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Relative uniformly positive entropy of induced amenable group actions

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Abstract. Let G be a countably infinite discrete amenable group. It should be noted that a G-system (X, G) naturally induces a G-system $(\mathcal{M}(X), G)$, where $\mathcal{M}(X)$ denotes the space of Borel probability measures on the compact metric space X endowed with the weak*-topology. A factor map $\pi : (X, G) \to (Y, G)$ between two G-systems induces a factor map $\widetilde{\pi} : (\mathcal{M}(X), G) \to (\mathcal{M}(Y), G)$. It turns out that $\widetilde{\pi}$ is open if and only if π is open. When Y is fully supported, it is shown that π has relative uniformly positive entropy if and only if $\widetilde{\pi}$ has relative uniformly positive entropy.

Key words: relative uniformly positive entropy, induced system, relative independence
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1. *Introduction*

In the process of studying the classification of topological dynamical systems, entropy as a conjugacy invariant plays an important role which divides them into two classes. For $\mathbb{Z}$-systems, the notion of uniformly positive entropy (u.p.e. for short) was introduced by Blanchard in [6] as an analogue in topological dynamics for the notion of a K-process in ergodic theory. He then naturally defined the notion of entropy pairs and used it to show that a u.p.e. system is disjoint from all minimal zero entropy systems [7]. Further research concerning u.p.e. systems and entropy pairs can be found in [8, 9, 13, 16, 17, 27].

Recently, there has been a lot of significant progress in studying relative entropy via local relative entropy theory for $\mathbb{Z}$-systems. For a factor map between two $\mathbb{Z}$-systems, Glasner and Weiss [14] introduced the relative uniformly positive entropy (rel-u.p.e.) and the notion of relative topological Pinsker factor based on the idea of u.p.e. extensions. Later, Park and Siemaszko [30] interpreted another relative topological Pinsker factor, defined by Lemańczyk and Siemaszko [27], using relative measure-theoretical entropy and discussed the relative product. In [19], Huang, Ye and Zhang introduced the notions of relative entropy tuples in both topological and measure-theoretical settings. They showed that the finite product of rel-u.p.e. extensions has rel-u.p.e. if and only if the factors are fully supported (for definitions see §2.3). They also proved some classical results about the rel-u.p.e. extension. We will refer readers to [10, 11, 18, 26] for more results related to local relative entropy theory.

Bauer and Sigmund [3] initiated a systematic study of the connections between dynamical properties of a $\mathbb{Z}$-system and its induced system (whose phase space consists of all Borel probability measures on the original space, for details see §2). A well-known result due to Glasner and Weiss [15] in 1995 reveals that if a system has zero topological entropy, then so does its induced system. Later, this connection was further developed by Kerr and Li in [23]. They obtained that a system is null if and only if its induced system is null. More research concerning relations of these systems was developed in [1, 2, 33, 37]. Recently, Bernardes *et al* [4] proved that a $\mathbb{Z}$-system has u.p.e. if and only if its induced system does.

After Ornstein and Weiss's pioneering work for amenable group actions in 1987 [29], there have been many developments in the process of studying the amenable group action systems. We will refer the reader to the related papers [20, 28, 31, 35, 36, 38]. In this paper, we always assume that G is a countably infinite discrete amenable group. By a *G-system* (X, G), we mean a compact metric space X together with G acting on X by homeomorphisms, that is, there exists a continuous map $\Gamma : G \times X \to X$, satisfying:

- $\Gamma(e_G, x) = x$ for every $x \in X$;
- $\Gamma(g, \Gamma(h, x)) = \Gamma(gh, x)$ for each $g, h \in G$ and $x \in X$.

We write $\Gamma(g, x)$ as gx for every $g \in G$ and $x \in X$.

Motivated by those works which were previously mentioned for $\mathbb{Z}$-systems and the local entropy theory developed for countable discrete amenable group action systems due to Huang, Ye and Zhang [20], and Kerr and Li [24], the present paper aims to investigate the properties of the relative uniformly positive entropy (rel-u.p.e.) for an induced factor map of a factor map between two G-systems (see §2 for definitions).

More precisely, let (X, G) be a G-system, $\mathcal{B}_X$ be the set of Borel subsets of X and $\mathcal{M}(X)$ be the space of Borel probability measures on the compact metric space X endowed with the weak*-topology. Then the G-system (X, G) induces a system $(\mathcal{M}(X), G)$ (see §2 for details). For any $x \in X$, let δ_x denote the Dirac measure on x and

$$\mathcal{M}_n(X) = \left\{ \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} : x_1, x_2, \ldots, x_n \in X \right\}$$

for each $n \in \mathbb{N}$. Then $\mathcal{M}_n(X)$ is closed and invariant under G (that is, $g\mathcal{M}_n(X) = \mathcal{M}_n(X)$ for every $g \in G$). Hence, we can consider the subsystems $(\mathcal{M}_n(X), G)$ of $(\mathcal{M}(X), G)$ for each $n \in \mathbb{N}$. For a factor map $\pi : (X, G) \to (Y, G)$ between two G-systems, when $\mathrm{supp}(Y) = Y$ (for definitions see §2.3), we have the following result.

THEOREM 1.1. *Let $\pi : (X, G) \to (Y, G)$ be a factor map between two G-systems, $\widetilde{\pi} : (\mathcal{M}(X), G) \to (\mathcal{M}(Y), G)$ be the factor map induced by π and $\widetilde{\pi}_n : (\mathcal{M}_n(X), G) \to (\mathcal{M}_n(Y), G)$ be the restriction of $\widetilde{\pi}$ on $\mathcal{M}_n(X)$. When $\mathrm{supp}(Y) = Y$, the following are equivalent:*

(1) *π has relative uniformly positive entropy;*
(2) *$\widetilde{\pi}_n$ has relative uniformly positive entropy for some $n \in \mathbb{N}$;*
(3) *$\widetilde{\pi}_n$ has relative uniformly positive entropy for every $n \in \mathbb{N}$;*
(4) *$\widetilde{\pi}$ has relative uniformly positive entropy.*

Notice that when Y is a singleton, we obtain that (X, G) has u.p.e. if and only if the induced system $(\mathcal{M}(X), G)$ has u.p.e. (when $G = \mathbb{Z}$, see [**4**, Theorem 4]).

We say a map $\pi : X \to Y$ between two topological spaces is *open* if the images of open sets are open. Then we have the following result.

THEOREM 1.2. *Let $\pi : X \to Y$ be a surjective continuous map between two compact metrizable spaces, and $\widetilde{\pi} : \mathcal{M}(X) \to \mathcal{M}(Y)$ be the induced map of π. Then π is open if and only if $\widetilde{\pi}$ is open.*

This paper is organized as follows. In §2, we will list some basic notions and results needed in our argument. In §§3 and 4, we will give a proof of Theorem 1.1. Finally, we prove Theorem 1.2 in §5.

2. *Preliminaries*

In this section, we recall some basic notation and results which will be used repeatedly in our paper. Denote by $\mathbb{N}$ and $\mathbb{R}$ the set of natural numbers and real numbers, respectively. For $n \in \mathbb{N}$, we write $[n]$ for $\{1, 2, \ldots, n\}$.

2.1. *Amenable group.*

We say a countably infinite discrete group G is *amenable* if there always exists an invariant Borel probability measure when it acts on any compact metric space. In the case where G is a countably infinite discrete group, amenability is equivalent to the existence of a *Fϕlner sequence*: a sequence of non-empty finite subsets $\{F_n\}_{n=1}^{\infty}$ of G such that

$$\lim_{n\to\infty} \frac{|F_n \Delta g F_n|}{|F_n|} = 0$$

for all $g \in G$. One should refer to Ornstein and Weiss' paper [**29**] for more details about an amenable group. In this paper, we always assume that G is a countably infinite discrete amenable group and denote by $\mathcal{F}(G)$ the collection of non-empty finite subsets of G. The following result is well known (see [**25**, Theorem 4.48]).

THEOREM 2.1. *Let ϕ be a real-valued function on $\mathcal{F}(G)$ satisfying:*

(1) *$\phi(Fs) = \phi(F)$ for all $F \in \mathcal{F}(G)$ and $s \in G$; and*
(2) *$\phi(F) \le (1/k) \sum_{E\in\mathcal{E}} \phi(E)$ for every $k \in \mathbb{N}$, $F \in \mathcal{F}(G)$ and finite collection $\mathcal{E} \subseteq \mathcal{F}(G)$ with $\bigcup_{E\in\mathcal{E}} E \subseteq F$ and $\sum_{E\in\mathcal{E}} 1_E \ge k 1_F$.*

Then $\phi(F)/|F|$ converges to a limit as F becomes more and more invariant and this limit is equal to $\inf_F \phi(F)/|F|$, where F ranges over all non-empty finite subsets of G.

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2.2. *Induced systems.* Assume that X is a compact metric space. Let $\mathcal{B}_X$ be the collection of Borel subsets of X, $C(X)$ be the space of continuous maps from X to $\mathbb{R}$ endowed with the supremum norm $\|\cdot\|_\infty$ and $\mathcal{M}(X)$ be the set of Borel probability measures on X endowed with the *weak*-topology*, which is the smallest topology making the map

$$D_g : \mathcal{M}(X) \to \mathbb{R}, \quad \mu \mapsto \int_X g\, d\mu$$

continuous for every $g \in C(X)$, and the topology basis of weak*-topology consists of the following sets:

$$\mathbb{V}(\mu; f_1, \ldots, f_k; \epsilon) := \left\{\nu \in \mathcal{M}(X) : \left| \int_X f_i\, d\mu - \int_X f_i\, d\nu \right| < \epsilon \text{ for all } i \in [k]\right\}, \tag{2.1}$$

where $\mu \in \mathcal{M}(X)$, $k \geq 1$, $\epsilon > 0$ and $f_i : X \to \mathbb{R}$ are continuous functions for $i \in [k]$. The *Prohorov metric* on $\mathcal{M}(X)$,

$$d_P(\mu, \nu) := \inf\{\delta > 0 : \mu(A) \leq \nu(A^\delta) + \delta \text{ and } \nu(A) \leq \mu(A^\delta) + \delta \text{ for all } A \in \mathcal{B}_X\},$$

where $A^\delta = \{x \in X : d(x, A) < \delta\}$, is compatible with the weak*-topology. We will refer the readers to the books [5, 12, 22] for the knowledge of space $\mathcal{M}(X)$. Moreover,

$$d_P(\mu, \nu) = \inf\{\delta > 0 : \mu(A) \leq \nu(A^\delta) + \delta \text{ for all } A \in \mathcal{B}_X\}$$

(see [5, p. 72]). Proposition 2.2 describes a basis for the weak*-topology on $\mathcal{M}(X)$ due to Bernardes *et al* (see [4, Lemma 1]).

PROPOSITION 2.2. *The set of the form*

$$\mathbb{W}(U_1, U_2, \ldots, U_k : \eta_1, \eta_2, \ldots, \eta_k) := \{\mu \in \mathcal{M}(X) : \ \mu(U_i) > \eta_i \textit{ for } i \in [k]\},$$

where $k \geq 1$, $U_1, U_2, \ldots, U_k$ *are non-empty disjoint open sets in* X *and* $\eta_1, \eta_2, \ldots, \eta_k$ *are positive real numbers with* $\eta_1 + \eta_2 + \cdots + \eta_k < 1$, *form a basis for the weak*-topology on* $\mathcal{M}(X)$.

A G-system (X, G) induces a system $(\mathcal{M}(X), G)$, where $g : \mathcal{M}(X) \to \mathcal{M}(X)$ is defined by $(g\mu)(A) := \mu(g^{-1}A)$ for every $g \in G$, $\mu \in \mathcal{M}(X)$ and $A \in \mathcal{B}_X$. We call $(\mathcal{M}(X), G)$ the *induced system* of (X, G).

Let (X, G) and (Y, G) be two G-systems. A continuous map $\pi : (X, G) \to (Y, G)$ is called a *factor map* between (X, G) and (Y, G) if it is onto and $\pi \circ g = g \circ \pi$ for every $g \in G$. Here, π can induce a factor map $\widetilde{\pi} : (\mathcal{M}(X), G) \to (\mathcal{M}(Y), G)$ by

$$(\widetilde{\pi}\mu)(B) = \mu(\pi^{-1}B)$$

for every $\mu \in \mathcal{M}(X)$ and $B \in \mathcal{B}_Y$. For every $n \in \mathbb{N}$, we denote

$$\widetilde{\pi}_n := \tilde{\pi}|_{\mathcal{M}_n(X)} : \mathcal{M}_n(X) \to \mathcal{M}_n(Y)$$

by the restriction of $\tilde{\pi}$ on $\mathcal{M}_n(X)$. Note that $\widetilde{\pi}_n$ is also a factor map for each $n \in \mathbb{N}$.

2.3. *Support.* Let (X, G) be a G-system, $(\mathcal{M}(X), G)$ be the induced G-system of (X, G). We denote by $\mathcal{M}(X, G)$ the set of all G-invariant measures. For $\mu \in \mathcal{M}(X)$, we denote by $\mathrm{supp}(\mu)$ the *support of* μ, that is, the smallest closed subset $W \subseteq X$ such that $\mu(W) = 1$. We denote by $\mathrm{supp}(X, G)$ the *support of* (X, G), that is,

$$\mathrm{supp}(X, G) = \bigcup_{\mu \in \mathcal{M}(X,G)} \mathrm{supp}(\mu).$$

Here, (X, G) is called *fully supported* if there is an invariant measure $\mu \in \mathcal{M}(X, G)$ with full support (that is, $\mathrm{supp}(\mu) = X$), equivalently, $\mathrm{supp}(X, G) = X$.

2.4. *Relative uniformly positive topological entropy.* For a given G-system (X, G), a *cover* of X is a family of Borel subsets of X, whose union is X. Denote the set of finite covers by $\mathcal{C}_X$. For $n \in \mathbb{N}$ and $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n \in \mathcal{C}_X$, we denote

$$\bigvee_{i=1}^{n} \mathcal{U}_i = \{A_1 \cap A_1 \cap \cdots \cap A_n : A_i \in \mathcal{U}_i,\ i \in [n]\}.$$

Let $\pi : (X, G) \to (Y, G)$ be a factor map between two G-systems and $\mathcal{U} \in \mathcal{C}_X$. For any non-empty subset E of X, let $N(\mathcal{U}, E)$ be the minimum among the cardinalities of the subsets of $\mathcal{U}$ which cover E, and define

$$N(\mathcal{U}|\pi) = \sup_{y \in Y} N(\mathcal{U}, \pi^{-1}(y)).$$

The *topological conditional entropy of $\mathcal{U}$ with respect to π* is defined by

$$h_{\mathrm{top}}(\mathcal{U}, G|\pi) = \lim_{n \to \infty} \frac{1}{|F_n|} \log N(\mathcal{U}_{F_n}|\pi),$$

where $\mathcal{U}_{F_n} = \bigvee_{g \in F_n} g^{-1}\mathcal{U}$ and $\{F_n\}_{n=1}^{\infty}$ is a Fϕlner sequence of G. It is well known that $h_{\mathrm{top}}(\mathcal{U}, G|\pi)$ is well defined and is independent of the choice of the Fϕlner sequences of G.

Let $\pi : (X, G) \to (Y, G)$ be a factor map between G-systems. Here, $\mathcal{U} = \{U_1, \ldots, U_n\} \in \mathcal{C}_X$ is said to be *non-dense-on-π-fibre* if there is $y \in Y$ such that $\pi^{-1}(y)$ is not contained in any element of $\overline{\mathcal{U}}$ which consists of the closures of elements of $\mathcal{U}$ in X. Clearly, if an open cover $\mathcal{U} = \{U_1, U_2\}$ is non-dense-on-π-fibre, then $\pi(U_1) \cap \pi(U_2) \neq \emptyset$. We say (X, G) or π has *relative uniformly positive entropy* (rel-u.p.e. for short) if for any non-dense-on-π-fibre open cover $\mathcal{U}$ of X with two elements, we have $h_{\mathrm{top}}(\mathcal{U}, G|\pi) > 0$.

For $n \in \mathbb{N}$ and G-systems (Z_i, G), $i \in [n]$, we set

$$\prod_{i \in [n]} Z_i = \{(z_1, z_2, \ldots, z_n) : z_i \in Z_i;\ i \in [n]\}$$

and

$$g(z_1, z_2, \ldots, z_n) = (gz_1, gz_2, \ldots, gz_n)$$

for every $g \in G$ and $z_i \in Z_i$ for $i \in [n]$. Clearly, $(\prod_{i \in [n]} Z_i, G)$ is also a G-system. When $Z_i = Z$ for all $i \in [n]$, we write $\prod_{i \in [n]} Z_i$ as $Z^{(n)}$. Let $\pi_i : (X_i, G) \to (Y_i, G)$ be factor

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maps between G-systems for $i \in [n]$. Then $\{\pi_i\}_{i\in[n]}$ induce a factor map

$$\prod_{i\in[n]} \pi_i : \bigg(\prod_{i\in[n]} X_i, G\bigg) \to \bigg(\prod_{i\in[n]} Y_i, G\bigg)$$

by

$$\prod_{i\in[n]} \pi_i(x_1, x_2, \ldots, x_n) = (\pi_1 x_1, \pi_2 x_2, \ldots, \pi_n x_n)$$

for every $(x_1, x_2, \ldots, x_n) \in \prod_{i\in[n]} X_i$. When $\pi_i = \pi$ for all $i \in [n]$, we write $\prod_{i\in[n]} \pi_i$ as $\pi^{(n)}$. In [19], Huang, Ye and Zhang showed that the finite product of rel-u.p.e. factor maps between $\mathbb{Z}$-systems has rel-u.p.e. It also holds for G-systems.

THEOREM 2.3. *Let $\pi_i : (X_i, G) \to (Y_i, G)$ be a factor map between two G-systems and* $\mathrm{supp}(Y_i) = Y_i$ *for $i = 1, 2$. Then π_1 and π_2 have rel-u.p.e. if and only if $\pi_1 \times \pi_2 : (X_1 \times X_2, G) \to (Y_1 \times Y_2, G)$ has rel-u.p.e.*

We will give a proof of Theorem 2.3 in Appendix A (see Theorem A.5).

3. *π has rel-u.p.e. if and only if $\widetilde{\pi}_n$ has rel-u.p.e.*

Let X be a compact metric space and ρ_X be a compatible metric for X. We denote $B_{\rho_X}(x, \delta) = \{y \in X : \rho_X(x, y) < \delta\}$ for $x \in X$ and $\delta > 0$, and denote

$$\Delta(X) = \{(x, x) : x \in X\}.$$

For $(x_1, x_2) \in X \times X \backslash \Delta(X)$ and $\mathcal{U} = \{U_1, U_2\} \in \mathcal{C}_X$, we say $\mathcal{U}$ is an *admissible cover of X with respect to* (x_1, x_2) if for any $i \in [2]$, one has $\{x_1, x_2\} \nsubseteq \overline{U_i}$. Let $\pi : (X, G) \to (Y, G)$ be a factor map between two G-systems. Here, $(x_1, x_2) \in X \times X \backslash \Delta(X)$ is called an *entropy pair relevant to* π if for any admissible cover $\mathcal{U}$ with respect to (x_1, x_2), we have $h_{\mathrm{top}}(\mathcal{U}, G|\pi) > 0$. Denote by $E(X, G|\pi)$ the set of all entropy pairs relevant to π. Let

$$R_\pi = \{(x_1, x_2) \in X \times X : \pi(x_1) = \pi(x_2)\}.$$

It is easy to see that $E(X, G|\pi) \subseteq R_\pi \setminus \Delta(X)$, and π has rel-u.p.e. if and only if $E(X, G|\pi) = R_\pi \setminus \Delta(X)$.

The concept of dynamical independence is introduced in [24, Definition 2.1]. Now we consider its relative version. Let $\pi : (X, G) \to (Y, G)$ be a factor map between two G-systems. For any $n \in \mathbb{N}$ and a tuple $\mathcal{V} = (V_1, V_2, \ldots, V_n)$ of subsets of X, we say $J \subseteq G$ is an *independence set of $\mathcal{V}$ with respect to* π if for every non-empty finite subset $I \subset J$, there exists $y \in Y$ such that

$$\pi^{-1}(y) \cap \bigcap_{g\in I} g^{-1} V_{\sigma(g)} \neq \emptyset$$

holds for every $\sigma \in [n]^I$. We denote by $\mathcal{P}^\pi_{\mathcal{V}}$ the set of all independence sets of $\mathcal{V}$ with respect to π.

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Remark 3.1. For every $n \in \mathbb{N}$ and a tuple $\mathcal{V} = (V_1, V_2, \ldots, V_n)$ of subsets of X, if we set

$$\mathcal{I}_{\mathcal{V}} : \mathcal{F}(G) \to \mathbb{R}; \quad \mathcal{I}_{\mathcal{V}}(F) := \max_{I \subseteq F, I \in \mathcal{P}^{\pi}_{\mathcal{V}}} |I|,$$

then by Theorem 2.1, $\mathcal{I}_{\mathcal{V}}(F)/|F|$ converges as F becomes increasingly more invariant and this limit is equal to $\inf_F(\mathcal{I}_{\mathcal{V}}(F)/|F|)$, where F ranges over $\mathcal{F}(G)$. When this limit is positive, we say $\mathcal{V}$ is *independent with respect to* π.

The next lemma follows [**24**, Lemma 3.4] (see also [**17**, Theorem 7.4]).

LEMMA 3.2. *Let $\pi : (X, G) \to (Y, G)$ be a factor map between two G-systems, and V_1, V_2 be two disjoint subsets of X. If we set $\mathcal{U} = \{X \setminus V_1, X \setminus V_2\}$, then $h_{\text{top}}(\mathcal{U}, G|\pi) > 0$ if and only if $\{V_1, V_2\}$ is independent with respect to π.*

Let $\pi : (X, G) \to (Y, G)$ be a factor map between two G-systems. For any $(x_1, x_2) \in X \times X \setminus \Delta(X)$, disjoint open subsets V_1, V_2 of X with $x_i \in V_i$ for $i \in [2]$, $\mathcal{V} = \{X \setminus V_1, X \setminus V_2\}$ is an admissible cover of X with respect to (x_1, x_2). Then by Lemma 3.2, we immediately have the following corollary.

COROLLARY 3.3. *Let $\pi : (X, G) \to (Y, G)$ be a factor map between two G-systems and $(x_1, x_2) \in X \times X \setminus \Delta(X)$. Then $(x_1, x_2) \in E(X, G|\pi)$ if and only if for any disjoint open subsets V_1, V_2 of X with $x_i \in V_i$ for $i = 1, 2$, $\{V_1, V_2\}$ is independent with respect to π.*

We note that for any two non-empty finite sets H, W, if $H \subseteq W$ and $S \subset \{1, 2\}^W$, one has

$$|S|_H| \geq \frac{|S|}{2^{|W|-|H|}}, \tag{3.1}$$

where $S|_H$ is the restriction of S on H, that is,

$$S|_H = \{\sigma \in \{1, 2\}^H : \text{there exists } \sigma' \in S \text{ such that } \sigma(h) = \sigma'(h) \text{ for all } h \in H\}.$$

The following consequence of Karpovsky and Milman's generalization of the Sauer–Perles–Shelah lemma [**21**, **32**, **34**] is well known, and one can also refer to [**24**, Lemma 3.5].

LEMMA 3.4. *Given $k \geq 2$ and $\lambda > 1$, there exists a constant $c > 0$ such that for all $n \in \mathbb{N}$, if $S \subseteq [k]^{[n]}$ satisfies $|S| \geq ((k-1)\lambda)^n$, then there is an $I \subseteq [n]$ with $|I| \geq cn$ and $S|_I = [k]^I$.*

Theorem 1.1 follows from Theorems 3.5, 4.2 and 4.3.

THEOREM 3.5. *Let $n \in \mathbb{N}$, $\pi : (X, G) \to (Y, G)$ be a factor map between two G-systems, $\widetilde{\pi} : (\mathcal{M}(X), G) \to (\mathcal{M}(Y), G)$ be the factor map induced by π and $\widetilde{\pi}_n : (\mathcal{M}_n(X), G) \to (\mathcal{M}_n(Y), G)$ be the restriction of $\widetilde{\pi}$ on $\mathcal{M}_n(X)$. When* $\text{supp}(Y) = Y$*, the following are equivalent:*

(1) *π has rel-u.p.e.;*
(2) *$\widetilde{\pi}_n$ has rel-u.p.e. for some $n \in \mathbb{N}$;*
(3) *$\widetilde{\pi}_n$ has rel-u.p.e. for every $n \in \mathbb{N}$.*

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Proof. (3) ⇒ (2) is trivial. We will prove (1) ⇒ (3) and (2) ⇒ (1).

(1) ⇒ (3). Assume that π has rel-u.p.e. For every fixed $1 \le n < \infty$, to obtain that $\widetilde{\pi}_n$ has rel-u.p.e., it is sufficient to prove that $E(\mathcal{M}_n(X), G|\widetilde{\pi}_n) \supseteq R_{\widetilde{\pi}_n}(\mathcal{M}_n(X), G) \setminus \Delta(\mathcal{M}_n(X))$.

Let $(\mu_1, \mu_2) \in R_{\widetilde{\pi}_n}(\mathcal{M}_n(X), G) \setminus \Delta(\mathcal{M}_n(X))$, and $\widetilde{V}_1$ and $\widetilde{V}_2$ be two disjoint open subsets of $\mathcal{M}_n(X)$ with $\mu_i \in \widetilde{V}_i$ for $i \in [2]$. By Corollary 3.3, we shall show that $\{\widetilde{V}_1, \widetilde{V}_2\}$ is independent with respect to $\widetilde{\pi}_n$.

For $i \in [2]$ and $j \in [n]$, there exist points $x_j^i \in X$ such that $\mu_i = (1/n) \sum_{j=1}^n \delta_{x_j^i}$. We note that the map $\Phi : X^{(n)} \to \mathcal{M}(X)$, defined by

$$\Phi(z_1, z_2, \ldots, z_n) = \frac{1}{n} \sum_{i=1}^{n} \delta_{z_i}$$

is continuous. Thus, for every $i \in [2]$ and $j \in [n]$, there exists open neighbourhoods V_j^i of x_j^i such that

$$\mu_i \in \left\{ \frac{1}{n} \sum_{j=1}^{n} \delta_{z_j} : z_j \in V_j^i, j \in [n] \right\} \subseteq \widetilde{V}_i.$$

Since $\widetilde{V}_1 \cap \widetilde{V}_2 = \emptyset$, if we set $W_i = V_1^i \times V_2^i \times \cdots \times V_n^i$ for $i = 1, 2$, one has $W_1 \cap W_2 = \emptyset$. Without loss of generality, we can assume that $\pi(x_j^1) = \pi(x_j^2)$ for all $j \in [n]$ since $\widetilde{\pi}_n(\mu_1) = \widetilde{\pi}_n(\mu_2)$. Let $\omega_i = (x_1^i, x_2^i, \ldots, x_n^i) \in W_i$ for $i = 1, 2$. Then

$$(\omega_1, \omega_2) \in R_{\pi^{(n)}} \setminus \Delta(X^{(n)}) = E(X^{(n)}, G|\pi^{(n)})$$

as $\pi^{(n)}$ has rel-u.p.e. by Theorem 2.3. Thus, $\{W_1, W_2\}$ is independent with respect to $\pi^{(n)}$. We note that $\mathcal{P}^{\pi^{(n)}}_{\{W_1, W_2\}} \subseteq \mathcal{P}^{\widetilde{\pi}_n}_{\{\widetilde{V}_1, \widetilde{V}_2\}}$. This implies $\{\widetilde{V}_1, \widetilde{V}_2\}$ is independent with respect to $\widetilde{\pi}_n$.

(2) ⇒ (1). We assume that $\widetilde{\pi}_n$ has rel-u.p.e. for some positive integer $1 \le n < \infty$. In the following, we prove that $R_\pi \setminus \Delta(X) \subseteq E(X, G|\pi)$. Let $(x_1, x_2) \in R_\pi \setminus \Delta(X)$, V_1 and V_2 be two disjoint open subsets of X with $x_i \in V_i$, $i = 1, 2$. By Corollary 3.3, we only need to show that $\{V_1, V_2\}$ is independent with respect to π.

We set

$$\widetilde{V}_i = \left\{ \mu \in \mathcal{M}_n(X) : \mu(V_i) > 1 - \frac{1}{2n} \right\}$$

for $i = 1, 2$. Clearly, $\widetilde{V}_1$ and $\widetilde{V}_2$ are disjoint open subsets of $\mathcal{M}_n(X)$ with $\delta_{x_i} \in \widetilde{V}_i$ for $i = 1, 2$. Since $\widetilde{\pi}_n$ has rel-u.p.e., and $(\delta_{x_1}, \delta_{x_2}) \in R_{\widetilde{\pi}_n} \setminus \Delta(\mathcal{M}_n(X)) = E(\mathcal{M}_n(X), G|\widetilde{\pi}_n)$, $\{\widetilde{V}_1, \widetilde{V}_2\}$ is independent with respect to $\widetilde{\pi}_n$. Then there exists a constant $c > 0$, such that for every fixed $F \in \mathcal{F}(G)$, there exist $I \subseteq F$ with $|I| > c|F|$ and $\nu = (1/n) \sum_{i=1}^n \delta_{y_i} \in \mathcal{M}_n(Y)$ for some $y_i \in Y$ such that

$$A_\sigma := \widetilde{\pi}_n^{-1}(\nu) \cap \bigcap_{g \in I} g^{-1} \widetilde{V}_{\sigma(g)} \neq \emptyset$$

for every $\sigma \in \{1, 2\}^I$.

For every $\sigma \in \{1,2\}^I$ and $\mu_\sigma = (1/n)\sum_{i=1}^n \delta_{z_i^\sigma} \in A_\sigma$, we can assume $\pi(z_i^\sigma) = y_i$ for $i \in [n]$. Moreover, for every $g \in I$, one has $g\mu_\sigma = (1/n)\sum_{i=1}^n \delta_{gz_i^\sigma} \in \widetilde{V}_{\sigma(g)}$. That is,

$$\frac{1}{n}\sum_{i=1}^{n} \delta_{gz_i^\sigma}(V_{\sigma(g)}) > 1 - \frac{1}{2n},$$

which implies $gz_i^\sigma \in V_{\sigma(g)}$ for every $i \in [n]$. In particular,

$$z_1^\sigma \in \pi^{-1}(y_1) \cap \bigcap_{g\in I} g^{-1}V_{\sigma(g)}$$

for every $\sigma \in \{1,2\}^I$. Thus, $\{V_1, V_2\}$ is independent with respect to π. This ends our proof. □

4. π *is rel-u.p.e. if and only if* $\widetilde{\pi}$ *is rel-u.p.e.*

In this section, we will prove π is rel-u.p.e. if and only if $\widetilde{\pi}$ is rel-u.p.e. We need the following lemma.

LEMMA 4.1. *Let $\pi : X \to Y$ be a continuous surjective map between two compact metric spaces, $\widetilde{\pi} : \mathcal{M}(X) \to \mathcal{M}(Y)$ be the map induced by π and $\widetilde{\pi}_n : \mathcal{M}_n(X) \to \mathcal{M}_n(Y)$ be the restriction of $\widetilde{\pi}$ on $\mathcal{M}_n(X)$. Then $\bigcup_{n\in\mathbb{N}} R_{\widetilde{\pi}_n}$ is dense in $R_{\widetilde{\pi}}$.*

Proof. Fix compatible metrics ρ_X for X and ρ_Y for Y. Let $(\mu_1, \mu_2) \in R_{\widetilde{\pi}}$. Without loss of generality, we can assume $\mu_1 \neq \mu_2$. For any two disjoint open subsets $\widetilde{V}_1, \widetilde{V}_2$ of $\mathcal{M}(X)$ with $\mu_i \in \widetilde{V}_i$ for $i \in [2]$, by (2.1), there exist a constant $r > 0$ small enough, integers L_1 and L_2, $f_1, \ldots, f_{L_1} \in C(X)$ and $g_1, \ldots, g_{L_2} \in C(X)$ such that

$$\mu_1 \in \widetilde{W}_1 := \bigg\{\mu \in \mathcal{M}(X) : \bigg|\int_X f_i\, d\mu - \int_X f_i\, d\mu_1\bigg| < r, i \in [L_1]\bigg\} \subseteq \widetilde{V}_1$$

and

$$\mu_2 \in \widetilde{W}_2 := \bigg\{\mu \in \mathcal{M}(X) : \bigg|\int_X g_j\, d\mu - \int_X g_j\, d\mu_2\bigg| < r, j \in [L_2]\bigg\} \subseteq \widetilde{V}_2.$$

It is sufficient to prove that $(\widetilde{W}_1 \times \widetilde{W}_2) \cap R_{\widetilde{\pi}_N} \neq \emptyset$ for some $N \in \mathbb{N}$.

Without loss of generality, we can assume $\|f_i\| \le 1$ and $\|g_j\| \le 1$ for $i \in [L_1]$ and $j \in [L_2]$. Moreover, since $f_i, g_j \in C(X)$ for $i \in [L_1]$ and $j \in [L_2]$, there exists $\varepsilon > 0$ such that for any $x, z \in X$ with $\rho_X(x, z) < \varepsilon$, one has

$$\begin{aligned} |f_i(x) - f_i(z)| &< \frac{r}{2} \quad \text{for every } i \in [L_1], \\ |g_j(x) - g_j(z)| &< \frac{r}{2} \quad \text{for every } j \in [L_2]. \end{aligned} \tag{4.1}$$

For every $y \in Y$, since π is continuous, one can find an open neighbourhood $V_y \subseteq Y$ such that

$$\pi^{-1}(y) \subseteq \pi^{-1}(V_y) \subseteq \overline{\pi^{-1}(V_y)} \subseteq (\pi^{-1}(y))^{\varepsilon/2},$$

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where $(\pi^{-1}(y))^{\varepsilon/2} = \{x \in X : \rho_X(x, \{\pi^{-1}(y)\}) < \varepsilon/2\}$. Moreover, since Y is compact, there exist $K \in \mathbb{N}$ and pairwise different points $y_1, \ldots, y_K$ of Y such that $Y = \bigcup_{i=1}^K V_{y_i}$. Then one can find $t > 0$ such that $y_i \in B_{\rho_Y}(y_i, t) \subset V_{y_i}$ for any $i \in [K]$ and $\{B_{\rho_Y}(y_1, t), \ldots, B_{\rho_Y}(y_K, t)\}$ are pairwise disjoint. We set

$$W_1 = V_{y_1} \setminus \bigcup_{i=2}^{K} B_{\rho_Y}(y_i, t) \quad \text{and} \quad W_i = V_{y_i} \setminus \bigg(\bigcup_{j=1}^{i-1} V_{y_j} \cup \bigcup_{j=i+1}^{K} B_{\rho_Y}(y_j, t) \bigg)$$

for $i = 2, \ldots, K$. Then $\{W_1, \ldots, W_K\}$ is a partition of Y and $y_i \in W_i \subseteq V_{y_i}$ for $i \in [K]$. Moreover, $\{\pi^{-1}(W_1), \ldots, \pi^{-1}(W_K)\}$ is a partition of X which satisfies

$$\pi^{-1}(y_i) \subseteq \pi^{-1}(W_i) \subseteq \overline{\pi^{-1}(V_{y_i})} \subseteq (\pi^{-1}(y_i))^{\varepsilon/2}$$

for every $i \in [K]$. Then for every $i \in [K]$, there exist $P_i \in \mathbb{N}$ and pairwise different $x_1^i, x_2^i, \ldots, x_{P_i}^i \in \pi^{-1}(y_i)$, such that $\{x_j^i : j \in [P_i]\}$ is a $\varepsilon/2$-net of $\pi^{-1}(W_i)$. Then one can choose Borel subsets A_j^i of X for $i \in [K]$ and $j \in [P_i]$, such that:

(i) $\operatorname{diam}(A_j^i) < \varepsilon$ for every $i \in [K]$, $j \in [P_i]$;

(ii) $x_j^i \in A_j^i$ for every $i \in [K]$, $j \in [P_i]$;

(iii) $\{A_j^i : j \in [P_i]\}$ is a partition of $\pi^{-1}(W_i)$ for every $i \in [K]$.

For every $i \in [K]$, $j \in [P_i]$, we set $a_{ij} = \mu_1(A_j^i)$ and $b_{ij} = \mu_2(A_j^i)$. Since $\widetilde{\pi}(\mu_1) = \widetilde{\pi}(\mu_2)$, we have

$$\sum_{j=1}^{P_i} a_{ij} = \sum_{j=1}^{P_i} \mu_1(A_j^i) = \mu_1(\pi^{-1}(W_i)) = \mu_2(\pi^{-1}(W_i)) = \sum_{j=1}^{P_i} \mu_2(A_j^i) = \sum_{j=1}^{P_i} b_{ij}$$

for $i \in [K]$. Then for any $i \in [K]$ and $j \in [P_i]$, there exist integers $q_{ij}, \tilde{q}_{ij}, Q_i$ and $N \in \mathbb{N}$ large enough satisfying the following conditions:

(i*) $q_{ij}/N \le a_{ij} < (q_{ij} + 1/N)$;

(ii*) $\tilde{q}_{ij}/N \le b_{ij} < \tilde{q}_{ij} + 1/N$;

(iii*) $Q_i/N \le \sum_{j=1}^{P_i} a_{ij} = \sum_{j=1}^{P_i} b_{ij} < (Q_i + 1)/N$.

Now, we choose an $x_0 \in X$ arbitrarily and set

$$\widetilde{\mu}_1 = \frac{1}{N} \bigg(\sum_{i=1}^{K} \bigg(\sum_{j=1}^{P_i - 1} q_{ij} \delta_{x_j^i} + \bigg(Q_i - \sum_{j=1}^{P_i - 1} q_{ij} \bigg) \delta_{x_{P_i}^i} \bigg) \bigg) + \frac{N - \sum_{i=1}^K Q_i}{N} \delta_{x_0}$$

and

$$\widetilde{\mu}_2 = \frac{1}{N} \bigg(\sum_{i=1}^{K} \bigg(\sum_{j=1}^{P_i - 1} \tilde{q}_{ij} \delta_{x_j^i} + \bigg(Q_i - \sum_{j=1}^{P_i - 1} \tilde{q}_{ij} \bigg) \delta_{x_{P_i}^i} \bigg) \bigg) + \frac{N - \sum_{i=1}^K Q_i}{N} \delta_{x_0}.$$

It is clear that $(\widetilde{\mu}_1, \widetilde{\mu}_2) \in R_{\widetilde{\pi}_N}$. Now we shall show that $\widetilde{\mu}_i \in \widetilde{W_i}$ for $i \in [2]$.

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In fact, for any $\ell \in [L_1]$, one has

$$\begin{aligned}\left|\int f_\ell\, d\mu_1 - \int f_\ell\, d\tilde{\mu}_1\right| &= \left|\sum_{i=1}^{K}\sum_{j=1}^{P_i}\int_{A_j^i} f_l\, d\mu_1 - \frac{1}{N}\sum_{i=1}^{K}\left(\sum_{j=1}^{P_i-1} q_{ij} f_l(x_j^i)\right.\right.\\ &\quad \left.\left.+\left(Q_i - \sum_{j=1}^{P_i-1} q_{ij}\right) f_l(x_{P_i}^i)\right) - \frac{N-\sum_{i=1}^{K} Q_i}{N} f_l(x_0)\right|\\ &\le \left|\sum_{i=1}^{K}\sum_{j=1}^{P_i}\int_{A_j^i} f_l\, d\mu_1 - \frac{1}{N}\sum_{i=1}^{K}\sum_{j=1}^{P_i} q_{ij} f_l(x_j^i)\right|\\ &\quad + \left|\frac{1}{N}\sum_{i=1}^{K}(Q_i - \sum_{j=1}^{P_i} q_{ij}) f_l(x_{P_i}^i)\right| + \left|\frac{N-\sum_{i=1}^{K} Q_i}{N} f_l(x_0)\right|.\end{aligned} \tag{4.2}$$

Since $\mathrm{diam}(A_j^i) < \varepsilon$ for $i \in [K]$ and $j \in [P_i]$, by (4.1) and (i*), we have

$$\begin{aligned}&\left|\sum_{i=1}^{K}\sum_{j=1}^{P_i}\int_{A_j^i} f_l(x)\, d\mu_1 - \frac{1}{N}\sum_{i=1}^{K}\sum_{j=1}^{P_i} q_{ij} f_l(x_j^i)\right|\\ &\quad \le \sum_{i=1}^{K}\sum_{j=1}^{P_i}\int_{A_j^i} |f_l(x) - f_l(x_j^i)|\, d\mu_1 + \sum_{i=1}^{K}\sum_{j=1}^{P_i}\left(a_{ij} - \frac{q_{ij}}{N}\right)|f_l(x_j^i)|\\ &\quad \le \frac{r}{2} + \frac{\sum_{i=1}^{K} P_i}{N}.\end{aligned} \tag{4.3}$$

By (i*) and (iii*), one has

$$\begin{aligned}\sum_{i=1}^{K}\left|\frac{Q_i}{N} - \frac{1}{N}\sum_{j=1}^{P_i} q_{ij}\right| &\le \sum_{i=1}^{K}\left|\frac{Q_i}{N} - \sum_{j=1}^{P_i} a_{ij}\right| + \sum_{i=1}^{K}\left|\sum_{j=1}^{P_i} a_{ij} - \frac{1}{N}\sum_{j=1}^{P_i} q_{ij}\right|\\ &\le \frac{K}{N} + \frac{\sum_{i=1}^{K} P_i}{N}\end{aligned} \tag{4.4}$$

and

$$\left|\frac{N-\sum_{i=1}^{K} Q_i}{N}\right| = \left|\sum_{i=1}^{K}\sum_{j=1}^{P_i} a_{ij} - \frac{1}{N}\sum_{i=1}^{K} Q_i\right| \le \sum_{i=1}^{K}\left|\sum_{j=1}^{P_i} a_{ij} - \frac{Q_i}{N}\right| \le \frac{K}{N}.$$

When N is large enough such that $K/N + \sum_{i=1}^{K} P_i/N \le r/6$, by (4.2), (4.3) and (4.4), we have $\widetilde{\mu}_1 \in \widetilde{W}_1$. Similarly, we can prove that $\widetilde{\mu}_2 \in \widetilde{W}_2$. This ends our proof. □

THEOREM 4.2. *Let $\pi : (X, G) \to (Y, G)$ be a factor map between two G-systems with* $\mathrm{supp}(Y) = Y$ *and $\widetilde{\pi} : (\mathcal{M}(X), G) \to (\mathcal{M}(Y), G)$ be the induced map of π. Suppose π has rel-u.p.e., then $\widetilde{\pi}$ also has rel-u.p.e.*

Proof. Assume that π has rel-u.p.e. To show $\widetilde{\pi}$ has rel-u.p.e., it suffices to prove that $R_{\widetilde{\pi}} \setminus \Delta(\mathcal{M}(X)) \subseteq E(\mathcal{M}(X), G|\widetilde{\pi})$. Let $(\mu_1, \mu_2) \in R_{\widetilde{\pi}} \setminus \Delta(\mathcal{M}(X))$ and $\widetilde{V}_1$, $\widetilde{V}_2$ be two

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disjoint open subsets of $\mathcal{M}(X)$ with $\mu_i \in \widetilde{V}_i$ for $i \in [2]$. By Lemma 4.1, there exist $n \in \mathbb{N}$ and $(\mu_1', \mu_2') \in R_{\widetilde{\pi}_n} \cap (\widetilde{V}_1 \times \widetilde{V}_2)$. Notice that, since π has rel-u.p.e., by Theorem 3.5, $\widetilde{\pi}_n$ has rel-u.p.e. Then $\{\widetilde{V}_1 \cap \mathcal{M}_n(X), \widetilde{V}_2 \cap \mathcal{M}_n(X)\}$ is independent with respect to $\widetilde{\pi}_n$, which implies $\{\widetilde{V}_1, \widetilde{V}_2\}$ is independent with respect to $\widetilde{\pi}$. This ends our proof. □

We note that for any non-empty finite subsets A, H of $\mathbb{N}$ with $A \subseteq H$ and $S \subseteq \{1,2\}^H$, one can find $S_0 \subset S$ with $|S_0| \geq |S|/2^{|H|-|A|}$ such that for every $\sigma_1 \neq \sigma_2 \in S_0$, there exists $a \in A$ with

$$\sigma_1(a) \neq \sigma_2(a). \tag{4.5}$$

In fact, if we let $\mathcal{W} = S|_A$, then $|\mathcal{W}| \geq |S|/2^{|H|-|A|}$. For each $w \in \mathcal{W}$, there exists $\sigma_w \in S$ such that $\sigma_w|_A = w$. Put $S_0 := \{\sigma_w : w \in \mathcal{W}\} \subseteq S$. Then $|S_0| = |\mathcal{W}| \geq |S|/2^{|H|-|A|}$, and for every $\sigma_1 \neq \sigma_2 \in S_0$, one has $\sigma_1|_A \neq \sigma_2|_A$.

THEOREM 4.3. *Let $\pi : (X, G) \to (Y, G)$ be a factor map between two G-systems and $\widetilde{\pi} : (\mathcal{M}(X), G) \to (\mathcal{M}(Y), G)$ be the induced map of π. If $\widetilde{\pi}$ has rel-u.p.e., then so does π.*

Proof. Assume that $\widetilde{\pi}$ has rel-u.p.e. To show π has rel-u.p.e., we shall show that $R_\pi \setminus \Delta(X) \subseteq E(X, G|\pi)$. Let $(x_1, x_2) \in R_\pi \setminus \Delta(X)$, V_1, V_2 be two non-empty disjoint open subsets of X with $x_i \in V_i$ for $i \in [2]$. By Corollary 3.3, it is sufficient to show that (V_1, V_2) is independent with respect to π.

Take $\epsilon \in (0, \frac{1}{2})$ with

$$2^{1-\epsilon^2} \cdot (1-\epsilon^2)^{(1-\epsilon^2)} \cdot (\epsilon^2)^{(\epsilon^2)} > 1. \tag{4.6}$$

We set

$$\widetilde{V}_i = \{\mu \in \mathcal{M}(X) : \mu(V_i) > 1 - \epsilon^4\} \tag{4.7}$$

for $i \in [2]$. Clearly, $\delta_{x_i} \in \widetilde{V}_i$. Since $(\delta_{x_1}, \delta_{x_2}) \in R_{\widetilde{\pi}}$ and $\widetilde{\pi}$ has rel-u.p.e., $(\widetilde{V}_1, \widetilde{V}_2)$ is independent with respect to $\widetilde{\pi}$. That is, there exists $c > 0$ such that for every $F \in \mathcal{F}(G)$, there exists an independence set $E \subseteq F$ of $(\widetilde{V}_1, \widetilde{V}_2)$ with respect to $\widetilde{\pi}$ with $|E| > c|F|$.

Fix an $F \in \mathcal{F}(G)$ and an independence set $E \subseteq F$ of $(\widetilde{V}_1, \widetilde{V}_2)$ with respect to $\widetilde{\pi}$ with $|E| > c|F|$. Then there exists $\nu \in \mathcal{M}(Y)$, such that for every $\sigma \in \{1,2\}^E$,

$$\widetilde{\boldsymbol{V}}_\sigma := \bigg(\bigcap_{g \in E} g^{-1}\widetilde{V}_{\sigma(g)}\bigg) \cap \widetilde{\pi}^{-1}(\nu) \neq \emptyset. \tag{4.8}$$

For every $\sigma \in \{1,2\}^E$, we take $\mu_\sigma \in \widetilde{\boldsymbol{V}}_\sigma$. Then $\mu_\sigma \in g^{-1}\widetilde{V}_{\sigma(g)}$ for every $g \in E$ and $\sigma \in \{1,2\}^E$, which implies $\mu_\sigma(g^{-1}V_\sigma) > 1 - \epsilon^4$ for every $g \in E$ and $\sigma \in \{1,2\}^E$. Thus,

$$\int_X \frac{1}{|E|} \sum_{g \in E} 1_{g^{-1}V_{\sigma(g)}}(x)\, d\mu_\sigma = \frac{1}{|E|} \sum_{g \in E} \mu_\sigma(g^{-1}V_{\sigma(g)}) > 1 - \epsilon^4$$

and $\mu_\sigma(\widetilde{X}_\sigma) > 1 - \epsilon^2$ for every $\sigma \in \{1,2\}^E$, where

$$\widetilde{X}_\sigma = \bigg\{x \in X : \frac{1}{|E|} \sum_{g \in E} 1_{g^{-1}V_{\sigma(g)}}(x) > 1 - \epsilon^2\bigg\}. \tag{4.9}$$

By the inner regular of measure, we can find a closed subset

$$X_\sigma \subseteq \widetilde{X}_\sigma \quad \text{with} \quad \mu_\sigma(X_\sigma) > 1 - \epsilon^2 \tag{4.10}$$

for every $\sigma \in \{1, 2\}^E$. Since π is continuous, for every $\sigma \in \{1, 2\}^E$, we have

$$Y_\sigma := \pi(X_\sigma) \tag{4.11}$$

is a closed subset of Y and

$$\nu(Y_\sigma) = \widetilde{\pi}\mu_\sigma(Y_\sigma) \geq \mu_\sigma(X_\sigma) > 1 - \epsilon^2.$$

Then

$$\int_Y \frac{1}{2^{|E|}} \sum_{\sigma \in \{1,2\}^E} 1_{Y_\sigma}(y)\, d\nu > 1 - \epsilon^2.$$

Put

$$Y' := \bigg\{ y \in Y : \frac{1}{2^{|E|}} \sum_{\sigma \in \{1,2\}^E} 1_{Y_\sigma}(y) > 1 - \epsilon \bigg\}, \tag{4.12}$$

then $\nu(Y') > 1 - \epsilon > \frac{1}{2}$.

Now, we fix a point $y_0 \in Y'$ and set

$$\mathcal{E} := \{\sigma \in \{1, 2\}^E : y_0 \in Y_\sigma\}. \tag{4.13}$$

Then $|\mathcal{E}| > (1 - \epsilon) \cdot 2^{|E|}$ by (4.12). For any $\sigma \in \mathcal{E}$, by (4.13), (4.11), (4.10) and (4.9), there is $x_\sigma \in X_\sigma$ with

$$\frac{1}{|E|} \sum_{g \in E} 1_{g^{-1}V_{\sigma(g)}}(x_\sigma) > 1 - \epsilon^2$$

such that $\pi(x_\sigma) = y_0$. For every $\sigma \in \mathcal{E}$, we set

$$A(\sigma) = \{g \in E : x_\sigma \in g^{-1}V_{\sigma(g)}\},$$

then $|A(\sigma)| > (1 - \epsilon^2)|E|$. Now we define

$$\Omega := \{H \subseteq E : |H| = \lfloor (1 - \epsilon^2) \cdot |E| \rfloor\}$$

and

$$\mathcal{Q}(H) = \{\sigma \in \mathcal{E} : H \subseteq \{g \in E : gx_\sigma \in V_{\sigma(g)}\}\}$$

for every $H \in \Omega$. Then $|\Omega| = \binom{|E|}{\lfloor (1-\epsilon^2)\cdot|E| \rfloor}$ and $\bigcup_{H \in \Omega} \mathcal{Q}(H) = \mathcal{E}$. Thus, there exists $H_0 \in \Omega$ such that $|\mathcal{Q}(H_0)| \geq |\mathcal{E}|/|\Omega| \geq (1 - \epsilon)2^{|E|}/\binom{|E|}{\lfloor (1-\epsilon^2)\cdot|E| \rfloor}$. By (4.5), we can choose $S \subseteq \mathcal{Q}(H_0)$ such that

$$|S| \geq \frac{(1 - \epsilon)2^{|E|}}{2^{|E| - \lfloor (1-\epsilon^2)\cdot|E| \rfloor} \cdot \binom{|E|}{\lfloor (1-\epsilon^2)\cdot|E| \rfloor}} \tag{4.14}$$

and for any $\sigma' \neq \sigma'' \in S$, there exists $g \in H_0$ that satisfies $\sigma'(g) \neq \sigma''(g)$. That is, $|S|_{H_0}| = |S|$. Let $t = 1 - \epsilon^2$ and $\lambda = \log_2(2^t \cdot t^t \cdot (1 - t)^{(1-t)}) > 0$. Then by Stirling's formula, when $|E|$ is large enough, one has

$$\begin{aligned}|S|_{H_0}| &\approx 2^{\lfloor t|E|\rfloor -1} \cdot \sqrt{2\pi t(1-t)|E|} \cdot t^{t|E|} \cdot (1-t)^{(1-t)|E|} \\ &\geq 2^{t|E|-2} \cdot \sqrt{2\pi t(1-t)|E|} \cdot t^{t|E|} \cdot (1-t)^{(1-t)|E|} \\ &\geq (2^t \cdot t^t \cdot (1-t)^{(1-t)})^{|E|} > 2^{\lambda |E|}.\end{aligned}$$

By Lemma 3.4, there exists a subset $H_1 \subseteq H_0$ with $|H_1| > d|H_0|$ such that $S|_{H_1} = \{1, 2\}^{H_1}$, where d is a positive constant independent with E when $|E|$ is large enough. By Remark 3.1, $(\mathcal{V}_1, \mathcal{V}_2)$ is independent with respect to π. This ends our proof. □

5. π *is open if and only if* $\widetilde{\pi}$ *is open*

In this section, we will prove Theorem 1.2. In fact, we have the following result.

THEOREM 5.1. *Let $\pi : X \to Y$ be a surjective continuous map between two compact metrizable spaces, $\widetilde{\pi} : \mathcal{M}(X) \to \mathcal{M}(Y)$ be the induced map of π and $\widetilde{\pi}_n : \mathcal{M}_n(X) \to \mathcal{M}_n(Y)$ be the restriction of $\widetilde{\pi}$ on $\mathcal{M}_n(X)$. Then the following are equivalent:*

(1) *π is open;*
(2) *$\widetilde{\pi}$ is open;*
(3) *$\widetilde{\pi}_n$ is open for each $n \in \mathbb{N}$;*
(4) *$\widetilde{\pi}_n$ is open for some $n \in \mathbb{N}$.*

Proof. (3) ⇒ (4) is trivial. We will show (2) ⇒ (1), (4) ⇒ (1), (1) ⇒ (3) and (1) ⇒ (2). Fix compatible metrics ρ_X for X and ρ_Y for Y.

(2) ⇒ (1). Suppose that $\widetilde{\pi}$ is open. For every non-empty open subset U of X, we shall show that $\pi(U)$ is an open subset of Y. That is, for every $y \in \pi(U)$, there exists $r > 0$ such that $B_{\rho_Y}(y, r) \subseteq \pi(U)$.

Now fix $y_0 \in \pi(U)$. Since U is open, there exist $x_0 \in U$ and $\delta > 0$ with $\pi(x_0) = y_0$ and $\overline{B_{\rho_X}(x_0, \delta)} \subseteq U$. Then by Urysohn's lemma, there exists a countinuous map $f : X \to [0, 1]$ with $f(z) = 1$ when $z \in B_{\rho_X}(x_0, \delta/2)$ and $f(z) = 0$ when $z \in X \backslash B_{\rho_X}(x_0, \delta)$. We set

$$\widetilde{U} := \left\{\mu \in \mathcal{M}(X) : \int f \, d\mu > \frac{2}{3}\right\}.$$

Clearly, $\widetilde{U}$ is an open subset of $\mathcal{M}(X)$ and $\delta_{x_0} \in \widetilde{U}$.

Since $\widetilde{\pi}$ is open, $\widetilde{\pi}(\widetilde{U})$ is an open subset of $\mathcal{M}(Y)$. Note that $\delta_{y_0} = \widetilde{\pi}(\delta_{x_0}) \in \widetilde{\pi}(\widetilde{U})$. Thus, there exists $r > 0$ such that

$$\{\delta_y : \ y \in B_{\rho_Y}(y_0, r)\} \subset \widetilde{\pi}(\widetilde{U}).$$

Then for every $y' \in B_{\rho_Y}(y_0, r)$, there exists $\mu_{y'} \in \widetilde{U}$ such that $\widetilde{\pi}(\mu_{y'}) = \delta_{y'}$. On the one hand, since $\mu_{y'}(\{\pi^{-1}(y')\}) = \delta_{y'}(\{y'\}) = 1$, we have

$$\operatorname{supp}(\mu_{y'}) \subseteq \pi^{-1}(\{y'\}). \tag{5.1}$$

On the other hand, since $\mu_{y'} \in \widetilde{U}$, we have $\int f \, d\mu_{y'} > \frac{2}{3}$. Thus,

$$\emptyset \neq \operatorname{supp}(\mu_{y'}) \cap B_{\rho_X}(x_0, \delta) \subseteq \operatorname{supp}(\mu_{y'}) \cap U.$$

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By (5.1), we have $U \cap \pi^{-1}(\{y'\}) \neq \emptyset$. That is, $y' \in \pi(U)$. Then by the arbitrariness of $y' \in B_{\rho_Y}(y_0, r)$, one has $B_{\rho_Y}(y_0, r) \subseteq \pi(U)$. Thus, $\pi(U)$ is an open subset of Y and π is open.

(4) $\Rightarrow$(1). We assume that there exists $n \in \mathbb{N}$ such that $\widetilde{\pi}_n$ is open. Let U be an open subset of X. We shall show that for every $y \in \pi(U)$, there exists $r > 0$ such that $y \in B_{\rho_Y}(y, r) \subseteq \pi(U)$.

Let $y \in \pi(U)$, there exists $x \in U$ with $\pi(x) = y$. We set

$$\widetilde{U} = \mathcal{M}_n(X) \cap \{\mu \in \mathcal{M}(X) : \mu(U) > 0\}.$$

Here, $\widetilde{U}$ is an open subset of $\mathcal{M}_n(X)$ which contains δ_x. Since $\widetilde{\pi}_n$ is open, $\widetilde{\pi}_n(\widetilde{U})$ is open which contains δ_y. Then there exists $r > 0$ such that $\{\delta_z : \rho_Y(z, y) < r\} \subseteq \widetilde{\pi}_n(\widetilde{U})$. Hence, for every $z \in B_{\rho_Y}(y, r)$, there exist $x_1, x_2, \ldots, x_n \in X$ such that

$$\mu = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i} \in \widetilde{U} \quad \text{and} \quad \widetilde{\pi}_n(\mu) = \delta_z.$$

Then one has $\pi(x_i) = z$ for every $i \in [n]$. Since $\mu \in \widetilde{U}$, there exists $i_0 \in [n]$ with $x_{i_0} \in U$. That is, $z = \pi(x_{i_0}) \in \pi(U)$. Hence, $B_{\rho_Y}(y, r) \subseteq \pi(U)$. This implies π is open.

(1) $\Rightarrow$(3). Now we assume that π is open. Let $n \in \mathbb{N}$ and $\widetilde{U}$ be an open subset of $\mathcal{M}_n(X)$. We shall show that for every $\nu \in \widetilde{\pi}_n(\widetilde{U}) \subseteq \mathcal{M}_n(Y)$, there exists an open neighbourhood of ν in $\mathcal{M}_n(Y)$ contained in $\widetilde{\pi}_n(\widetilde{U})$.

For any $\nu \in \widetilde{\pi}_n(\widetilde{U}) \subseteq \mathcal{M}_n(Y)$, there exist positive integers h, $k_1, k_2, \ldots, k_h$ with $\sum_{i\in[h]} k_i = n$ and pairwise distinct $y_1, y_2, \ldots, y_h \in Y$ such that

$$\nu = \frac{1}{n}(k_1\delta_{y_1} + k_2\delta_{y_2} + \cdots + k_h\delta_{y_h}).$$

Since $\nu \in \widetilde{\pi}_n(\widetilde{U})$, there exists $\mu \in \widetilde{U} \subseteq \mathcal{M}_n(X)$ such that $\widetilde{\pi}_n(\mu) = (1/n)\sum_{i=1}^{h} k_i\delta_{y_i}$. Then for every $i \in [h]$, there exist integers ℓ_i, $m_{i,j}$, and points $x_{i,j} \in X$ for $j \in [\ell_i]$ satisfying:

(a) $m_{i,1} + m_{i,2} + \cdots + m_{i,\ell_i} = k_i$ for every $i \in [h]$;

(b) $x_{i,1}, x_{i,2}, \ldots, x_{i,\ell_i}$ are pairwise distinct and $\pi(x_{i,j}) = y_i$ for every $i \in [h]$ and $j \in [\ell_i]$;

(c) $\mu = (1/n)\sum_{i\in[h]}\sum_{j\in[\ell_i]} m_{i,j}\delta_{x_{i,j}}$.

Since $\widetilde{U}$ is an open neighbourhood of μ, there exists $r_0 > 0$ such that if $z_{i,j}^1, z_{i,j}^2, \ldots, z_{i,j}^{m_{i,j}} \in B_{\rho_X}(x_{i,j}, r_0)$ for every $i \in [h]$, $j \in [\ell_i]$, then

$$\frac{1}{n}\sum_{i\in[h]}\sum_{j\in[\ell_i]}\bigg(\sum_{t\in[m_{i,j}]} \delta_{z_{i,j}^t}\bigg) \in \widetilde{U}. \tag{5.2}$$

Note that $y_1, y_2, \ldots, y_h$ are pairwise distinct, then there exists $\delta > 0$ such that $\{B_{\rho_Y}(y_i, \delta)\}_{i\in[h]}$ are pairwise disjoint. By item (b) and the continuity of π, there exists $r \in (0, r_0)$ such that

$$\pi(B_{\rho_X}(x_{i,j}, r)) \subseteq B_{\rho_X}(y_i, \delta) \quad \text{and} \quad B_{\rho_X}(x_{i,t}, r) \cap B_{\rho_X}(x_{i,j}, r) = \emptyset \tag{5.3}$$

for every $i \in [h]$ and different $j, t \in [\ell_i]$.

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Since π is open, $\bigcap_{j=1}^{\ell_i} \pi(B_{\rho_X}(x_{i,j}, r))$ for every $i \in [h]$ is open. We set

$$\widetilde{V} := \bigg\{\tau \in \mathcal{M}_n(Y) : \tau\bigg(\bigcap_{j=1}^{\ell_i} \pi(B_{\rho_X}(x_{i,j}, r))\bigg) > \frac{k_i}{n} - \frac{1}{2n},\ i \in [h]\bigg\}.$$

It is an open subset of $\mathcal{M}_n(Y)$. Moreover, for every $i_0 \in [h]$,

$$\begin{aligned}\nu\bigg(\bigcap_{j=1}^{\ell_{i_0}} \pi(B_{\rho_X}(x_{i_0,j}, r))\bigg) &= \frac{1}{n}\sum_{i\in[h]} k_i\delta_{y_i}\bigg(\bigcap_{j=1}^{\ell_{i_0}} \pi(B_{\rho_X}(x_{i_0,j}, r))\bigg)\\ &= \frac{1}{n}k_{i_0}\delta_{y_{i_0}}\bigg(\bigcap_{j=1}^{\ell_{i_0}} \pi(B_{\rho_X}(x_{i_0,j}, r))\bigg)\\ &= \frac{1}{n}k_{i_0} > \frac{k_{i_0}}{n} - \frac{1}{2n}.\end{aligned}$$

Thus, $\nu \in \widetilde{V}$. Next, we shall show that $\widetilde{V} \subseteq \widetilde{\pi}_n(\widetilde{U})$.

Now fix any $\tau \in \widetilde{V} \subseteq \mathcal{M}_n(Y)$. We have $\tau = (1/n)\sum_{s=1}^{n} \delta_{u_s}$ for some $u_s \in Y$. For every $i \in [h]$, we set

$$L(i) := \bigg\{s \in [n] : u_s \in \bigcap_{j=1}^{\ell_i} \pi(B_{\rho_X}(x_{i,j}, r))\bigg\}. \tag{5.4}$$

By $\tau \in \widetilde{V}$, one has

$$|L(i)| = n\cdot\tau\bigg(\bigcap_{j=1}^{\ell_i} \pi(B_{\rho_X}(x_{i,j}, r))\bigg) > k_i - \frac{1}{2} \tag{5.5}$$

for every $i \in [h]$. Since $|L(i)| \in \mathbb{N}$, by (5.5), $|L(i)| \geq k_i$. We note that $L(i)$, $i \in [h]$ are pairwise disjoint since $\bigcap_{j=1}^{\ell_i} \pi(B_{\rho_X}(x_{i,j}, r))$, $i \in [h]$ are pairwise disjoint. Moreover, by $\sum_{i\in[h]} k_i = n$, one has $|L(i)| = k_i$ for every $i \in [h]$. Hence,

$$\bigcup_{i\in[h]} L(i) = \bigsqcup_{i\in[h]} L(i) = [n] \quad\text{and}\quad \tau = \frac{1}{n}\sum_{i\in[h]}\bigg(\sum_{s\in L(i)} \delta_{u_s}\bigg). \tag{5.6}$$

For every $i \in [h]$, since $|L(i)| = k_i \overset{(a)}{=} \sum_{j\in[\ell_i]} m_{i,j}$, we can rewrite $L(i) = \{s_1, s_2, \ldots, s_{k_i}\}$. For every $i \in [h]$ and $j \in [\ell_i]$, we denote $R_i(j) = \sum_{t=1}^{j} m_{i,t}$ and $R_i(0) = 0$. Then $R_i(\ell_i) = k_i$. By (5.4), for every $j \in [\ell_i]$ and integer q with $R_i(j-1)+1 \leq q \leq R_i(j)$, there exists $x'_{i,q} \in B(x_{i,j}, r)$ such that $\pi(x'_{i,q}) = u_{s_q}$. Then by (5.2), one has

$$\mu' := \frac{1}{n}\sum_{i\in[h]}\sum_{j\in[\ell_i]}\sum_{q=R_i(j-1)+1}^{R(j)} \delta_{x'_{i,q}} \in \widetilde{U}$$

and

$$\widetilde{\pi}_n(\mu') = \frac{1}{n}\sum_{i\in[h]}\sum_{j\in[\ell_i]}\sum_{q=R_i(j-1)+1}^{R_i(j)} \delta_{u_{sq}} = \frac{1}{n}\sum_{i\in[h]}\sum_{q\in[k_i]} \delta_{u_{sq}}$$
$$= \frac{1}{n}\sum_{i\in[h]}\sum_{s\in L(i)} \delta_{u_s} \overset{(5.6)}{=} \tau.$$

This implies $\widetilde{V} \subset \widetilde{\pi}_n(\widetilde{U})$. Hence, $\widetilde{\pi}_n(\widetilde{U})$ is an open subset of $\mathcal{M}_n(Y)$ and $\widetilde{\pi}_n$ is open.

(1) $\Rightarrow$ (2). Now we assume that π is open. Let $\widetilde{U}$ be an open subset of $\mathcal{M}(X)$. We shall show $\widetilde{\pi}(\widetilde{U})$ is open in $\mathcal{M}(Y)$.

For every $\nu \in \widetilde{\pi}(\widetilde{U})$, there exists $\mu \in \widetilde{U}$ such that $\nu = \widetilde{\pi}(\mu)$. Next we shall show that there exists $\delta > 0$ small enough such that if we set

$$\widetilde{V} := \{\tau \in \mathcal{M}(Y) : d_P(\nu, \tau) < \delta\},$$

where

$$d_P(\tau, \nu) := \inf\{\delta > 0 : \tau(A) \le \nu(A^\delta) + \delta \text{ and } \nu(A) \le \tau(A^\delta) + \delta \text{ for all } A \in \mathcal{B}_Y\},$$

then $\widetilde{V}$ is an open neighbourhood of ν contained in $\widetilde{\pi}(\widetilde{U})$.

Since $\widetilde{U}$ is open, by Proposition 2.2, there exist $k \in \mathbb{N}$ and an open set of the form $\mathbb{W}(U_1, U_2, \ldots, U_k;\ \eta_1, \eta_2, \ldots, \eta_k)$ of $\mathcal{M}(X)$, where $U_1, U_2, \ldots, U_k$ are disjoint non-empty open subsets of X and $\eta_1, \eta_2, \ldots, \eta_k$ are positive real numbers with $\eta_1 + \eta_2 + \cdots + \eta_k < 1$, such that

$$\mu \in \mathbb{W}(U_1, U_2, \ldots, U_k; \eta_1, \eta_2, \ldots, \eta_k) \subset \overline{\mathbb{W}(U_1, U_2, \ldots, U_k; \eta_1, \eta_2, \ldots, \eta_k)} \subset \widetilde{U}.$$

For any $t_1, t_2 \in \{0, 1\}^{[k]}$, we denote $t_1 > t_2$ if $t_1 \neq t_2$ and $t_1(i) \ge t_2(i)$ for every $i \in [k]$. For every $\sigma \in \{0, 1\}^{[k]}$, we set

$$V_\sigma := \bigcap_{\substack{i\in[k]\\ \sigma(i)=1}} \pi(U_i), \qquad V'_\sigma := V_\sigma \setminus \bigcup_{\substack{\alpha\in\{0,1\}^{[k]}\\ \alpha>\sigma}} V_\alpha$$

and

$$\mathcal{E} := \{t \in \{0, 1\}^{[k]} : \nu(V'_t) > 0\}. \tag{5.7}$$

Recall that for any subset A of Y and $a > 0$, we denote $A^a = \{y \in Y : \rho_Y(y, A) < a\}$, where ρ_Y is the compatible metric on Y. For every $i \in [k]$, since π is open and U_i is open in X, then $\pi(U_i)$ is an open subset of Y. Then by inner regularity, there exist $\varepsilon > 0$ small enough, $\delta \in (0, \varepsilon)$ and compact subsets C_i of Y for $i \in [k]$ such that:

(c1) $\mu(U_i) > \eta_i + 6^k\varepsilon$ for every $i \in [k]$;

(c2) $\nu(V'_\sigma) > 5k\varepsilon$, for every $\sigma \in \mathcal{E}$;

(c3) $C_i \subseteq C_i^\delta \subseteq C_i^{2\delta} \subseteq \pi(U_i)$ for every $i \in [k]$;

(c4) $\nu(C_i) > \nu(\pi(U_i)) - \varepsilon$ for $i \in [k]$.

Now we set

$$\widetilde{V} := \{\tau \in \mathcal{M}(Y) : d_P(\nu, \tau) < \delta\}.$$

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Clearly, $\widetilde{V}$ is an open subset of $\mathcal{M}(Y)$ containing ν. Now it is sufficient to prove that $\widetilde{V} \subset \widetilde{\pi}(\widetilde{U})$.

For every $\sigma \in \{0, 1\}^{[k]}$, we set

$$C_\sigma(\delta) := \bigcap_{\substack{i\in[k]\\ \sigma(i)=1}} C_i^\delta \quad \text{and} \quad C'_\sigma := C_\sigma(\delta)\setminus \bigcup_{\substack{\alpha\in\{0,1\}^{[k]}\\ \alpha>\sigma}} C_\alpha(\delta).$$

Then for every $\sigma \in \{0, 1\}^{[k]}$, by items (c3) and (c4), we have

$$\begin{aligned}\nu(V_\sigma) &= \nu\bigg(\bigcap_{\substack{i\in[k]\\ \sigma(i)=1}} \pi(U_i)\bigg)\\ &\overset{(c3)}{=} \nu\bigg(\bigcap_{\substack{i\in[k]\\ \sigma(i)=1}} ((\pi(U_i)\setminus C_i)\cup C_i)\bigg) \overset{(c4)}{\le} \nu\bigg(\bigcap_{\substack{i\in[k]\\ \sigma(i)=1}} C_i\bigg) + k\varepsilon. \end{aligned} \tag{5.8}$$

We note that for any $t_1 \neq t_2 \in \{0, 1\}^{[k]}$, one has $C'_{t_1} \cap C'_{t_2} = \emptyset$. In fact, if we define $t_1 \vee t_2 \in \{0, 1\}^{[k]}$ by

$$(t_1 \vee t_2)(i) = \max\{t_1(i), t_2(i)\}$$

for every $i \in [k]$, then it is clear that $t_1 \vee t_2 > t_1$ or $t_1 \vee t_2 > t_2$. Without loss of generality, we can assume $t_1 \vee t_2 > t_1$, then $C'_{t_1} \subseteq C_{t_1}(\delta)\setminus C_{t_1\vee t_2}(\delta)$. However,

$$C'_{t_1} \cap C'_{t_2} \subseteq C_{t_1}(\delta) \cap C_{t_2}(\delta) = C_{t_1\vee t_2}(\delta).$$

Hence, $C'_{t_1} \cap C'_{t_2} = \emptyset$.

Now for any fixed $\tau \in \widetilde{V}$, we shall show $\tau \in \widetilde{\pi}(\widetilde{U})$. By $d_P(\nu, \tau) < \delta$, one has $\tau(A^\delta) \geq \nu(A) - \delta$ for every $A \in \mathcal{B}_Y$. Then for every $\sigma \in \mathcal{E}$,

$$\begin{aligned}\tau(C_\sigma(\delta)) &= \tau\bigg(\bigcap_{\substack{i\in[k]\\ \sigma(i)=1}} C_i^\delta\bigg) \geq \tau\bigg(\bigg(\bigcap_{\substack{i\in[k]\\ \sigma(i)=1}} C_i\bigg)^\delta\bigg)\\ &\geq \nu\bigg(\bigcap_{\substack{i\in[k]\\ \sigma(i)=1}} C_i\bigg) - \delta \overset{(5.8)}{\geq} \nu(V_\sigma) - k\varepsilon - \delta. \end{aligned} \tag{5.9}$$

Moreover, for every $\sigma \in \{0, 1\}^{[k]}$,

$$\begin{aligned}\bigg(\bigcup_{\substack{\alpha\in\{0,1\}^{[k]}\\ \alpha>\sigma}} C_\alpha(\delta)\bigg)^\delta &= \bigcup_{\substack{\alpha\in\{0,1\}^{[k]}\\ \alpha>\sigma}} (C_\alpha(\delta))^\delta = \bigcup_{\substack{\alpha\in\{0,1\}^{[k]}\\ \alpha>\sigma}} \bigg(\bigcap_{\substack{i\in[k]\\ \alpha(i)=1}} C_i^\delta\bigg)^\delta\\ &\subseteq \bigg(\bigcup_{\substack{\alpha\in\{0,1\}^{[k]}\\ \alpha>\sigma}} \bigg(\bigcap_{\substack{i\in[k]\\ \alpha(i)=1}} C_i^{2\delta}\bigg)\bigg) \overset{(c3)}{\subseteq} \bigg(\bigcup_{\substack{\alpha\in\{0,1\}^{[k]}\\ \alpha>\sigma}} \bigg(\bigcap_{\substack{i\in[k]\\ \alpha(i)=1}} \pi(U_i)\bigg)\bigg)\\ &= \bigcup_{\substack{\alpha\in\{0,1\}^{[k]}\\ \alpha>\sigma}} V_\alpha. \end{aligned} \tag{5.10}$$

Since $d_P(\nu, \tau) < \delta$, one has

$$\tau(A) \le \nu(A^\delta) + \delta \quad \text{for every } A \in \mathcal{B}_Y. \tag{5.11}$$

Note that for every $\alpha, \sigma \in \{0,1\}^{[k]}$ with $\alpha > \sigma$, one has $C_\alpha(\delta) \subseteq C_\sigma(\delta)$ and $V_\alpha \subseteq V_\sigma$. Then for every $\sigma \in \mathcal{E}$,

$$\begin{aligned}
\tau(C'_\sigma) &= \tau(C_\sigma(\delta)) - \tau\bigg(\bigcup_{\substack{\alpha\in\{0,1\}^{[k]}\\ \alpha>\sigma}} C_\alpha(\delta)\bigg) \\
&\overset{(5.9)}{\ge} \nu(V_\sigma) - k\varepsilon - \delta - \tau\bigg(\bigcup_{\substack{\alpha\in\{0,1\}^{[k]}\\ \alpha>\sigma}} C_\alpha(\delta)\bigg) \\
&\overset{(5.11)}{\ge} \nu(V_\sigma) - k\varepsilon - \delta - \nu\bigg(\bigg(\bigcup_{\substack{\alpha\in\{0,1\}^{[k]}\\ \alpha>\sigma}} C_\alpha(\delta)\bigg)^\delta\bigg) - \delta \\
&\overset{(5.10)}{\ge} \nu(V_\sigma) - k\varepsilon - 2\delta - \nu\bigg(\bigcup_{\substack{\alpha\in\{0,1\}^{[k]}\\ \alpha>\sigma}} V_\alpha\bigg) \\
&= \nu(V'_\sigma) - k\varepsilon - 2\delta \ge \nu(V'_\sigma) - 3k\varepsilon > 0.
\end{aligned} \tag{5.12}$$

By $\overline{\bigcup_{n\in N} \mathcal{M}_n(Y)} = \mathcal{M}(Y)$, there exist $\tau_n = (1/n)\sum_{j=1}^n \delta_{y_{n,j}} \in \mathcal{M}_n(Y)$ for $n \in \mathbb{N}$ and some $y_{n,j} \in Y$, $j \in [n]$, such that $\tau_n \to \tau$ as $n \to \infty$. Moreover, since $\widetilde{V}$ is open in $\mathcal{M}(Y)$, we can find $N_0 \in \mathbb{N}$ such that $\tau_n \in \widetilde{V}$ for $n \ge N_0$.

Let $n \ge N_0$. For every $\sigma \in \{0,1\}^{[k]}$, we set

$$S^n_\sigma := \{h \in [n] : y_{n,h} \in C'_\sigma\}.$$

Since $\tau_n \in \widetilde{V}$, by (5.12) and recall that for any $t_1 \ne t_2 \in \{0,1\}^{[k]}$, one has $C'_{t_1} \cap C'_{t_2} = \emptyset$, then:

(i) $S^n_{t_1} \cap S^n_{t_2} = \emptyset$ for any $t_1 \ne t_2 \in \{0,1\}^{[k]}$;
(ii) $|S^n_\sigma| \ge n(\nu(V'_\sigma) - 3k\varepsilon)$ for every $\sigma \in \mathcal{E}$, where $\mathcal{E}$ is defined as (5.7).

Now, for every $\sigma \in \{0,1\}^{[k]}$ and $i \in [k]$, we set

$$U_{i,\sigma} := U_i \cap \pi^{-1}(V'_\sigma) \quad \text{and} \quad a_{i,\sigma} := \mu(U_{i,\sigma}). \tag{5.13}$$

Fix any $\sigma \in \{0,1\}^{[k]}$. We can rewrite $\{i \in [k] : \sigma(i) = 1\}$ as $\{i_1 < i_2 < \cdots < i_q\}$ for some $q \in \mathbb{N}$. For i_1, we choose arbitrarily a subset S^n_{σ,i_1} of S^n_σ with $|S^n_{\sigma,i_1}| = \lfloor a_{i_1,\sigma}/\sum_{\ell\in[k],\sigma(\ell)=1} a_{\ell,\sigma}|S^n_\sigma|\rfloor$, where we note: $\frac{0}{0} = 0$. For i_2, we choose arbitrarily a subset S^n_{σ,i_2} of $S^n_\sigma \setminus S^n_{\sigma,i_1}$ with $|S^n_{\sigma,i_2}| = \lfloor a_{i_2,\sigma}/\sum_{\ell\in[k],\sigma(\ell)=1} a_{\ell,\sigma}|S^n_\sigma|\rfloor$. We continue inductively obtaining

$$S^n_{\sigma,i_j} \subseteq S^n_\sigma \setminus (S^n_{\sigma,i_1} \cup S^n_{\sigma,i_2} \cup \cdots \cup S^n_{\sigma,i_{j-1}})$$

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for $j = 3, 4, \ldots, q-1$, with $|S^n_{\sigma,i_j}| = \lfloor a_{i_j,\sigma}/\sum_{\ell\in[k],\sigma(\ell)=1} a_{\ell,\sigma}|S^n_\sigma|\rfloor$. We set $S^n_{\sigma,i_q} = S^n_\sigma \setminus (\bigcup_{j=1}^{q-1} S^n_{\sigma,i_j})$. Additionally, we note that

$$y_{n,h} \in C'_\sigma \subseteq \bigcap_{\substack{i\in[k]\\ \sigma(i)=1}} \pi(U_i) = \bigcap_{\ell=1}^{q} \pi(U_{i_\ell})$$

for every $h \in S^n_\sigma$. Then we have the following properties for $S^n_{\sigma,i}$, $i \in [k]$.

(i*) $|S^n_{\sigma,i}| \geq \lfloor a_{i,\sigma}/\sum_{\ell\in[k],\sigma(\ell)=1} a_{\ell,\sigma}|S^n_\sigma|\rfloor$ for every $i \in [k]$ with $\sigma(i) = 1$.

(ii*) For every $i \in [k]$ with $\sigma(i) = 1$, if $h \in S^n_{\sigma,i}$, then there exists $x^\sigma_{n,h} \in U_i$ satisfying $\pi(x^\sigma_{n,h}) = y_{n,h}$.

(iii*) $S^n_{\sigma,i'} \cap S^n_{\sigma,i''} = \emptyset$ for every $i' \neq i'' \in \{i \in [k] : \sigma(i) = 1\}$ and $\bigcup_{i\in[k],\sigma(i)=1} S^n_{\sigma,i} = S^n_\sigma$.

Since π is surjective, for every $h' \in S^n_0 := [n]\setminus(\bigcup_{\sigma\in\{0,1\}^{[k]}} S^n_\sigma)$, there exists $x_{n,h'} \in X$ such that $\pi(x_{n,h'}) = y_{n,h'}$. Now we set

$$\begin{aligned}\mu_n &:= \frac{1}{n}\bigg(\sum_{\sigma\in\{0,1\}^{[k]}} \sum_{h\in S^n_\sigma} \delta_{x^\sigma_{n,h}} + \sum_{h'\in S^n_0} \delta_{x_{n,h'}}\bigg)\\ &\overset{\text{(iii*)}}{=} \frac{1}{n}\bigg(\sum_{\sigma\in\{0,1\}^{[k]}} \sum_{\substack{i\in[k]\\ \sigma(i)=1}} \sum_{h\in S^n_{\sigma,i}} \delta_{x^\sigma_{n,h}} + \sum_{h'\in S^n_0} \delta_{x_{n,h'}}\bigg).\end{aligned} \tag{5.14}$$

Clearly, $\widetilde{\pi}(\mu_n) = \tau_n$. We claim that $\mu_n(U_{i_0}) > \eta_{i_0}$ for every $i_0 \in [k]$ when n is sufficiently large. Once it is true, we have

$$\mu_n \in \mathbb{W}(U_1, U_2, \ldots, U_k; \eta_1, \eta_2, \ldots, \eta_k).$$

Then we can find a sequence $n_1 < n_2 < \cdots$ such that $\lim_{i\to\infty} \mu_{n_i} = \mu'$ for some $\mu' \in \mathcal{M}(X)$. Thus,

$$\mu' \in \overline{\mathbb{W}(U_1, U_2, \ldots, U_k; \eta_1, \eta_2, \ldots, \eta_k)} \subset \widetilde{U}$$

and $\widetilde{\pi}(\mu') = \lim_{i\to\infty} \widetilde{\pi}(\mu_{n_i}) = \lim_{i\to\infty} \tau_{n_i} = \tau$. By the arbitrariness of τ, one has $\widetilde{V} \subseteq \widetilde{\pi}(\widetilde{U})$. This will end our proof.

Now, we shall show the claim: $\mu_n(U_{i_0}) > \eta_{i_0}$ for every $i_0 \in [k]$ when n is sufficiently large. To show that, for any fixed $i_0 \in \{1, 2, \ldots, k\}$, we need the following facts.

Fact 1: $\sum_{\sigma\in\mathcal{E},\sigma(i_0)=1} \mu(U_{i_0} \cap \pi^{-1}(V'_\sigma)) = \mu(U_{i_0} \cap \bigcup_{\sigma\in\mathcal{E},\sigma(i_0)=1} \pi^{-1}(V'_\sigma))$. In fact, for any $t_1 \neq t_2 \in \{0, 1\}^{[k]}$, if $y \in V'_{t_1} \cap V'_{t_2} \subseteq V_{t_1} \cap V_{t_2}$, then $y \in V_{t_1\vee t_2}$. Since $t_1 \vee t_2 > t_1$ or t_2, one has $y \notin V'_{t_1}$ or $y \notin V'_{t_2}$, which is a contradiction of $y \in V'_{t_1} \cap V'_{t_2}$. Hence, $V'_{t_1} \cap V'_{t_2} = \emptyset$. Then Fact 1 follows.

Fact 2: $\nu((\bigcup_{\substack{\sigma\in\mathcal{E}\\ \sigma(i_0)=1}} V'_\sigma)\Delta(\bigcup_{\substack{\sigma\in\{0,1\}^{[k]}\\ \sigma(i_0)=1}} V_\sigma)) = 0$, where $A\Delta B$ denotes $(A \setminus B) \cup (B \setminus A)$ for every $A, B \in \mathcal{B}_Y$. In fact, it is clear that

$$\bigcup_{\substack{\sigma\in\mathcal{E}\\ \sigma(i_0)=1}} V'_\sigma \subseteq \bigcup_{\substack{\sigma\in\{0,1\}^{[k]}\\ \sigma(i_0)=1}} V_\sigma. \tag{5.15}$$

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Since $\sigma \in \{0,1\}^{[k]} \backslash \mathcal{E}$ implies $\nu(V'_\sigma) = 0$, one has

$$\nu\bigg(\bigcup_{\substack{\sigma\in\mathcal{E}\\ \sigma(i_0)=1}} V'_\sigma\bigg) = \nu\bigg(\bigcup_{\substack{\sigma\in\{0,1\}^{[k]}\\ \sigma(i_0)=1}} V'_\sigma\bigg). \tag{5.16}$$

Clearly, $\bigcup_{\substack{\sigma\in\{0,1\}^{[k]}\\ \sigma(i_0)=1}} V_\sigma \supseteq \bigcup_{\substack{\sigma\in\{0,1\}^{[k]}\\ \sigma(i_0)=1}} V'_\sigma$. Moreover, for any given $x \in \bigcup_{\substack{\sigma\in\{0,1\}^{[k]}\\ \sigma(i_0)=1}} V_\sigma$, if we define σ' as

$$\sigma'(i) = \max\{t(i) : t \in \{0,1\}^{[k]} \text{ with } x \in V_t\}$$

for every $i \in [k]$, then $\sigma'(i_0) = 1$ and $x \in V'_{\sigma'} \subseteq \bigcup_{\substack{\sigma\in\{0,1\}^{[k]}\\ \sigma(i_0)=1}} V'_\sigma$. Hence,

$$\bigcup_{\substack{\sigma\in\{0,1\}^{[k]}\\ \sigma(i_0)=1}} V_\sigma = \bigcup_{\substack{\sigma\in\{0,1\}^{[k]}\\ \sigma(i_0)=1}} V'_\sigma. \tag{5.17}$$

By (5.15), (5.16) and (5.17), Fact 2 holds.

Fact 3: For every $\sigma \in \mathcal{E}$, $\sum_{\ell\in[k],\sigma(\ell)=1} a_{\ell,\sigma} \le \nu(V'_\sigma)$. Note that $U_1, U_2, \ldots, U_k$ are disjoint. Then by (5.13), we have

$$\begin{aligned}
\sum_{\ell\in[k],\sigma(\ell)=1} a_{\ell,\sigma} &\overset{(5.13)}{=} \sum_{\ell\in[k],\sigma(\ell)=1} \mu(U_{\ell,\sigma}) \\
&\overset{(5.13)}{=} \sum_{\ell\in[k],\sigma(\ell)=1} \mu(U_\ell \cap \pi^{-1}(V'_\sigma)) = \mu\bigg(\bigg(\bigcup_{\ell\in[k],\sigma(\ell)=1} U_\ell\bigg) \cap \pi^{-1}(V'_\sigma)\bigg) \\
&\le \mu(\pi^{-1}(V'_\sigma)) = \nu(V'_\sigma).
\end{aligned}$$

Thus, Fact 3 holds.

Now by Facts 1–3, we have

$$\begin{aligned}
\sum_{\sigma\in\mathcal{E},\sigma(i_0)=1} a_{i_0,\sigma} &\overset{(5.13)}{=} \sum_{\sigma\in\mathcal{E},\sigma(i_0)=1} \mu(U_{i_0} \cap \pi^{-1} V'_\sigma) \\
&\overset{\text{(Fact 1)}}{=} \mu\bigg(U_{i_0} \cap \bigcup_{\sigma\in\mathcal{E},\sigma(i_0)=1} \pi^{-1}(V'_\sigma)\bigg) \\
&\overset{\text{(Fact 2)}}{=} \mu\bigg(U_{i_0} \cap \pi^{-1}\bigg(\bigcup_{\substack{\sigma\in\{0,1\}^{[k]}\\ \sigma(i_0)=1}} V_\sigma\bigg)\bigg).
\end{aligned} \tag{5.18}$$

We define $t \in \{0,1\}^{[k]}$ as $t(i_0) = 1$ and $t(i) = 0$ for each $i \in [k]\backslash\{i_0\}$. Then $\pi(U_{i_0}) = V_t \subseteq \bigcup_{\substack{\sigma\in\{0,1\}^{[k]}\\ \sigma(i_0)=1}} V_\sigma$. Thus, $U_{i_0} \subseteq \pi^{-1}(\bigcup_{\substack{\sigma\in\{0,1\}^{[k]}\\ \sigma(i_0)=1}} V_\sigma)$ and by (5.18), we have

$$\sum_{\sigma\in\mathcal{E},\sigma(i_0)=1} a_{i_0,\sigma} = \mu(U_{i_0}). \tag{5.19}$$

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Then for any $n > N_0$, we have

$$\begin{aligned}\mu_n(U_{i_0}) &\overset{(5.14)}{\geq} \frac{1}{n}\sum_{\sigma\in\{0,1\}^{[k]}}\sum_{\substack{i\in[k]\\ \sigma(i)=1}}\sum_{h\in S^n_{\sigma,i}}\delta_{x^\sigma_{n,h}}(U_{i_0}) \geq \frac{1}{n}\sum_{\sigma\in\mathcal{E}}\sum_{\substack{i\in[k]\\ \sigma(i)=1}}\sum_{h\in S^n_{\sigma,i}}\delta_{x^\sigma_{n,h}}(U_{i_0})\\ &\overset{(\text{ii*})}{\geq} \frac{1}{n}\sum_{\substack{\sigma\in\mathcal{E}\\ \sigma(i_0)=1}}|S^n_{\sigma,i_0}| \overset{(\text{i*})}{\geq} \frac{1}{n}\sum_{\substack{\sigma\in\mathcal{E}\\ \sigma(i_0)=1}}\left\lfloor\frac{a_{i_0,\sigma}}{\sum_{\ell\in[k],\sigma(\ell)=1}a_{\ell,\sigma}}|S^n_\sigma|\right\rfloor\\ &\geq \frac{1}{n}\left(\sum_{\substack{\sigma\in\mathcal{E}\\ \sigma(i_0)=1}}\left(\frac{a_{i_0,\sigma}}{\sum_{\ell\in[k],\sigma(\ell)=1}a_{\ell,\sigma}}|S^n_\sigma|\right)\right) - \frac{2^k}{n}\\ &\overset{(\text{ii})}{\geq} \frac{1}{n}\left(\sum_{\substack{\sigma\in\mathcal{E}\\ \sigma(i_0)=1}}\left(\frac{a_{i_0,\sigma}}{\sum_{\ell\in[k],\sigma(\ell)=1}a_{\ell,\sigma}}n(\nu(V'_\sigma)-3k\varepsilon)\right)\right) - \frac{2^k}{n}\\ &\geq \sum_{\substack{\sigma\in\mathcal{E}\\ \sigma(i_0)=1}}\left(\frac{a_{i_0,\sigma}}{\sum_{\ell\in[k],\sigma(\ell)=1}a_{\ell,\sigma}}\nu(V'_\sigma)\right) - 2^k\cdot 3k\varepsilon - \frac{2^k}{n}\\ &\overset{(\text{Fact 3})}{\geq} \sum_{\substack{\sigma\in\mathcal{E}\\ \sigma(i_0)=1}} a_{i_0,\sigma} - 2^k\cdot 3k\varepsilon - \frac{2^k}{n} \overset{(5.19)}{=} \mu(U_{i_0}) - 2^k\cdot 3k\varepsilon - \frac{2^k}{n}.\end{aligned}$$

Then by letting $n\to\infty$, for every $i_0\in[k]$ since $\mu(U_{i_0}) > \eta_{i_0} + 6^k\varepsilon$ by (c1), we have $\mu_n(U_{i_0}) > \eta_{i_0}$. This ends the proof of the claim. □

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A. *Appendix. Proof of Theorem 2.3*

Let $\pi : (X, G) \to (Y, G)$ be a factor map between two G-systems. For any $n \in \mathbb{N}$ and a tuple $\mathcal{V} = (V_1, V_2, \ldots, V_n)$ of subsets of X, recall that we denote by $\mathcal{P}^\pi_\mathcal{V}$ the set of all independence sets of $\mathcal{V}$ with respect to π.

Identifying subsets of G with elements of $\{0, 1\}^G$ by taking indicator functions, we may think of $\mathcal{P}^\pi_\mathcal{V}$ as a subset of $\{0, 1\}^G$. Endow $\{0, 1\}^G$ with the shift given by

$$(s\sigma)(t) = \sigma(ts)$$

for all $\sigma \in \{0, 1\}^G$ and $s, t \in G$. It is clear that $\mathcal{P}^\pi_\mathcal{V}$ is shift-invariant. Moreover, when $V_1, V_2, \ldots, V_n$ are closed subsets of X, $\mathcal{P}^\pi_\mathcal{V}$ is also closed in $\{0, 1\}^G$.

We say a closed and shift-invariant subset $\mathcal{P} \subseteq \{0, 1\}^G$ *has positive density* if there exists constant $c > 0$ such that for every non-empty subset F of G, there exists $I \in \mathcal{P}$ with $I \subseteq F$ such that $|I| > c|F|$. Then by Corollary 3.3, we immediately have the following property.

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PROPOSITION A.1. *Let $\pi : (X, G) \to (Y, G)$ be a factor map between two G-systems, $(x_1, x_2) \in X \times X\backslash\Delta(X)$. Then $(x_1, x_2) \in E(X, G|\pi)$ if and only if for any disjoint open subsets V_1, V_2 of X with $x_i \in V_i$ for $i = 1, 2$, $\mathcal{P}^{\pi}_{\{V_1,V_2\}}$ has positive density.*

The following lemma is useful.

LEMMA A.2. [25, Lemma 12.6] *Let A be a closed subset of X. Then $\mathcal{P}_A := \{I \subseteq G : \bigcap_{g\in I} g^{-1}A \neq \emptyset\}$ has positive density if and only if there exists $\mu \in \mathcal{M}(X, G)$ with $\mu(A) > 0$.*

The following lemma is proved when $G = \mathbb{Z}$ in [19, Proposition 3.9]. We omit the proof.

LEMMA A.3. *Let $\pi : (X, G) \to (Z, G)$, $\pi_1 : (X, G) \to (Y, G)$ and $\pi_2 : (Y, G) \to (Z, G)$ be three factor maps such that $\pi = \pi_2 \cdot \pi_1$. Then π has rel-u.p.e. implies π_2 has rel-u.p.e.*

For a factor map $\pi : (X, \mathbb{Z}) \to (Y, \mathbb{Z})$ between two $\mathbb{Z}$-systems, the authors in [19] proved that if π has rel-u.p.e., then $\mathrm{supp}(Y) = Y$ implies $\mathrm{supp}(X) = X$ (see [19, Theorem 5.4]). For discrete countable amenable group G, we have the same result.

PROPOSITION A.4. *Let $\pi : (X, G) \to (Y, G)$ be a factor map between two G-systems. If π has rel-u.p.e. and* $\mathrm{supp}(Y) = Y$, *then* $\mathrm{supp}(X) = X$.

Proof. Assume that $\mathrm{supp}(X) \neq X$, then there exist $x_1 \in X$ and an open neighbourhood V of x_1 such that $V \cap \mathrm{supp}(X) = \emptyset$. Let $U = \bigcup_{g\in G} g^{-1}V$, then U is open and $\mu(U) = 0$ for every $\mu \in \mathcal{M}(X, G)$. Thus, $\mathrm{supp}(X) \subseteq U^c$, where $U^c = X \setminus U$.

Let $y = \pi(x_1)$. We note that $\pi^{-1}\{y\} \cap U^c \neq \emptyset$. In fact, since $\mathrm{supp}(Y) = Y$, there exits $\nu \in \mathcal{M}(Y, G)$ such that $y \in \mathrm{supp}(\nu)$. Then there exists $\tilde{\mu} \in \mathcal{M}(X, G)$ such that $\widetilde{\pi}(\tilde{\mu}) = \nu$. If $\pi^{-1}\{y\} \subseteq U$, there exists $\delta > 0$ such that $\pi^{-1}B(y, \delta) \subseteq U$. Then $\nu(B(y, \delta)) = \tilde{\mu}(\pi^{-1}B(y, \delta)) = 0$. This contradicts $y \in \mathrm{supp}(\nu)$. Thus, there exists $x_2 \in U^c$ such that $\pi(x_2) = y$.

By Urysohn's lemma, there exists continuous function $f : X \to [0, 1]$ such that $f(x_1) = 0$ and $f(x) = 1$ for any $x \in U^c$. We set

$$F : X \to [0, 1]^G \text{ by } (F(x))(g) = f(gx).$$

Consider the G-action on $[0, 1]^G$ defined by $(g\omega)(h) = \omega(hg)$ for every $\omega \in [0, 1]^G$ and $g, h \in G$. We define a factor map

$$\phi : (X, G) \to ([0, 1]^G \times Y, G) \text{ by } \phi(x) = (F(x), \pi(x)).$$

Let $W = \phi(X)$ and $\pi_2 : (W, G) \to (Y, G)$ be the projection map to the second coordinate. Then $\pi = \pi_2 \circ \phi$. By Proposition A.3, π_2 has rel-u.p.e. Note that $\pi_2(\phi(x_1)) = \pi(x_1) = \pi(x_2) = \pi_2(\phi(x_2))$ and $\phi(x_1) \neq \phi(x_2)$. Thus,

$$(\phi(x_1), \phi(x_2)) \in R_{\pi_2} \setminus \Delta(W) = E(W, G|\pi_2).$$

Then, by Lemma A.2, one has $\phi(x_1) \in \mathrm{supp}(W)$. However, $\phi(x_1) \notin \{1^G\} \times Y$ and for every $\mu \in \mathcal{M}(X, G)$, one has $\mathrm{supp}(\mu) \subseteq U^c$, which implies $\mathrm{supp}(W) \subseteq \phi(U^c) \subseteq \{1^G\} \times Y$. Thus, $\phi(x_1) \notin \mathrm{supp}(W)$. This is a contradiction. □

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Now we are ready to give the proof of Theorem 2.3.

THEOREM A.5. *Let $\pi_i : (X_i, G) \to (Y_i, G)$ be two factor maps between G-systems and $\mathrm{supp}(Y_i) = Y_i$ for $i = 1, 2$. Then π_1 and π_2 has rel-u.p.e. if and only if $\pi_1 \times \pi_2 : (X_1 \times X_2, G) \to (Y_1 \times Y_2, G)$ has rel-u.p.e.*

Proof. For the non-trivial direction, if π_1 and π_2 have rel-u.p.e., for any $u_1 = (x_1, z_1)$ and $u_2 = (x_2, z_2)$ in $X_1 \times X_2$ with $(u_1, u_2) \in R_{\pi_1 \times \pi_2} \setminus \Delta(X_1 \times X_2)$, we shall prove $(u_1, u_2) \in E(X_1 \times X_2, G|\pi_1 \times \pi_2)$. Without loss of generality, we assume $x_1 \neq x_2$.

Let $\widetilde{U}_1 = U_1 \times V_1$, $\widetilde{U}_2 = U_2 \times V_2$ be neighbourhoods of u_1 and u_2, respectively. Note that $(x_1, x_2) \in R_{\pi_1} \setminus \Delta(X_1) = E(X_1, G|\pi_1)$ since π_1 has rel-u.p.e. Then by Corollary 3.3, there exists $c_1 > 0$ such that for every $F \in \mathcal{F}(G)$, there exists $E \subseteq F$ with $|E| > c_1|F|$, which is an independence set of $\{U_1, U_2\}$ with respect to π_1. For z_1 and z_2, there are two cases.

Case 1: $z_1 \neq z_2$. In this case, $(z_1, z_2) \in R_{\pi_2} \setminus \Delta(X_2) = E(X_2, G|\pi_2)$ since π_2 has rel-u.p.e. Then there exists $c_2 > 0$ such that for every $F \in \mathcal{F}(G)$, there exists $F_0 \subseteq F$ with $|F_0| > c_1 \cdot c_2|F|$, which is an independence set of $\{\widetilde{U}_1, \widetilde{U}_2\}$ with respect to $\pi_1 \times \pi_2$. This implies $(u_1, u_2) \in E(X_1 \times X_2, G|\pi_1 \times \pi_2)$.

Case 2: $z_1 = z_2 = z$ for some $z \in X_2$. We set $V = V_1 \cap V_2$. Then V is an open neighbourhood of z. Since $\mathrm{supp}(Y_2) = Y_2$ and π_2 has rel-u.p.e., by Proposition A.4, we have $\mathrm{supp}(X_2) = X_2$. Thus, there exists $\nu \in \mathcal{M}(X_2, G)$ such that $\nu(V) > 0$. By Lemma A.2, $\mathcal{P}_V^{\pi_2}$ has positive density. Then by similar analysis in Case 1, we can also obtain that $(u_1, u_2) \in E(X_1 \times X_2, G|\pi_1 \times \pi_2)$. This ends our proof. □

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