classical quotient ring of a PWD is again a PWD.

Suppose R has a classical right semiprimary quotient ring Q. Then by Theorem 3.16 $Q_i = h_iQh_i$ is the classical right quotient ring of $R_i = h_iRh_i$.

Conversely suppose that each R_i has a semiprimary classical right quotient ring Q_i. We may identify $e_3Q_1e_3$ with $e_3Q_2e_3$ since each is the classical quotient ring of e_3Re_3. Consider the collection Q of "matrices" of the following form:

$$Q = \begin{bmatrix} e_1Q_1e_1 & & \\ 0 & e_2Q_2e_2 & \\ e_3Q_1e_1 & e_3Q_2e_2 & e_3Q_1e_3\ (= e_3Q_2e_3) \end{bmatrix}.$$

Q is a semiprimary ring since it is merely the "matrix" representation of the split subdirect product of Q_1 and Q_2 via the identification of $e_3Q_1e_3$ with $e_3Q_2e_3$. Let $d = \begin{bmatrix} a_1 & & \\ 0 & a_2 & \\ c_1 & c_2 & a_3 \end{bmatrix}$ be a regular element of R. Then $\begin{bmatrix} a_1 & \\ c_1 & a_3 \end{bmatrix}$ is a right regular element of R_1, for if $\begin{bmatrix} a_1 & \\ c_1 & a_3 \end{bmatrix}\begin{bmatrix} x_1 & \\ x_2 & x_3 \end{bmatrix} = 0$, then $\begin{bmatrix} a_1 & & \\ 0 & a_2 & \\ c_1 & c_2 & a_3 \end{bmatrix}$ $\begin{bmatrix} x_1 & & \\ 0 & 0 & \\ x_2 & 0 & x_3 \end{bmatrix} = 0$. Similarly $\begin{bmatrix} a_2 & \\ c_2 & a_3 \end{bmatrix}$ is a right regular element of R_2. Since R_1 and R_2 have classical *right* quotient rings Q_1 and Q_2 respectively, $\begin{bmatrix} a_1 & \\ c_1 & a_3 \end{bmatrix}$

and $\begin{bmatrix} a_2 & \\ c_2 & a_3 \end{bmatrix}$ are right regular in Q_1 and Q_2 respectively. They are then invertible since each Q_i is semiprimary. Let their respective inverses be $\begin{bmatrix} x_1 & \\ d_1 & x_3 \end{bmatrix}$ and $\begin{bmatrix} x_2 & \\ d_2 & x_3 \end{bmatrix}$ (we are using the identification of $e_3Q_1e_3$ and $e_3Q_2e_3$). Then $\begin{bmatrix} a_1 & & \\ 0 & a_2 & \\ c_1 & c_2 & a_3 \end{bmatrix}\begin{bmatrix} x_1 & & \\ 0 & x_2 & \\ d_1 & d_2 & x_3 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 0 & 1 \end{bmatrix}$ in Q; Q is semiprimary so right invertible elements are invertible and d has an inverse in Q.

Let $x = \begin{bmatrix} x_{11} & & \\ 0 & x_{22} & \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$ be an element of Q. Applying Theorem 3.16 separately to Q_1 and Q_2 yields that $e_iQe_j = e_iRe_je_jQe_j$ for all i and j; hence $x_{ij} = a_{ij}b_{ij}^{-1}$ where b_{ij} is a regular element of e_jRe_j and $a_{ij} \in e_iRe_j$. Let d_1 be a common denominator for b_{11} and b_{31} (an element such that $b_{11}^{-1}d_1$ and $b_{31}^{-1} d_1$ are elements of R), let d_2 be a common denominator for b_{22} and b_{32}, and let $d = \begin{bmatrix} d_1 & & \\ 0 & d_2 & \\ 0 & 0 & b_{33} \end{bmatrix}$. The element d is invertible in Q and hence must be regular in R. By the choice of d, $xd = r$ an element of R. Hence $x = \mathrm{r}d^{-1}$.

We've now shown that every regular element of R is invertible in Q and that every element of Q is of the form rd^{-1} for $r \in R$ and d a regular element of R; hence Q is the classical right quotient ring of R. Hence if each R_i has a semiprimary classical right quotient ring, then so does R; furthermore, if R is a split subdirect product, then so is its classical quotient ring.

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