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HORN PROBLEM FOR QUASI-HERMITIAN LIE GROUPS

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Abstract In this paper, we prove some convexity results associated to orbit projection of noncompact real reductive Lie groups.

Contents

1. Introduction

This paper is concerned with convexity properties associated to orbit projection.

Let us consider two Lie groups $G \subset \tilde{G}$ with Lie algebras $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ and corresponding projection $\pi_{\mathfrak{g},\tilde{\mathfrak{g}}} : \tilde{\mathfrak{g}}^* \to \mathfrak{g}^*$. A longstanding problem has been to understand how a coadjoint orbit $\tilde{\mathcal{O}} \subset \tilde{\mathfrak{g}}^*$ decomposes under the projection $\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}$. For this purpose, we may define

$$\Delta_G(\tilde{\mathcal{O}}) = \{\mathcal{O} \in \mathfrak{g}^*/G;\ \mathcal{O} \subset \pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{\mathcal{O}})\}.$$

When the Lie group G is compact and connected, the set $\mathfrak{g}^*/G$ admits a natural identification with a Weyl chamber $\mathfrak{t}^*_{\geq 0}$. In this context, we have the well-known convexity theorem [12, 1, 10, 16, 13, 35, 22].

Theorem 1.1. *Suppose that G is compact connected and that the projection $\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}$ is proper when restricted to $\tilde{\mathcal{O}}$. Then $\Delta_G(\tilde{\mathcal{O}}) = \{\xi \in \mathfrak{t}^*_{\geq 0};\ G\xi \subset \pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{\mathcal{O}})\}$ is a closed convex locally polyhedral subset of $\mathfrak{t}^*$.*

When the Lie group $\tilde{G}$ is also compact and connected, we may consider

$$\Delta(\tilde{G},G) := \left\{(\tilde{\xi},\xi) \in \tilde{\mathfrak{t}}^*_{\geq 0} \times \mathfrak{t}^*_{\geq 0};\ G\xi \subset \pi_{\mathfrak{g},\tilde{\mathfrak{g}}}\left(\tilde{G}\tilde{\xi}\right)\right\}. \tag{1}$$

Here is another convexity theorem [14, 17, 4, 2, 3, 25, 19, 20, 36].

Theorem 1.2. *Suppose that $G \subset \tilde{G}$ are compact connected Lie groups. Then $\Delta(\tilde{G},G)$ is a closed convex polyhedral cone and we can parametrize its facets by cohomological means (i.e., Schubert calculus).*

In this article, we obtain an extension of Theorems 1.1 and 1.2 in a case where G and $\tilde{G}$ are both noncompact real reductive Lie groups.

Let us explain what framework we are considering. Let $\tilde{K}$ be a maximal compact subgroup of $\tilde{G}$. We suppose that $\tilde{G}/\tilde{K}$ is a Hermitian symmetric space of a noncompact type. Among the elliptic coadjoint orbits of $\tilde{G}$, some of them are naturally Kähler $\tilde{K}$-manifolds. These orbits are called the holomorphic coadjoint orbits of $\tilde{G}$. They are the strongly elliptic coadjoint orbits closely related to the holomorphic discrete series of Harish–Chandra. These orbits intersect the Weyl chamber $\tilde{\mathfrak{t}}^*_{\geq 0}$ of $\tilde{K}$ into a subchamber $\tilde{\mathcal{C}}_{\mathrm{hol}}$ called the holomorphic chamber. The basic fact here is that the union

$$\mathcal{C}^0_{\tilde{G}/\tilde{K}} := \bigcup_{\tilde{a} \in \tilde{\mathcal{C}}_{\mathrm{hol}}} \tilde{G}\tilde{a}$$

is an open invariant convex cone of $\tilde{\mathfrak{g}}^*$. See §2.1 for more details.

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In this article, we work in the context where $\tilde{G}/\tilde{K}$ admits a sub-Hermitian symmetric space of a noncompact type G/K. For the convenience of the reader, we list below some examples of the pairs $(\tilde{G},G)$:

$\tilde{G}$	G
$U(p,q)^s, s \geq 2$	$U(p,q)$
$Sp(n,\mathbb{R})$	$Sp(p,\mathbb{R}) \times Sp(n-p,\mathbb{R})$
$Sp(n,\mathbb{R})$	$U(p,n-p)$
$SO(2,2n)$	$U(1,n)$
$SO(2,n)$	$SO(2,p) \times SO(n-p)$
$SO^*(2n)$	$U(p,n-p)$
$SO^*(2n)$	$SO^*(2p) \times SO^*(2n-2p)$
$U(n,n)$	$Sp(n,\mathbb{R})$
$U(n,n)$	$SO^*(2n)$
$U(p,q)$	$U(i,j) \times U(p-i,q-j)$.

As the projection $\pi_{\mathfrak{g},\tilde{\mathfrak{g}}} : \tilde{\mathfrak{g}}^* \to \mathfrak{g}^*$ sends the convex cone $\mathcal{C}^0_{\tilde{G}/\tilde{K}}$ inside the convex cone $\mathcal{C}^0_{G/K}$, it is natural to study the following object reminiscent of equation (1):

$$\Delta_{\mathrm{hol}}(\tilde{G},G) := \left\{(\tilde{\xi},\xi) \in \tilde{\mathcal{C}}_{\mathrm{hol}} \times \mathcal{C}_{\mathrm{hol}};\ G\xi \subset \pi_{\mathfrak{g},\tilde{\mathfrak{g}}}\left(\tilde{G}\tilde{\xi}\right)\right\}. \tag{2}$$

Let $\tilde{\mu} \in \tilde{\mathcal{C}}_{\mathrm{hol}}$. We will also give a particular attention to the intersection of $\Delta_{\mathrm{hol}}(\tilde{G},G)$ with the linear subspace $\tilde{\xi} = \tilde{\mu}$, that is to say

$$\Delta_G(\tilde{G}\tilde{\mu}) := \left\{\xi \in \mathcal{C}_{\mathrm{hol}};\ G\xi \subset \pi_{\mathfrak{g},\tilde{\mathfrak{g}}}\left(\tilde{G}\tilde{\mu}\right)\right\}. \tag{3}$$

Consider the case where G is embedded diagonally in $\tilde{G} := G^s$ for $s \geq 2$. The corresponding set $\Delta_{\mathrm{hol}}(G^s,G)$ is called the holomorphic Horn cone, and it is defined as follows:

$$\mathrm{Horn}^s_{\mathrm{hol}}(G) := \left\{(\xi_1,\cdots,\xi_{s+1}) \in \mathcal{C}^{s+1}_{\mathrm{hol}};\ G\xi_{s+1} \subset \sum_{j=1}^{s} G\xi_j\right\}.$$

The first result of this article is the following theorem.

Theorem A.

- *$\Delta_{\mathrm{hol}}(\tilde{G},G)$ is a closed convex cone of $\tilde{\mathcal{C}}_{\mathrm{hol}} \times \mathcal{C}_{\mathrm{hol}}$.*
- *$\mathrm{Horn}^s_{\mathrm{hol}}(G)$ is a closed convex cone of $\mathcal{C}^{s+1}_{\mathrm{hol}}$ for any $s \geq 2$.*

We obtain the following corollary which corresponds to a result of A. Weinstein [38].

Corollary. *For any $\tilde{\mu} \in \tilde{\mathcal{C}}_{\mathrm{hol}}$, $\Delta_G(\tilde{G}\tilde{\mu})$ is a closed and convex subset of $\mathcal{C}_{\mathrm{hol}}$.*

A first description of the closed convex cone $\Delta_{\mathrm{hol}}(\tilde{G},G)$ goes as follows. The quotient $\mathfrak{q}$ of the tangent spaces $\mathbf{T}_e G/K$ and $\mathbf{T}_e\tilde{G}/\tilde{K}$ has a natural structure of a Hermitian

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K-vector space. The symmetric algebra $\mathrm{Sym}(\mathfrak{q})$ of $\mathfrak{q}$ defines an admissible K-module. The irreducible representations of K (resp. $\tilde{K}$) are parametrized by a semi-group $\wedge^*_+$ (resp. $\tilde{\wedge}^*_+$). For any $\lambda \in \wedge^*_+$ (resp. $\tilde{\lambda} \in \tilde{\wedge}^*_+$), we denote by V^K_λ (resp. $V^{\tilde{K}}_{\tilde{\lambda}}$) the irreducible representation of K (resp. $\tilde{K}$) with highest weight λ (resp. $\tilde{\lambda}$). If E is a representation of K, we denote by $\left[V^K_\lambda : E\right]$ the multiplicity of V^K_λ in E.

Definition 1.3.

1. $\Pi^{\mathbb{Z}}_{\mathfrak{q}}(\tilde{K},K)$ is the semigroup of $\tilde{\wedge}^*_+ \times \wedge^*_+$ defined by the conditions:
$$(\tilde{\lambda},\lambda) \in \Pi^{\mathbb{Z}}_{\mathfrak{q}}(\tilde{K},K) \quad \Longleftrightarrow \quad \left[V^K_\lambda : V^{\tilde{K}}_{\tilde{\lambda}} \otimes \mathrm{Sym}(\mathfrak{q})\right] \neq 0.$$
2. $\Pi_{\mathfrak{q}}(\tilde{K},K)$ is the convex cone defined as the closure of $\mathbb{Q}^{>0} \cdot \Pi^{\mathbb{Z}}_{\mathfrak{q}}(\tilde{K},K)$.

The second result of this article is the following theorem.

Theorem B. *We have the equality*
$$\Delta_{\mathrm{hol}}(\tilde{G},G) = \Pi_{\mathfrak{q}}(\tilde{K},K) \bigcap \tilde{\mathcal{C}}_{\mathrm{hol}} \times \mathcal{C}_{\mathrm{hol}}. \tag{4}$$

A natural question is the description of the facets of the convex cone $\Delta_{\mathrm{hol}}(\tilde{G},G)$. In order to do that, we consider the group $\tilde{K}$ endowed with the following $\tilde{K} \times K$-action: $(\tilde{k},k) \cdot \tilde{a} = \tilde{k}\tilde{a}k^{-1}$. The cotangent space $\mathbf{T}^*\tilde{K}$ is then a symplectic manifold equipped with a Hamiltonian action of $\tilde{K} \times K$. We consider now the Hamiltonian $\tilde{K} \times K$-manifold $\mathbf{T}^*\tilde{K} \times \mathfrak{q}$, and we denote by $\Delta(\mathbf{T}^*\tilde{K} \times \mathfrak{q})$ the corresponding Kirwan polyhedron.

Let $W = N(T)/T$ be the Weyl group of (K,T), and let w_0 be the longest Weyl group element. Define an involution $* : \mathfrak{t}^* \to \mathfrak{t}^*$ by $\xi^* = -w_0\xi$. A standard result permits to affirm that $(\tilde{\xi},\xi) \in \Pi_{\mathfrak{q}}(\tilde{K},K)$ if and only if $(\tilde{\xi},\xi^*) \in \Delta(\mathbf{T}^*\tilde{K} \times \mathfrak{q})$ (see §4.2).

We obtain then another version of Theorem B.

Theorem B, second version. *An element $(\tilde{\xi},\xi)$ belongs to $\Delta_{\mathrm{hol}}(\tilde{G},G)$ if and only if*
$$(\tilde{\xi},\xi) \in \tilde{\mathcal{C}}_{\mathrm{hol}} \times \mathcal{C}_{\mathrm{hol}} \quad \text{and} \quad (\tilde{\xi},\xi^*) \in \Delta(\boldsymbol{T}^*\tilde{K} \times \mathfrak{q}).$$

Thanks to the second version of Theorem B, a natural way to describe the facets of the cone $\Delta_{\mathrm{hol}}(\tilde{G},G)$ is to exhibit those of the Kirwan polyhedron $\Delta(\mathbf{T}^*\tilde{K} \times \mathfrak{q})$. In this later case, it can be done using Ressayre's data (see §5).

The second version of Theorem B permits also the following description of the convex subsets $\Delta_G(\tilde{G}\tilde{\mu})$, $\tilde{\mu} \in \tilde{\mathcal{C}}_{\mathrm{hol}}$. Let $\Delta_K(\tilde{K}\tilde{\mu} \times \overline{\mathfrak{q}})$ be the Kirwan polyhedron associated to the Hamiltonian action of K on $\tilde{K}\tilde{\mu} \times \overline{\mathfrak{q}}$, where $\overline{\mathfrak{q}}$ denotes the K-module $\mathfrak{q}$ with opposite complex structure.

Theorem C. *For any $\tilde{\mu} \in \tilde{\mathcal{C}}_{\mathrm{hol}}$, we have $\Delta_G(\tilde{G}\tilde{\mu}) = \Delta_K(\tilde{K}\tilde{\mu} \times \overline{\mathfrak{q}})$.*

Let us detail Theorem C in the case where G is embedded in $\tilde{G} = G \times G$ diagonally. We denote by $\mathfrak{p}$ the K-Hermitian space $\mathbf{T}_e G/K$. Let κ be the Killing form of the Lie algebra $\mathfrak{g}$. The vector space $\overline{\mathfrak{p}}$ is equipped with the symplectic 2-form $\Omega_{\overline{\mathfrak{p}}}(X,Y) = -\kappa(z,[X,Y])$ and the compatible complex structure $-\mathrm{ad}(z)$.

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Let us denote by $\Delta_K(K\mu_1 \times K\mu_2 \times \overline{\mathfrak{p}})$ and by $\Delta_K(\overline{\mathfrak{p}})$ the Kirwan polyhedrons relative to the Hamiltonian actions of K on $K\mu_1 \times K\mu_2 \times \overline{\mathfrak{p}}$ and on $\overline{\mathfrak{p}}$. Theorem C says that, for any $\mu_1,\mu_2 \in \mathcal{C}_{\mathrm{hol}}$, the convex set $\Delta_G(G\mu_1 \times G\mu_2)$ is equal to the Kirwan polyhedron $\Delta_K(K\mu_1 \times K\mu_2 \times \overline{\mathfrak{p}})$.

To any nonempty subset $\mathcal{C}$ of a real vector space E, we may associate its asymptotic cone $\mathrm{As}(\mathcal{C}) \subset E$ which is the set formed by the limits $y = \lim_{k\to\infty} t_k y_k$, where (t_k) is a sequence of nonnegative reals converging to 0 and $y_k \in \mathcal{C}$.

We finally get the following description of the asymptotic cone of $\Delta_G(G\mu_1 \times G\mu_2)$.

Corollary D. *For any $\mu_1,\mu_2 \in \mathcal{C}_{\mathrm{hol}}$, the asymptotic cone of $\Delta_G(G\mu_1 \times G\mu_2)$ is equal to $\Delta_K(\overline{\mathfrak{p}})$.*

In [29] §5, we explained how to describe the cone $\Delta_K(\overline{\mathfrak{p}})$ in terms of strongly orthogonal roots.

Let us finish this introduction with few remarks on related works:

- When G is compact, equal to the maximal compact subgroup $\tilde{K}$ of $\tilde{G}$, the results of Theorems B and C were already obtained by G. Deltour in his thesis [6, 7]. He proved the equality $\Delta_{\tilde{K}}(\tilde{G}\tilde{\mu}) = \Delta_{\tilde{K}}(\tilde{K}\tilde{\mu} \times \overline{\tilde{\mathfrak{p}}})$ by showing that the coadjoint orbit $\tilde{G}\tilde{\mu}$ admits a $\tilde{K}$-equivariant symplectomorphism with $\tilde{K}\tilde{\mu} \times \overline{\tilde{\mathfrak{p}}}$, thus generalizing an earlier result of D. McDuff [26]. We explain in §7 a conjectural symplectomorphism that would lead to the relation $\Delta_G(\tilde{G}\tilde{\mu}) = \Delta_K(\tilde{K}\tilde{\mu} \times \overline{\mathfrak{q}})$.
- In [9], A. Eshmatov and P. Foth proposed a description of the set $\Delta_G(G\mu_1 \times G\mu_2)$. **But their computations do not give the same result as ours**. From their main result (Theorem 3.2), it follows that the asymptotic cone of $\Delta_G(G\mu_1 \times G\mu_2)$ is equal to the intersection of the Kirwan polyhedron $\Delta_T(\overline{\mathfrak{p}})$ with the Weyl chamber $\mathfrak{t}^*_{\geq 0}$. But since $\Delta_K(\overline{\mathfrak{p}}) \neq \Delta_T(\overline{\mathfrak{p}}) \cap \mathfrak{t}^*_{\geq 0}$ in general, it is in contradiction with Corollary D.

Notations

In this paper, we take the convention of A. Knapp [18]: A connected real reductive Lie group G means that we have a Cartan involution Θ on G such that the fixed point set $K := G^\Theta$ is a connected maximal compact subgroup. We have Cartan decompositions at the level of Lie algebras $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and at the level of the group $G \simeq K \times \exp(\mathfrak{p})$. We denote by b a G-invariant nondegenerate bilinear form on $\mathfrak{g}$ that is equal to the Killing form on $[\mathfrak{g},\mathfrak{g}]$, and that defines a K-invariant scalar product $(X,Y) := -b(X,\Theta(Y))$. We will use the K-equivariant identification $\xi \mapsto \tilde{\xi}$, $\mathfrak{g}^* \simeq \mathfrak{g}$ defined by $(\tilde{\xi},X) := \langle \xi, X\rangle$ for $\xi \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$.

When a Lie group H acts on a manifold N, the stabilizer subgroup of $n \in N$ is denoted by $H_n = \{g \in G, gn = n\}$ and its Lie algebra by $\mathfrak{h}_n$. Let us define

$$\dim_H(\mathcal{X}) = \min_{n\in\mathcal{X}} \dim(\mathfrak{h}_n) \tag{5}$$

for any subset $\mathcal{X} \subset N$.

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2. The cone $\Delta_{\mathrm{hol}}(\tilde{G},G)$: first properties

We assume here that G/K is a Hermitian symmetric space of a noncompact type, that is to say, there exists a G-invariant complex structure on the manifold G/K or, equivalently, there exists a K-invariant element $z\in\mathfrak{k}$ such that $\mathrm{ad}(z)|_{\mathfrak{p}}$ defines a complex structure on $\mathfrak{p}$: $(\mathrm{ad}(z)|_{\mathfrak{p}})^2=-\mathrm{Id}_{\mathfrak{p}}$. This condition imposes that the ranks of G and K are equal.

We are interested in the following closed invariant convex cone of $\mathfrak{g}^*$:

$$\mathcal{C}_{G/K}=\{\xi\in\mathfrak{g}^*,\langle\xi,gz\rangle\geq 0,\ \forall g\in G\}.$$

2.1. The holomorphic chamber

Let T be a maximal torus of K, with Lie algebra $\mathfrak{t}$. Its dual $\mathfrak{t}^*$ can be seen as the subspace of $\mathfrak{g}^*$ fixed by T. Let us denote by $\mathfrak{g}^*_e$ the set formed by the elliptic elements: In other words, $\mathfrak{g}^*_e:=\mathrm{Ad}^*(G)\cdot\mathfrak{t}^*$.

Following [38], we consider the invariant open subset $\mathfrak{g}^*_{se}=\{\xi\in\mathfrak{g}^*\,|\,G_\xi \text{ is compact}\}$ of *strongly elliptic* elements. It is nonempty since the groups G and K have the same rank.

We start with the following basic facts.

Lemma 2.1.

- *$\mathfrak{g}^*_{se}$ is contained in $\mathfrak{g}^*_e$.*
- *The interior $\mathcal{C}^0_{G/K}$ of the cone $\mathcal{C}_{G/K}$ is contained in $\mathfrak{g}^*_{se}$.*

Proof. The first point is due to the fact that every compact subgroup of G is conjugate to a subgroup of K.

Let $\xi\in\mathcal{C}^0_{G/K}$. There exists $\epsilon>0$ so that

$$\langle\xi+\eta,gz\rangle\geq 0,\quad\forall g\in G,\quad\forall\|\eta\|\leq\epsilon.$$

It implies that $|\langle\eta,gz\rangle|\leq\langle\xi,z\rangle$, $\forall g\in G_\xi$ and $\forall\|\eta\|\leq\epsilon$. In other words, the adjoint orbit $G_\xi\cdot z\subset\mathfrak{g}$ is bounded. For any $g=e^Xk$, with $(X,k)\in\mathfrak{p}\times K$, a direct computation shows that $\|gz\|=\|e^Xz\|\geq\|[z,X]\|=\|X\|$. Then, there exists $\rho>0$ such that $\|X\|\leq\rho$ if $e^Xk\in G_\xi$ for some $k\in K$. It follows that the stabilizer subgroup G_ξ is compact. □

Let $\wedge^*\subset\mathfrak{t}^*$ be the weight lattice: By definition, $\alpha\in\wedge^*$ if and only if $i\alpha$ is the differential of a character of T. Let $\mathfrak{R}\subset\wedge^*$ be the set of roots for the action of T on $\mathfrak{g}\otimes\mathbb{C}$. We have $\mathfrak{R}=\mathfrak{R}_c\cup\mathfrak{R}_n$, where $\mathfrak{R}_c$ and $\mathfrak{R}_n$ are, respectively, the set of roots for the action of T on $\mathfrak{k}\otimes\mathbb{C}$ and $\mathfrak{p}\otimes\mathbb{C}$. We fix a system of positive roots $\mathfrak{R}^+_c$ in $\mathfrak{R}_c$, and we denote by $\mathfrak{t}^*_{\geq 0}$ the corresponding Weyl chamber.

We have $\mathfrak{p}\otimes\mathbb{C}=\mathfrak{p}^+\oplus\mathfrak{p}^-$, where the K-module $\mathfrak{p}^\pm$ is equal to $\ker(\mathrm{ad}(z)\mp i)$. Let $\mathfrak{R}^{\pm,z}_n$ be the set of roots for the action of T on $\mathfrak{p}^\pm$. The union

$$\mathfrak{R}^+=\mathfrak{R}^+_c\cup\mathfrak{R}^{+,z}_n \tag{6}$$

defines then a system of positive roots in $\mathfrak{R}$. We notice that $\mathfrak{R}^{+,z}_n$ is the set of roots $\beta\in\mathfrak{R}$ satisfying $\langle\beta,z\rangle=1$. Hence, $\mathfrak{R}^{+,z}_n$ is invariant relatively to the action of the Weyl group $W=N(T)/T$.

Let us recall the following classical fact concerning the parametrization of the G-orbits in $\mathcal{C}^0_{G/K}$ via the holomorphic chamber

$$\mathcal{C}_{\rm hol} := \{\xi \in \mathfrak{t}^*_{\geq 0}, (\xi,\beta) > 0,\ \forall \beta \in \mathfrak{R}^{+,z}_n\}.$$

The elliptic coadjoint orbits of G, i.e., those contained in $\mathfrak{g}^*_e$, are parameterized by the Weyl chamber $\mathfrak{t}^*_{\geq 0}$. Thus, we have a projection $\mathrm{p} : \mathfrak{g}^*_e \to \mathfrak{t}^*_{\geq 0}$, defined by the relations $G\xi \cap \mathfrak{t}^*_{\geq 0} = \{\mathrm{p}(\xi)\}$, and that induces a bijection $\mathfrak{g}^*_e/G \simeq \mathfrak{t}^*_{\geq 0}$.

Proposition 2.2. *The set* $\mathrm{p}(\mathcal{C}^0_{G/K})$ *is equal to* $\mathcal{C}_{\rm hol}$. *In other words, the map* p *induces a bijective map between the set of* G*-orbits in* $\mathcal{C}^0_{G/K}$ *and the holomorphic chamber* $\mathcal{C}_{\rm hol}$.

Proof. Let us first prove that $\mathrm{p}(\mathcal{C}^0_{G/K}) = \mathfrak{t}^*_{\geq 0} \cap \mathcal{C}^0_{G/K}$ is contained in $\mathcal{C}_{\rm hol}$. Let $\xi \in \mathfrak{t}^*_{\geq 0} \cap \mathcal{C}^0_{G/K}$: We have to check that $(\xi,\beta) > 0$ for any $\beta \in \mathfrak{R}^{+,z}_n$. Let $X_\beta, Y_\beta \in \mathfrak{p}$ such that $X_\beta + iY_\beta \in (\mathfrak{p}\otimes\mathbb{C})_\beta$. We choose the following normalization: The vector $h_\beta := [X_\beta, Y_\beta]$ satisfies $\langle \beta, h_\beta\rangle = 1$. We see then that $(\xi,\beta) = \frac{1}{\|h_\beta\|^2}\langle \xi, h_\beta\rangle$ for any $\xi \in \mathfrak{g}^*$. Standard computation [28] gives: $e^{t\,\mathrm{ad}(X_\beta)}z = z + (\cosh(t)-1)h_\beta + \sinh(t)Y_\beta$, $\forall t \in \mathbb{R}$. By definition, we must have $\langle \xi+\eta, e^{t\,\mathrm{ad}(X_\beta)}z\rangle \geq 0, \forall t \in \mathbb{R}$, for any $\eta \in \mathfrak{t}^*$ small enough. It imposes that $\langle \xi, h_\beta\rangle > 0$. The first point is settled.

The other inclusion $\mathcal{C}_{\rm hol} \subset \mathfrak{t}^*_{\geq 0} \cap \mathcal{C}^0_{G/K}$ is a consequence of the next lemma. □

Lemma 2.3. *For any compact subset* $\mathcal{K}$ *of* $\mathcal{C}_{\rm hol}$, *there exists* $c_{\mathcal{K}} > 0$ *such that* $\langle \xi, gz\rangle \geq c_{\mathcal{K}}\|gz\|$, $\forall g \in G$, $\forall \xi \in \mathcal{K}$.

Proof. Let us choose some maximal strongly orthogonal system $\Sigma \subset \mathfrak{R}^{+,z}_n$. The real span $\mathfrak{a}$ of the $X_\beta, \beta \in \Sigma$ is a maximal abelian subspace of $\mathfrak{p}$. Hence, any element $g \in G$ can be written $g = ke^{X(t)}k'$ with $X(t) = \sum_{\beta\in\Sigma} t_\beta X_\beta$ and $k,k' \in K$. We get

$$gz = k\left(z + \sum_{\beta\in\Sigma}(\cosh(t_\beta)-1)h_\beta + \sum_{\beta\in\Sigma}\sinh(t_\beta)Y_\beta\right) \tag{7}$$

and

$$\langle \xi, gz\rangle = \langle k^{-1}\xi, z\rangle + \sum_{\beta\in\Sigma}(\cosh(t_\beta)-1)\langle k^{-1}\xi, h_\beta\rangle.$$

For any $\xi \in \mathcal{C}_{\rm hol}$, we define $c_\xi := \min_{\beta\in\mathfrak{R}^{+,z}_n}\langle \xi, h_\beta\rangle > 0$. Let $\pi : \mathfrak{k}^* \to \mathfrak{t}^*$ be the projection. We have $\langle k^{-1}\xi, z\rangle = \langle \pi(k^{-1}\xi), z\rangle$ and $\langle k^{-1}\xi, h_\beta\rangle = \langle \pi(k^{-1}\xi), h_\beta\rangle$. The convexity theorem of Kostant tell us that $\pi(k^{-1}\xi)$ belongs to the convex hull of $\{w\xi, w \in W\}$. It follows that $\langle k^{-1}\xi, z\rangle \geq \langle \xi, z\rangle > 0$ and $\langle k^{-1}\xi, h_\beta\rangle \geq c_\xi > 0$ for any $k \in K$. We obtain then that $\langle \xi, gz\rangle \geq \frac{1}{2}\min(\langle \xi, z\rangle, c_\xi)e^{\|X(t)\|}$ for any $\xi \in \mathcal{C}_{\rm hol}$, where $\|X(t)\| = \sup_\beta |t_\beta|$. From equation (7), it is not difficult to see that there exists $C > 0$ such that $\|gz\| \leq Ce^{\|X(t)\|}$ for any $g = ke^{X(t)}k' \in G$.

Let $\mathcal{K}$ be a compact subset of $\mathcal{C}_{\rm hol}$. Take $c_{\mathcal{K}} = \frac{1}{2C}\min(\min_{\xi\in\mathcal{K}}\langle \xi, z\rangle, \min_{\xi\in\mathcal{K}} c_\xi) > 0$. The previous computations show that $\langle \xi, gz\rangle \geq c_{\mathcal{K}}\|gz\|$, $\forall g \in G$, $\forall \xi \in \mathcal{K}$. □

The following result is needed in §4.1.

Lemma 2.4. *The map* $\mathrm{p}:\mathcal{C}^0_{G/K}\to\mathcal{C}_{\mathrm{hol}}$ *is continuous.*

Proof. Let (ξ_n) be a sequence of $\mathcal{C}^0_{G/K}$ converging to $\xi_\infty\in\mathcal{C}^0_{G/K}$. Let $\xi'_n=\mathrm{p}(\xi_n)$ and $\xi'_\infty=\mathrm{p}(\xi_\infty)$: We have to prove that the sequence (ξ'_n) converges to ξ'_∞. We choose elements $g_n,g_\infty\in G$ such that $\xi_n=g_n\xi'_n,\forall n$ and $\xi_\infty=g_\infty\xi'_\infty$.

First, we notice that $-b(\xi_n,\xi_n)=\|\xi'_n\|^2$; hence, the sequence (ξ'_n) is bounded. We will now prove that the sequence (g_n) is bounded. Let $\epsilon>0$ such that $\langle\xi_\infty+\eta,gz\rangle\geq 0$, $\forall g\in G$, $\forall\|\eta\|\leq\epsilon$. If $\|\xi-\xi_\infty\|\leq\epsilon/2$, we write $\xi=\frac{1}{2}(\xi_\infty+2(\xi-\xi_\infty))+\frac{1}{2}\xi_\infty$, and then

$$\langle\xi,gz\rangle=\frac{1}{2}\langle\xi_\infty+2(\xi-\xi_\infty),gz\rangle+\frac{1}{2}\langle\xi_\infty,gz\rangle\geq\frac{1}{2}\langle\xi_\infty,gz\rangle,\quad\forall g\in G.$$

Now we have $\langle\xi'_n,z\rangle=\langle\xi_n,g_nz\rangle\geq\frac{1}{2}\langle\xi_\infty,g_nz\rangle$ if n is large enough. This shows that the sequence $\langle\xi_\infty,g_nz\rangle$ is bounded. If we use Lemma 2.3, we can conclude that the sequence (g_n) is bounded.

Let $(\xi'_{\phi(n)})$ be a subsequence converging to $\ell\in\mathfrak{t}^*_{\geq 0}$. Since $(g_{\phi(n)})$ is bounded, there exists a subsequence $(g_{\phi\circ\psi(n)})$ converging to $h\in G$. From the relations $\xi_{\phi\circ\psi(n)}=g_{\phi\circ\psi(n)}\xi'_{\phi\circ\psi(n)},\forall n\in\mathbb{N}$, we obtain $\xi_\infty=h\ell$. Then $\ell=\mathrm{p}(\xi_\infty)=\xi'_\infty$. Since every subsequence of (ξ'_n) has a subsequential limit to ξ'_∞, then the sequence (ξ'_n) converges to ξ'_∞. □

2.2. The cone $\Delta_{\mathrm{hol}}(\tilde{G},G)$ is closed

We suppose that G/K is a complex submanifold of a Hermitian symmetric space $\tilde{G}/\tilde{K}$. In other words, $\tilde{G}$ is a reductive real Lie group such that $G\subset\tilde{G}$ is a closed connected subgroup preserved by the Cartan involution, and $\tilde{K}$ is a maximal compact subgroup of $\tilde{G}$ containing K. We denote by $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{k}}$ the Lie algebras of $\tilde{G}$ and $\tilde{K}$, respectively. We suppose that there exists a $\tilde{K}$-invariant element $z\in\mathfrak{k}$ such that $\mathrm{ad}(z)|_{\tilde{\mathfrak{p}}}$ defines a complex structure on $\tilde{\mathfrak{p}}$: $(\mathrm{ad}(z)|_{\tilde{\mathfrak{p}}})^2=-Id_{\tilde{\mathfrak{p}}}$.

Let $\mathcal{C}_{\tilde{G}/\tilde{K}}\subset\tilde{\mathfrak{g}}^*$ be the closed invariant cone associated to the Hermitian symmetric space $\tilde{G}/\tilde{K}$. We start with the following key fact.

Lemma 2.5. *The projection* $\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}:\tilde{\mathfrak{g}}^*\to\mathfrak{g}^*$ *sends* $\mathcal{C}^0_{\tilde{G}/\tilde{K}}$ *into* $\mathcal{C}^0_{G/K}$.

Proof. Let $\tilde{\xi}\in\mathcal{C}^0_{\tilde{G}/\tilde{K}}$ and $\xi=\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{\xi})$. Then $\langle\tilde{\xi}+\tilde{\eta},\tilde{g}z\rangle\geq 0$, $\forall\tilde{g}\in\tilde{G}$ if $\tilde{\eta}\in\tilde{\mathfrak{g}}^*$ is small enough. It follows that $\langle\xi+\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{\eta}),gz\rangle=\langle\tilde{\xi}+\tilde{\eta},gz\rangle\geq 0$, $\forall g\in G$ if $\tilde{\eta}$ is small enough. Since $\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}$ is an open map, we can conclude that $\xi\in\mathcal{C}^0_{G/K}$. □

Let $\tilde{T}$ be a maximal torus of $\tilde{K}$, with Lie algebra $\tilde{\mathfrak{t}}$. The $\tilde{G}$-orbits in the interior of $\mathcal{C}_{\tilde{G}/\tilde{K}}$ are parametrized by the holomorphic chamber $\tilde{\mathcal{C}}_{\mathrm{hol}}\subset\tilde{\mathfrak{t}}^*$. The previous lemma says that the projection $\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{\mathcal{O}})$ of any $\tilde{G}$-orbit $\tilde{\mathcal{O}}\subset\mathcal{C}^0_{\tilde{G}/\tilde{K}}$ is the union of G-orbits $\mathcal{O}\subset\mathcal{C}^0_{G/K}$. So it is natural to study the following object:

$$\Delta_{\mathrm{hol}}(\tilde{G},G):=\left\{(\tilde{\xi},\xi)\in\tilde{\mathcal{C}}_{\mathrm{hol}}\times\mathcal{C}_{\mathrm{hol}};\ G\xi\subset\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}\left(\tilde{G}\tilde{\xi}\right)\right\}. \tag{8}$$

Here is a first result.

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Proposition 2.6. *$\Delta_{\mathrm{hol}}(\tilde{G},G)$ is a closed cone of $\tilde{\mathcal{C}}_{\mathrm{hol}}\times\mathcal{C}_{\mathrm{hol}}$.*

Proof. Suppose that a sequence $(\tilde{\xi}_n,\xi_n)\in\Delta_{\mathrm{hol}}(\tilde{G},G)$ converges to $(\tilde{\xi}_\infty,\xi_\infty)\in\tilde{\mathcal{C}}_{\mathrm{hol}}\times\mathcal{C}_{\mathrm{hol}}$. By definition, there exists a sequence $(\tilde{g}_n,g_n)\in\tilde{G}\times G$ such that $g_n\xi_n=\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{g}_n\tilde{\xi}_n)$. Let $\tilde{h}_n:=g_n^{-1}\tilde{g}_n$ so that $\xi_n=\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{h}_n\tilde{\xi}_n)$ and $\langle\tilde{h}_n\tilde{\xi}_n,z\rangle=\langle\xi_n,z\rangle$. We use now that the sequence $\langle\xi_n,z\rangle$ is bounded and that the sequence $\tilde{\xi}_n$ belongs to a compact subset of $\tilde{\mathcal{C}}_{\mathrm{hol}}$. Thanks to Lemma 2.3, these facts imply that $\|\tilde{h}_n^{-1}z\|$ is a bounded sequence. Hence, $\tilde{h}_n$ admits a subsequence converging to $\tilde{h}_\infty$. So we get $\xi_\infty=\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{h}_\infty\tilde{\xi}_\infty)$, and that proves that $(\tilde{\xi}_\infty,\xi_\infty)\in\Delta_{\mathrm{hol}}(\tilde{G},G)$. □

2.3. Rational and weakly regular points

Let (M,Ω) be a symplectic manifold. We suppose that there exists a line bundle $\mathcal{L}$ with connection ∇ that prequantizes the 2-form Ω: In other words, $\nabla^2=-i\Omega$. Let K be a compact connected Lie group acting on $\mathcal{L}\to M$, and leaving the connection invariant. Let $\Phi_K:M\to\mathfrak{k}^*$ be the moment map defined by Kostant's relations

$$L_X-\nabla_X=i\langle\Phi_K,X\rangle,\quad \forall X\in\mathfrak{k}. \tag{9}$$

Here L_X is the Lie derivative acting on the sections of $\mathcal{L}\to M$.

Remark that relations (9) imply, via the equivariant Bianchi formula, the relations

$$\iota(X_M)\Omega=-d\langle\Phi_K,X\rangle,\quad \forall X\in\mathfrak{k}, \tag{10}$$

where $X_M(m):=\frac{d}{dt}|_{t=0}e^{-tX}m$ is the vector field on M generated by $X\in\mathfrak{k}$.

Definition 2.7. Let $\dim_K(M):=\min_{m\in M}\dim\mathfrak{k}_m$. An element $\xi\in\mathfrak{k}^*$ is a weakly regular value of Φ_K if for all $m\in\Phi_K^{-1}(\xi)$ we have $\dim\mathfrak{k}_m=\dim_K(M)$.

When $\xi\in\mathfrak{k}^*$ is a weakly regular value of Φ_K, the constant rank theorem tells us that $\Phi_K^{-1}(\xi)$ is a submanifold of M stable under the action of the stabilizer subgroup K_ξ. We see then that the reduced space

$$M_\xi:=\Phi_K^{-1}(\xi)/K_\xi \tag{11}$$

is a symplectic orbifold.

Let $T\subset K$ be a maximal torus with Lie algebra $\mathfrak{t}$. We consider the lattice $\wedge:=\frac{1}{2\pi}\ker(\exp:\mathfrak{t}\to T)$ and the dual lattice $\wedge^*\subset\mathfrak{t}^*$ defined by $\wedge^*=\hom(\wedge,\mathbb{Z})$. We remark that $i\eta$ is a differential of a character of T if and only if $\eta\in\wedge^*$. The $\mathbb{Q}$-vector space generated by the lattice $\wedge^*$ is denoted by $\mathfrak{t}^*_{\mathbb{Q}}$: The vectors belonging to $\mathfrak{t}^*_{\mathbb{Q}}$ are designed as rational. An affine subspace $V\subset\mathfrak{t}^*$ is called rational if it is affinely generated by its rational points.

We also fix a closed positive Weyl chamber $\mathfrak{t}^*_{\geq 0}$. For each relatively open face $\sigma\subset\mathfrak{t}^*_{\geq 0}$, the stabilizer K_ξ of points $\xi\in\sigma$ under the coadjoint action does not depend on ξ and will be denoted by K_σ. The Lie algebra $\mathfrak{k}_\sigma$ decomposes into its semisimple and central parts $\mathfrak{k}_\sigma=[\mathfrak{k}_\sigma,\mathfrak{k}_\sigma]\oplus\mathfrak{z}_\sigma$. The subspace $\mathfrak{z}^*_\sigma$ is defined to be the annihilator of $[\mathfrak{k}_\sigma,\mathfrak{k}_\sigma]$ or, equivalently, the fixed point set of the coadjoint K_σ action. Notice that $\mathfrak{z}^*_\sigma$ is a rational subspace of $\mathfrak{t}^*$ and that the face σ is an open cone of $\mathfrak{z}^*_\sigma$,

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We suppose that the moment map Φ_K is *proper*. The convexity theorem [1, 10, 16, 35, 22] tells us that $\Delta_K(M) := \mathrm{Image}(\Phi_K) \bigcap \mathfrak{t}^*_{\geq 0}$ is a closed, convex, locally polyhedral set.

Definition 2.8. We denote by $\Delta_K(M)^0$ the subset of $\Delta_K(M)$ formed by the *weakly regular values* of the moment map Φ_K contained in $\Delta_K(M)$.

We will use the following remark in the next sections.

Lemma 2.9. *The subset $\Delta_K(M)^0 \cap \mathfrak{t}^*_{\mathbb{Q}}$ is dense in $\Delta_K(M)$.*

Proof. Let us first explain why $\Delta_K(M)^0$ is a dense open subset of $\Delta_K(M)$. There exists a unique open face τ of the Weyl chamber $\mathfrak{t}^*_{\geq 0}$ such as $\Delta_K(M) \cap \tau$ is dense in $\Delta_K(M)$: τ is called the *principal* face in [22]. The principal-cross-section theorem [22] tells us that $Y_\tau := \Phi^{-1}(\tau)$ is a symplectic K_τ-manifold, with a trivial action of $[K_\tau, K_\tau]$. The line bundle $\mathcal{L}_\tau := \mathcal{L}|_{Y_\tau}$ prequantizes the symplectic structure on Y_τ, and relations (10) show that $[K_\tau, K_\tau]$ acts trivially on $\mathcal{L}_\tau$. Moreover, the restriction of Φ_K on Y_τ is the moment map $\Phi_\tau : Y_\tau \to \mathfrak{z}^*_\tau$ associated to the action of the torus $Z_\tau = \exp(\mathfrak{z}_\tau)$ on $\mathcal{L}_\tau$.

Let $I \subset \mathfrak{z}^*_\tau$ be the smallest affine subspace containing $\Delta_K(M)$. Let $\mathfrak{z}_I \subset \mathfrak{z}_\tau$ be the annihilator of the subspace parallel to I: Relations (10) show that $\mathfrak{z}_I$ is the generic infinitesimal stabilizer of the $\mathfrak{z}_\tau$-action on Y_τ. Hence, $\mathfrak{z}_I$ is the Lie algebra of the torus $Z_I := \exp(\mathfrak{z}_I)$.

We see then that any regular value of $\Phi_\tau : Y_\tau \to I$, viewed as a map with codomain I, is a weakly regular value of the moment map Φ_K. It explains why $\Delta_K(M)^0$ is a dense open subset of $\Delta_K(M)$.

As the convex set $\Delta_K(M) \cap \tau$ is equal to $\Delta_{Z_\tau}(Y_\tau) := \mathrm{Image}(\Phi_\tau)$, it is sufficient to check that $\Delta_{Z_\tau}(Y_\tau)^0 \cap \mathfrak{t}^*_{\mathbb{Q}}$ is dense in $\Delta_{Z_\tau}(Y_\tau)$. The subtorus $Z_I \subset Z_\tau$ acts trivially on Y_τ, and it acts on the line bundle $\mathcal{L}_\tau$ through a character χ. Let $\eta \in \wedge^* \cap \mathfrak{t}^*_\tau$ such that $d\chi = i\eta|_{\mathfrak{z}_I}$. The affine subspace I which is equal to $\eta + (\mathfrak{z}_I)^\perp$ is rational. Since the open subset $\Delta_{Z_\tau}(Y_\tau)^0$ generates the rational affine subspace I, we can conclude that $\Delta_{Z_\tau}(Y_\tau)^0 \cap \mathfrak{t}^*_{\mathbb{Q}}$ is dense in $\Delta_{Z_\tau}(Y_\tau)$. □

2.4. Weinstein's theorem

Let $\tilde{a} \in \tilde{\mathcal{C}}_{\mathrm{hol}}$. Consider the Hamiltonian action of the group G on the coadjoint orbit $\tilde{G}\tilde{a}$. The moment map $\Phi^{\tilde{a}}_G : \tilde{G}\tilde{a} \to \mathfrak{g}^*$ corresponds to the restriction of the projection $\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}$ to $\tilde{G}\tilde{a}$. In this setting, the following conditions holds:

1. The action of G on $\tilde{G}\tilde{a}$ is proper.
2. The moment map $\Phi^{\tilde{a}}_G$ is a proper map since the map $\langle \Phi^{\tilde{a}}_G, z\rangle$ is proper (see Lemma 2.3).

Conditions 1 and 2 impose that the image of $\Phi^{\tilde{a}}_G$ is contained in the open subset $\mathfrak{g}^*_{se}$ of strongly elliptic elements [31]. Thus, the G-orbits contained in the image of $\Phi^{\tilde{a}}_G$ are parametrized by the following subset of the holomorphic chamber $\mathcal{C}_{\mathrm{hol}}$:

$$\Delta_G(\tilde{G}\tilde{a}) := \mathrm{Image}(\Phi^{\tilde{a}}_G) \bigcap \mathfrak{t}^*_{\geq 0}.$$

We notice that $\Delta_{\mathrm{hol}}(\tilde{G}, G) = \bigcup_{\tilde{a} \in \tilde{\mathcal{C}}_{\mathrm{hol}}} \{\tilde{a}\} \times \Delta_G(\tilde{G}\tilde{a})$.

Like in Definition 2.7, an element $\xi \in \mathfrak{g}^*$ is a *weakly regular* value of $\Phi_G^{\tilde{a}}$ if for all $m \in (\Phi_G^{\tilde{a}})^{-1}(\xi)$ we have $\dim \mathfrak{g}_m = \min_{x\in \tilde{G}\tilde{a}} \dim(\mathfrak{g}_x)$. We denote by $\Delta_G(\tilde{G}\tilde{a})^0$ the set of elements $\xi \in \Delta_G(\tilde{G}\tilde{a})$ that are weakly regular for $\Phi_G^{\tilde{a}}$.

Theorem 2.10 (Weinstein). *For any $\tilde{a} \in \tilde{\mathcal{C}}_{\mathrm{hol}}$, $\Delta_G(\tilde{G}\tilde{a})$ is a closed convex subset contained in $\mathcal{C}_{\mathrm{hol}}$.*

Proof. We recall briefly the arguments of the proof (see [38] or [31][§2]). Under Conditions 1 and 2, one checks easily that $Y_{\tilde{a}} := (\Phi_G^{\tilde{a}})^{-1}(\mathfrak{k}^*)$ is a smooth K-invariant symplectic submanifold of $\tilde{G}\tilde{a}$ such that

$$\tilde{G}\tilde{a} \simeq G \times_K Y_{\tilde{a}}. \tag{12}$$

The moment map $\Phi_K^{\tilde{a}} : Y_{\tilde{a}} \to \mathfrak{k}^*$, which corresponds to the restriction of the map $\Phi_G^{\tilde{a}}$ to $Y_{\tilde{a}}$, is a proper map. Hence, the convexity theorem tells us that $\Delta_K(Y_{\tilde{a}}) := \mathrm{Image}(\Phi_K^{\tilde{a}}) \bigcap \mathfrak{t}^*_{\geq 0}$ is a closed, convex, locally polyhedral set. Thanks to the isomorphism (12), we see that $\Delta_G(\tilde{G}\tilde{a})$ coincides with the closed convex subset $\Delta_K(Y_{\tilde{a}})$. The proof is completed. $\square$

The next lemma is used in §4.1.

Lemma 2.11. *Let $\tilde{a} \in \tilde{\mathcal{C}}_{\mathrm{hol}}$ be a rational element. Then $\Delta_G(\tilde{G}\tilde{a})^0 \cap \mathfrak{t}^*_{\mathbb{Q}}$ is dense in $\Delta_G(\tilde{G}\tilde{a})$.*

Proof. Thanks to equation (12), we know that $\Delta_G(\tilde{G}\tilde{a}) = \Delta_K(Y_{\tilde{a}})$. Relation (12) shows also that $\Delta_G(\tilde{G}\tilde{a})^0 = \Delta_K(Y_{\tilde{a}})^0$. Let $N \geq 1$ such that $\tilde{\mu} = N\tilde{a} \in \wedge^* \cap \mathcal{C}_{\mathrm{hol}}$. The stabilizer subgroup $\tilde{G}_{\tilde{\mu}}$ is compact, equal to $\tilde{K}_{\tilde{\mu}}$. Let us denote by $\mathbb{C}_{\tilde{\mu}}$ the one-dimensional representation of $\tilde{K}_{\tilde{\mu}}$ associated to $\tilde{\mu}$. The convex set $\Delta_G(\tilde{G}\tilde{a})$ is equal to $\frac{1}{N}\Delta_G(\tilde{G}\tilde{\mu})$, so it is sufficient to check that $\Delta_G(\tilde{G}\tilde{\mu})^0 \cap \mathfrak{t}^*_{\mathbb{Q}} = \Delta_K(Y_{\tilde{\mu}})^0 \cap \mathfrak{t}^*_{\mathbb{Q}}$ is dense in $\Delta_G(\tilde{G}\tilde{\mu}) = \Delta_K(Y_{\tilde{\mu}})$. The coadjoint orbit $\tilde{G}\tilde{\mu}$ is prequantized by the line bundle $\tilde{G} \times_{K_{\tilde{\mu}}} \mathbb{C}_{\tilde{\mu}}$, and the symplectic slice $Y_{\tilde{\mu}}$ is prequantized by the line bundle $\mathcal{L}_{\tilde{\mu}} := \tilde{G} \times_{K_{\tilde{\mu}}} \mathbb{C}_{\tilde{\mu}}|_{Y_{\tilde{\mu}}}$. Thanks to Lemma 2.9, we know that $\Delta_K(Y_{\tilde{\mu}})^0 \cap \mathfrak{t}^*_{\mathbb{Q}}$ is dense in $\Delta_K(Y_{\tilde{\mu}})$: The proof is complete. $\square$

3. Holomorphic discrete series

3.1. Definition

We return to the framework of §2.1. We recall the notion of holomorphic discrete series representations associated to a Hermitian symmetric spaces G/K. Let us introduce

$$\mathcal{C}^{\rho}_{\mathrm{hol}} := \left\{\xi \in \mathfrak{t}^*_{\geq 0} \middle|\ (\xi,\beta) \geq (2\rho_n,\beta),\ \forall \beta \in \mathfrak{R}_n^{+,z}\right\},$$

where $2\rho_n = \sum_{\beta\in\mathfrak{R}_n^{+,z}} \beta$ is W-invariant.

Lemma 3.1.

1. *We have $\mathcal{C}^{\rho}_{\mathrm{hol}} \subset \mathcal{C}_{\mathrm{hol}}$.*
2. *For any $\xi \in \mathcal{C}_{\mathrm{hol}}$, there exists $N \geq 1$ such that $N\xi \in \mathcal{C}^{\rho}_{\mathrm{hol}}$.*

Proof. The first point is due to the fact that $(\beta_0,\beta_1) \geq 0$ for any $\beta_0,\beta_1 \in \mathfrak{R}_n^{+,z}$. The second point is obvious. $\square$

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We will be interested in the following subset of dominant weights:

$$\widehat{G}_{\rm hol} := \wedge^*_+ \bigcap \mathcal{C}^{\rho}_{\rm hol}.$$

Let $\mathrm{Sym}(\mathfrak{p})$ be the symmetric algebra of the complex K-module $(\mathfrak{p}, \mathrm{ad}(z))$.

Theorem 3.2 (Harish–Chandra). *For any $\lambda \in \widehat{G}_{\rm hol}$, there exists an irreducible unitary representation of G, denoted by V^G_λ, such that the vector space of K-finite vectors is $V^G_\lambda|_K := V^K_\lambda \otimes \mathrm{Sym}(\mathfrak{p})$.*

The set $V^G_\lambda, \lambda \in \widehat{G}_{\rm hol}$ corresponds to the holomorphic discrete series representations associated to the complex structure $\mathrm{ad}(z)$.

3.2. Restriction

We come back to the framework of §2.2. We consider two compatible Hermitian symmetric spaces $G/K \subset \tilde{G}/\tilde{K}$, and we look after the restriction of holomorphic discrete series representations of $\tilde{G}$ to the subgroup G.

Let $\tilde{\lambda} \in \widehat{\tilde{G}}_{\rm hol}$. Since the representation $V^{\tilde{G}}_{\tilde{\lambda}}$ is discretely admissible relatively to the circle group $\exp(\mathbb{R}z)$, it is also discretely admissible relatively to G. We can be more precise [15, 24, 21]:

Proposition 3.3. *We have an Hilbertian direct sum*

$$V^{\tilde{G}}_{\tilde{\lambda}}|_G = \bigoplus_{\lambda \in \widehat{G}_{\rm hol}} m^{\lambda}_{\tilde{\lambda}}\ V^G_\lambda,$$

where the multiplicity $m^{\lambda}_{\tilde{\lambda}} := [V^G_\lambda : V^{\tilde{G}}_{\tilde{\lambda}}]$ is finite for any λ.

The Hermitian $\tilde{K}$-vector space $\tilde{\mathfrak{p}}$, when restricted to the K-action, admits an orthogonal decomposition $\tilde{\mathfrak{p}} = \mathfrak{p} \oplus \mathfrak{q}$. Notice that the symmetric algebra $\mathrm{Sym}(\mathfrak{q})$ is an admissible K-module.

In [15], H. P. Jakobsen and M. Vergne obtained the following nice characterization of the multiplicities $[V^G_\lambda : V^{\tilde{G}}_{\tilde{\lambda}}]$. Another proof is given in [31], §4.4.

Theorem 3.4 (Jakobsen–Vergne). *Let $(\tilde{\lambda}, \lambda) \in \widehat{\tilde{G}}_{\rm hol} \times \widehat{G}_{\rm hol}$. The multiplicity $[V^G_\lambda : V^{\tilde{G}}_{\tilde{\lambda}}]$ is equal to the multiplicity of the representation V^K_λ in $\mathrm{Sym}(\mathfrak{q}) \otimes V^{\tilde{K}}_{\tilde{\lambda}}|_K$.*

3.3. Discrete analogues of $\Delta_{\rm hol}(\tilde{G}, G)$

We define the following discrete analogues of the cone $\Delta_{\rm hol}(\tilde{G}, G)$:

$$\Pi^{\mathbb{Z}}_{\rm hol}(\tilde{G}, G) := \left\{ (\tilde{\lambda}, \lambda) \in \widehat{\tilde{G}}_{\rm hol} \times \widehat{G}_{\rm hol}\ \ [V^G_\lambda : V^{\tilde{G}}_{\tilde{\lambda}}] \neq 0 \right\}, \tag{13}$$

and

$$\Pi^{\mathbb{Q}}_{\rm hol}(\tilde{G}, G) := \left\{ (\tilde{\xi}, \xi) \in \tilde{\mathcal{C}}_{\rm hol} \times \mathcal{C}_{\rm hol}\ \ \exists N \geq 1,\ (N\xi, N\tilde{\xi}) \in \Pi^{\mathbb{Z}}_{\rm hol}(\tilde{G}, G) \right\}. \tag{14}$$

We have the following key fact.

Proposition 3.5.

- $\Pi^{\mathbb{Z}}_{\rm hol}(\tilde{G},G)$ *is a subset of* $\tilde{\wedge}^* \times \wedge^*$ *stable under the addition.*
- $\Pi^{\mathbb{Q}}_{\rm hol}(\tilde{G},G)$ *is a* $\mathbb{Q}$*-convex cone of the* $\mathbb{Q}$*-vector space* $\tilde{\mathfrak{t}}^*_{\mathbb{Q}} \times \mathfrak{t}^*_{\mathbb{Q}}$.

Proof. Suppose that $a_1 := (\tilde{\lambda}_1, \lambda_1)$ and $a_2 := (\tilde{\lambda}_2, \lambda_2)$ belongs to $\Pi^{\mathbb{Z}}_{\rm hol}(\tilde{G},G)$. Thanks to Theorem 3.4, we know that the K-modules $\mathrm{Sym}(\mathfrak{q}) \otimes (V^K_{\lambda_j})^* \otimes V^{\tilde{K}}_{\tilde{\lambda}_j}|_K$ possess a nonzero invariant vector ϕ_j, for $j = 1,2$.

Let $\mathbb{X} := \overline{K/T} \times \tilde{K}/\tilde{T}$ be the product of flag manifolds. The complex structure is normalized so that $\mathbf{T}_{([e],[\tilde{e}])}\mathbb{X} \simeq \mathfrak{n}_- \oplus \tilde{\mathfrak{n}}_+$, where $\mathfrak{n}_- = \sum_{\alpha<0} (\mathfrak{k}_{\mathbb{C}})_\alpha$ and $\tilde{\mathfrak{n}}_+ = \sum_{\tilde{\alpha}>0} (\tilde{\mathfrak{k}}_{\mathbb{C}})_{\tilde{\alpha}}$. We associate to each data a_j, the holomorphic line bundle $\mathcal{L}_j := K \times_T \mathbb{C}_{-\lambda_j} \boxtimes \tilde{K} \times_{\tilde{T}} \mathbb{C}_{-\tilde{\lambda}_j}$ on $\mathbb{X}$. Let $H^0(\mathbb{X}, \mathcal{L}_j)$ be the space of holomorphic sections of the line bundle $\mathcal{L}_j$. The Borel–Weil theorem tells us that $H^0(\mathbb{X}, \mathcal{L}_j) \simeq (V^K_{\lambda_j})^* \otimes V^{\tilde{K}}_{\tilde{\lambda}_j}|_K$, $\forall j \in \{1,2\}$.

We have $\phi_j \in \left[\mathrm{Sym}(\mathfrak{q}) \otimes H^0(\mathbb{X}, \mathcal{L}_j)\right]^K$, $\forall j$, and then $\phi_1\phi_2 \in \mathrm{Sym}(\mathfrak{q}) \otimes H^0(\mathbb{X}, \mathcal{L}_1 \otimes \mathcal{L}_2)$ is a nonzero invariant vector. Hence, $[\mathrm{Sym}(\mathfrak{q}) \otimes (V^K_{\lambda_1+\lambda_2})^* \otimes V^{\tilde{K}}_{\tilde{\lambda}_1+\tilde{\lambda}_2}|_K]^K \neq 0$. Thanks to Theorem 3.4, we can conclude that $a_1 + a_2 = (\tilde{\lambda}_1 + \tilde{\lambda}_2, \lambda_1 + \lambda_2)$ belongs to $\Pi^{\mathbb{Z}}_{\rm hol}(\tilde{G},G)$. The first point is proved. From the first point, one checks easily that

- $\Pi^{\mathbb{Q}}_{\rm hol}(\tilde{G},G)$ is stable under addition,
- $\Pi^{\mathbb{Q}}_{\rm hol}(\tilde{G},G)$ is stable by expansion by a nonnegative rational number.

The second point is settled. □

3.4. Riemann–Roch numbers

We come back to the framework of §2.3.

We associate to a dominant weight $\mu \in \wedge^*_+$ the (possibly singular) symplectic reduced space $M_\mu := \Phi_K^{-1}(\mu)/K_\mu$ and the (possibly singular) line bundle over M_μ:

$$\mathcal{L}_\mu := \left(\mathcal{L}|_{\Phi_K^{-1}(\mu)} \otimes \mathbb{C}_{-\mu}\right)/K_\mu.$$

Suppose first that μ is a weakly regular value of Φ_K. Then M_μ is an orbifold equipped with a symplectic structure Ω_μ, and $\mathcal{L}_\mu$ is a line orbi-bundle over M_μ that prequantizes the symplectic structure. By choosing an almost complex structure on M_μ compatible with Ω_μ, we get a decomposition $\wedge\, \mathbf{T}^* M_\mu \otimes \mathbb{C} = \oplus_{i,j} \wedge^{i,j}\, \mathbf{T}^* M_\mu$ of the bundle of differential forms. Using Hermitian structures in the tangent bundle $\mathbf{T}M_\mu$ of M_μ and in the fibers of $\mathcal{L}_\mu$, we define a Dolbeault–Dirac operator

$$D^+_\mu : \mathcal{A}^{0,+}(M_\mu, \mathcal{L}_\mu) \longrightarrow \mathcal{A}^{0,-}(M_\mu, \mathcal{L}_\mu),$$

where $\mathcal{A}^{i,j}(M_\mu, \mathcal{L}_\mu) = \Gamma(M_\mu, \wedge^{i,j}\, \mathbf{T}^* M_\mu \otimes \mathcal{L}_\mu)$.

Definition 3.6. Let $\mu \in \wedge^*_+$ be a weakly regular value of the moment map Φ_K. The Riemann–Roch number $RR(M_\mu, \mathcal{L}_\mu) \in \mathbb{Z}$ is defined as the index of the elliptic operator D^+_μ: $RR(M_\mu, \mathcal{L}_\mu) = \dim(ker(D^+_\mu)) - \dim(coker(D^+_\mu))$.

Suppose that $\mu \notin \Delta_K(M)$. Then $M_\mu = \emptyset$, and we take $RR(M_\mu, \mathcal{L}_\mu) = 0$.

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Suppose now that $\mu\in\Delta_K(M)$ is not (necessarily) a weakly regular value of Φ_K. Take a small element $\epsilon\in\mathfrak{t}^*$ such that $\mu+\epsilon$ is a weakly regular value of Φ_K belonging to $\Delta_K(M)$. We consider the symplectic orbifold $M_{\mu+\epsilon}$: If ϵ is small enough,

$$\mathcal{L}_{\mu,\epsilon}:=\left(\mathcal{L}|_{\Phi_K^{-1}(\mu+\epsilon)}\otimes\mathbb{C}_{-\mu}\right)/K_{\mu+\epsilon}.$$

is a line orbi-bundle over $M_{\mu+\epsilon}$.

We have the following important result (see §3.4.3 in [34]).

Proposition 3.7. *Let $\mu\in\Delta_K(M)\cap\wedge^*$. The Riemann–Roch number $RR(M_{\mu+\epsilon},\mathcal{L}_{\mu,\epsilon})$ do not depend on the choice of ϵ small enough so that $\mu+\epsilon\in\Delta_K(M)$ is a weakly regular value of Φ_K.*

We can now introduce the following definition.

Definition 3.8. Let $\mu\in\wedge^*_+$. We define

$$\mathcal{Q}(M_\mu,\Omega_\mu)=\begin{cases}0 & \text{if}\quad \mu\notin\Delta_K(M),\\ RR(M_{\mu+\epsilon},\mathcal{L}_{\mu,\epsilon}) & \text{if}\quad \mu\in\Delta_K(M).\end{cases}$$

Above, ϵ is chosen small enough so that $\mu+\epsilon\in\Delta_K(M)$ is a weakly regular value of Φ_K.

Let $n\geq 1$. The manifold M, equipped with the symplectic structure $n\Omega$, is prequantized by the line bundle $\mathcal{L}^{\otimes n}$: The corresponding moment map is $n\Phi_K$. For any dominant weight $\mu\in\wedge^*_+$, the symplectic reduction of $(M,n\Omega)$ relatively to the weight $n\mu$ is $(M_\mu,n\Omega_\mu)$. Like in Definition 3.8, we consider the following Riemann–Roch numbers

$$\mathcal{Q}(M_\mu,n\Omega_\mu)=\begin{cases}0 & \text{if}\quad \mu\notin\Delta_K(M),\\ RR(M_{\mu+\epsilon},(\mathcal{L}_{\mu,\epsilon})^{\otimes n}) & \text{if}\quad \mu\in\Delta_K(M)\text{ and }\|\epsilon\|<<1.\end{cases}$$

The Kawasaki–Riemann–Roch formula shows that $n\geq 1\mapsto\mathcal{Q}(M_\mu,n\Omega_\mu)$ is a quasi-polynomial map [37, 23]. When μ is a weakly regular value of Φ_K, we denote by $\mathrm{vol}(M_\mu):=\frac{1}{d_\mu}\int_{M_\mu}(\frac{\Omega_\mu}{2\pi})^{\frac{\dim M_\mu}{2}}$ the symplectic volume of the symplectic orbifold (M_μ,Ω_μ). Here, d_μ is the generic value of the map $m\in\Phi_K^{-1}(\mu)\mapsto\mathrm{cardinal}(K_m/K_m^0)$.

The following proposition is a direct consequence of the Kawasaki–Riemann–Roch formula (see [23] or §1.3.4 in [30]).

Proposition 3.9. *Let $\mu\in\Delta_K(M)\cap\wedge^*_+$ be a weakly regular value of Φ_K. Then we have $\mathcal{Q}(M_\mu,n\Omega_\mu)\sim\mathrm{vol}(M_\mu)\,n^{\frac{\dim M_\mu}{2}}$ when $n\to\infty$. In particular, the map $n\geq 1\mapsto\mathcal{Q}(M_\mu,n\Omega_\mu)$ is nonzero.*

3.5. Quantization commutes with reduction

Let us explain the "*quantization commutes with reduction*" theorem proved in [31].

We fix $\tilde{\lambda}\in\widehat{\tilde{G}}_{\mathrm{hol}}$. The coadjoint orbit $\tilde{G}\tilde{\lambda}$ is prequantized by the line bundle $\tilde{G}\times_{K_{\tilde{\lambda}}}\mathbb{C}_{\tilde{\lambda}}$, and the moment map $\Phi_G^{\tilde{\lambda}}:\tilde{G}\tilde{\lambda}\to\mathfrak{g}^*$ corresponding to the G-action on $\tilde{G}\times_{K_{\tilde{\lambda}}}\mathbb{C}_{\tilde{\lambda}}$ is equal to the restriction of the map $\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}$ to $\tilde{G}\tilde{\lambda}$.

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The symplectic slice $Y_{\tilde{\lambda}} = (\Phi_G^{\tilde{\lambda}})^{-1}(\mathfrak{k}^*)$ is prequantized by the line bundle $\mathcal{L}_{\tilde{\lambda}} := \tilde{G} \times_{K_{\tilde{\lambda}}} \mathbb{C}_{\tilde{\lambda}}|_{Y_{\tilde{\lambda}}}$. The moment map $\Phi_K^{\tilde{\lambda}} : Y_{\tilde{\lambda}} \to \mathfrak{k}^*$ corresponding to the K-action is equal to the restriction of $\Phi_G^{\tilde{\lambda}}$ to $Y_{\tilde{\lambda}}$.

For any $\lambda \in \widehat{G}_{\mathrm{hol}}$, we consider the (possibly singular) symplectic reduced space

$$\mathbb{X}_{\tilde{\lambda},\lambda} := (\Phi_K^{\tilde{\lambda}})^{-1}(\lambda)/K_\lambda,$$

equipped with the reduced symplectic form $\Omega_{\tilde{\lambda},\lambda}$, and the (possibly singular) line bundle

$$\mathbb{L}_{\tilde{\lambda},\lambda} := \left(\mathcal{L}_{\tilde{\lambda}}|_{(\Phi_K^{\tilde{\lambda}})^{-1}(\lambda)} \otimes \mathbb{C}_{-\lambda}\right)/K_\lambda.$$

Thanks to Definition 3.8, the geometric quantization $\mathcal{Q}(\mathbb{X}_{\tilde{\lambda},\lambda},\Omega_{\tilde{\lambda},\lambda}) \in \mathbb{Z}$ of those compact symplectic spaces $(\mathbb{X}_{\tilde{\lambda},\lambda},\Omega_{\tilde{\lambda},\lambda})$ are well-defined even if they are singular. In particular, $\mathcal{Q}(\mathbb{X}_{\tilde{\lambda},\lambda},\Omega_{\tilde{\lambda},\lambda}) = 0$ when $\mathbb{X}_{\tilde{\lambda},\lambda} = \emptyset$.

The following theorem is proved in [31].

Theorem 3.10. *Let $\tilde{\lambda} \in \widehat{\tilde{G}}_{\mathrm{hol}}$. We have an Hilbertian direct sum*

$$V_{\tilde{\lambda}}^{\tilde{G}}|_G = \bigoplus_{\lambda \in \widehat{G}_{\mathrm{hol}}} \mathcal{Q}(\mathbb{X}_{\tilde{\lambda},\lambda},\Omega_{\tilde{\lambda},\lambda})\ V_\lambda^G.$$

It means that, for any $\lambda \in \widehat{G}_{\mathrm{hol}}$, the multiplicity of the representation V_λ^G in the restriction $V_{\tilde{\lambda}}^{\tilde{G}}|_G$ is equal to the geometric quantization $\mathcal{Q}(\mathbb{X}_{\tilde{\lambda},\lambda},\Omega_{\tilde{\lambda},\lambda})$ of the (compact) reduced space $\mathbb{X}_{\tilde{\lambda},\lambda}$.

Remark 3.11. Let $(\tilde{\lambda},\lambda) \in \widehat{\tilde{G}}_{\mathrm{hol}} \times \widehat{G}_{\mathrm{hol}}$. Theorem 3.10. shows that

$$\left[V_{n\lambda}^G : V_{n\tilde{\lambda}}^{\tilde{G}}\right] = \mathcal{Q}(\mathbb{X}_{\tilde{\lambda},\lambda}, n\Omega_{\tilde{\lambda},\lambda})$$

for any $n \geq 1$.

4. Proofs of the main results

We come back to the setting of §2.2: G/K is a complex submanifold of a Hermitian symmetric space $\tilde{G}/\tilde{K}$. It means that there exits a $\tilde{K}$-invariant element $z \in \mathfrak{k}$ such that $\mathrm{ad}(z)$ defines complex structures on $\tilde{\mathfrak{p}}$ and $\mathfrak{p}$. We consider the orthogonal decomposition $\tilde{\mathfrak{p}} = \mathfrak{p} \oplus \mathfrak{q}$, and we denote by $\mathrm{Sym}(\mathfrak{q})$ the symmetric algebra of the complex K-module $(\mathfrak{q}, \mathrm{ad}(z))$.

4.1. Proof of Theorem A

The set $\Delta_{\mathrm{hol}}(\tilde{G},G)$ is equal to $\bigcup_{\tilde{a} \in \tilde{\mathcal{C}}_{\mathrm{hol}}} \{\tilde{a}\} \times \Delta_G(\tilde{G}\tilde{a})$. We define

$$\Delta_{\mathrm{hol}}(\tilde{G},G)^0 := \bigcup_{\tilde{a} \in \tilde{\mathcal{C}}_{\mathrm{hol}}} \{\tilde{a}\} \times \Delta_G(\tilde{G}\tilde{a})^0.$$

We start with the following result.

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Lemma 4.1. *The set* $\Delta_{\mathrm{hol}}(\tilde{G},G)^0 \bigcap \tilde{\mathfrak{t}}^*_{\mathbb{Q}} \times \mathfrak{t}^*_{\mathbb{Q}}$ *is dense in* $\Delta_{\mathrm{hol}}(\tilde{G},G)$.

Proof. Let $(\tilde{\xi},\xi) \in \Delta_{\mathrm{hol}}(\tilde{G},G)$: take $\tilde{g} \in \tilde{G}$ such that $\xi = \pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{g}\tilde{\xi})$. We consider a sequence $\tilde{\xi}_n \in \tilde{\mathcal{C}}_{\mathrm{hol}} \cap \tilde{\mathfrak{t}}^*_{\mathbb{Q}}$ converging to $\tilde{\xi}$. Then $\xi_n := \pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{g}\tilde{\xi}_n)$ is a sequence of $\mathcal{C}^0_{G/K}$ converging to $\xi \in \mathcal{C}_{\mathrm{hol}}$. Since the map $\mathrm{p} : \mathcal{C}^0_{G/K} \to \mathcal{C}_{\mathrm{hol}}$ is continuous (see Lemma 2.4), the sequence $\eta_n := \mathrm{p}(\xi_n)$ converges to $\mathrm{p}(\xi) = \xi$. By definition, we have $\eta_n \in \Delta_G(\tilde{G}\tilde{\xi}_n)$ for any $n \in \mathbb{N}$. Since $\tilde{\xi}_n$ are rational, each subset $\Delta_G(\tilde{G}\tilde{\xi}_n)^0 \cap \mathfrak{t}^*_{\mathbb{Q}}$ is dense in $\Delta_G(\tilde{G}\tilde{\xi}_n)$ (see Lemma 2.11). Hence, $\forall n \in \mathbb{N}$, there exists $\zeta_n \in \Delta_G(\tilde{G}\tilde{\xi}_n)^0 \cap \mathfrak{t}^*_{\mathbb{Q}}$ such that $\|\zeta_n - \eta_n\| \leq 2^{-n}$. Finally, we see that $(\tilde{\xi}_n,\zeta_n)$ is a sequence of rational elements of $\Delta_{\mathrm{hol}}(\tilde{G},G)^0$ converging to $(\xi,\tilde{\xi})$. □

The main purpose of this section is the proof of the following theorem.

Theorem 4.2. *For any rational element* $(\tilde{\mu},\mu)$ *of the holomorphic chamber* $\tilde{\mathcal{C}}_{\mathrm{hol}} \times \mathcal{C}_{\mathrm{hol}}$, *the following statements hold:*

- *If* $\mu \in \Delta_G(\tilde{G}\tilde{\mu})^0$, *then* $(\tilde{\mu},\mu) \in \Pi^{\mathbb{Q}}_{\mathrm{hol}}(\tilde{G},G)$.
- *If* $(\tilde{\mu},\mu) \in \Pi^{\mathbb{Q}}_{\mathrm{hol}}(\tilde{G},G)$, *then* $\mu \in \Delta_G(\tilde{G}\tilde{\mu})$.

In other words, we have the following inclusions:

$$\Delta_{\mathrm{hol}}(\tilde{G},G)^0 \bigcap \tilde{\mathfrak{t}}^*_{\mathbb{Q}} \times \mathfrak{t}^*_{\mathbb{Q}} \underset{(1)}{\subset} \Pi^{\mathbb{Q}}_{\mathrm{hol}}(\tilde{G},G) \underset{(2)}{\subset} \Delta_{\mathrm{hol}}(\tilde{G},G).$$

Lemma 4.1 and Theorem 4.2 gives the important corollary.

Corollary 4.3. $\Pi^{\mathbb{Q}}_{\mathrm{hol}}(\tilde{G},G)$ *is dense in* $\Delta_{\mathrm{hol}}(\tilde{G},G)$.

Proof of Theorem 4.2. Let $(\tilde{\mu},\mu) \in \Pi^{\mathbb{Q}}_{\mathrm{hol}}(\tilde{G},G)$: There exists $N \geq 1$ such that $(N\tilde{\mu},N\mu) \in \Pi^{\mathbb{Z}}_{\mathrm{hol}}(\tilde{G},G)$. The multiplicity $[V^G_{N\mu} : V^{\tilde{G}}_{N\tilde{\mu}}]$ is nonzero, and thanks to Theorem 3.10, it implies that the reduced space $\mathbb{X}_{N\tilde{\mu},N\mu}$ is nonempty. In other words, $(N\tilde{\mu},N\mu) \in \Delta_{\mathrm{hol}}(\tilde{G},G)$. The inclusion (2) is proven.

Let $(\tilde{\mu},\mu) \in \Delta_{\mathrm{hol}}(\tilde{G},G)^0 \bigcap \mathfrak{t}^*_{\mathbb{Q}} \times \tilde{\mathfrak{t}}^*_{\mathbb{Q}}$. There exists $N_o \geq 1$ such that $\lambda := N_o\mu \in \widehat{G}_{\mathrm{hol}}$, $\tilde{\lambda} := N_o\tilde{\mu} \in \widehat{\tilde{G}}_{\mathrm{hol}}$ and $\lambda \in \Delta_G(\tilde{G}\tilde{\lambda})^0$: The element λ is a weakly regular value of the moment map $\tilde{G}\tilde{\lambda} \to \mathfrak{g}^*$. Theorem 3.10 tells us that, for any $n \geq 1$, the multiplicity $[V^G_{n\lambda} : V^{\tilde{G}}_{n\tilde{\lambda}}]$ is equal to Riemann–Roch number $\mathcal{Q}(\mathbb{X}_{\tilde{\lambda},\lambda}, n\Omega_{\tilde{\lambda},\lambda})$. Since the map $n \mapsto \mathcal{Q}(\mathbb{X}_{\tilde{\lambda},\lambda}, n\Omega_{\tilde{\lambda},\lambda})$ is nonzero (see Proposition 3.9), we can conclude that there exists $n_o \geq 1$ such that $[V^G_{n_o\lambda} : V^{\tilde{G}}_{n_o\tilde{\lambda}}] \neq 0$. In other words, we obtain $n_oN_o(\tilde{\mu},\mu) \in \Pi^{\mathbb{Z}}_{\mathrm{hol}}(\tilde{G},G)$ and so $(\tilde{\mu},\mu) \in \Pi^{\mathbb{Q}}_{\mathrm{hol}}(\tilde{G},G)$. The inclusion (1) is settled. □

Now we can terminate the proof of Theorem A.

Thanks to Proposition 3.5, we know that $\Pi^{\mathbb{Q}}_{\mathrm{hol}}(\tilde{G},G)$ is a $\mathbb{Q}$-convex cone. Since $\Delta_{\mathrm{hol}}(\tilde{G},G)$ is a closed subset of $\tilde{\mathcal{C}}_{\mathrm{hol}} \times \mathcal{C}_{\mathrm{hol}}$ (see Proposition 2.6), we can conclude, by a density argument, that $\Delta_{\mathrm{hol}}(\tilde{G},G)$ is a closed convex cone of $\tilde{\mathcal{C}}_{\mathrm{hol}} \times \mathcal{C}_{\mathrm{hol}}$.

4.2. The affine variety $\tilde{K}_{\mathbb{C}} \times \mathfrak{q}$

Let $\tilde{\kappa}$ be the Killing form on the Lie algebra $\tilde{\mathfrak{g}}$. We consider the $\tilde{K}$-invariant symplectic structures $\Omega_{\tilde{\mathfrak{p}}}$ on $\tilde{\mathfrak{p}}$, defined by the relation

$$\Omega_{\tilde{\mathfrak{p}}}(\tilde{Y},\tilde{Y}') = \tilde{\kappa}(z,[\tilde{Y},\tilde{Y}']), \quad \forall \tilde{Y},\tilde{Y}' \in \tilde{\mathfrak{p}}.$$

We notice that the complex structure $\mathrm{ad}(z)$ is adapted to $\Omega_{\tilde{\mathfrak{p}}}$: $\Omega_{\tilde{\mathfrak{p}}}(\tilde{Y},\mathrm{ad}(z)\tilde{Y}) > 0$ if $\tilde{Y} \neq 0$.

We denote by $\Omega_{\mathfrak{q}}$ the restriction of $\Omega_{\tilde{\mathfrak{p}}}$ on the symplectic subspace $\mathfrak{q}$. The moment map $\Phi_{\mathfrak{q}}$ associated to the K-action on $(\mathfrak{q},\Omega_{\mathfrak{q}})$ is defined by the relations $\langle \Phi_{\mathfrak{q}}(Y),X\rangle = \frac{-1}{2}\tilde{\kappa}([X,Y],[z,Y])$, $\forall (X,Y) \in \mathfrak{p}\times\mathfrak{q}$. In particular, $\langle \Phi_{\mathfrak{q}}(Y),z\rangle = \frac{-1}{2}\|Y\|^2$, $\forall Y \in \mathfrak{q}$, so the map $\langle \Phi_{\mathfrak{q}},z\rangle$ is proper.

The complex reductive group $\tilde{K}_{\mathbb{C}}$ is equipped with the following action of $\tilde{K}\times K$: $(\tilde{k},k)\cdot a = \tilde{k}ak^{-1}$. It has a canonical structure of a smooth affine variety. There is a diffeomorphism of the cotangent bundle $\mathbf{T}^*\tilde{K}$ with $\tilde{K}_{\mathbb{C}}$ defined as follows. We identify $\mathbf{T}^*\tilde{K}$ with $\tilde{K}\times\tilde{\mathfrak{k}}^*$ by means of left-translation and then with $\tilde{K}\times\tilde{\mathfrak{k}}$ by means of an invariant inner product on $\tilde{\mathfrak{k}}$. The map $\varphi : \tilde{K}\times\tilde{\mathfrak{k}} \to \tilde{K}_{\mathbb{C}}$ given by $\varphi(a,X) = ae^{iX}$ is a diffeomorphism. If we use φ to transport the complex structure of $\tilde{K}_{\mathbb{C}}$ to $\mathbf{T}^*\tilde{K}$, then the resulting complex structure on $\mathbf{T}^*\tilde{K}$ is compatible with the symplectic structure on $\mathbf{T}^*\tilde{K}$ so that $\mathbf{T}^*\tilde{K}$ becomes a Kähler Hamiltonian $\tilde{K}\times K$-manifold (see [11], §3). The moment map relative to the $\tilde{K}\times K$-action is the proper map $\Phi_{\tilde{K}}\oplus\Phi_K :\ \mathbf{T}^*\tilde{K} \to \tilde{\mathfrak{k}}^*\oplus\mathfrak{k}^*$ defined by $\Phi_{\tilde{K}}(\tilde{a},\tilde{\eta}) = -\tilde{a}\tilde{\eta}$ and $\Phi_K(\tilde{a},\tilde{\eta}) = \pi_{\mathfrak{k},\tilde{\mathfrak{k}}}(\tilde{\eta})$. Here $\pi_{\mathfrak{k},\tilde{\mathfrak{k}}} : \tilde{\mathfrak{k}}^* \to \mathfrak{k}^*$ is the projection dual to the inclusion $\mathfrak{k}\hookrightarrow\tilde{\mathfrak{k}}$ of Lie algebras.

Finally, we consider the Kähler Hamiltonian $\tilde{K}\times K$-manifold $\mathbf{T}^*\tilde{K}\times\mathfrak{q}$, where $\mathfrak{q}$ is equipped with the symplectic structure $\Omega_{\mathfrak{q}}$. Let us denote by $\Phi :\ \mathbf{T}^*\tilde{K}\times\mathfrak{q} \to \tilde{\mathfrak{k}}^*\oplus\mathfrak{k}^*$ the moment map relative to the $\tilde{K}\times K$-action:

$$\Phi(\tilde{a},\tilde{\eta},Y) = \Big(-\tilde{a}\tilde{\eta},\pi_{\mathfrak{k},\tilde{\mathfrak{k}}}(\tilde{\eta}) + \Phi_{\mathfrak{q}}(Y)\Big). \tag{15}$$

Since Φ is proper map, the convexity theorem tells us that

$$\Delta(\mathbf{T}^*\tilde{K}\times\mathfrak{q}) := \mathrm{Image}(\Phi)\bigcap \tilde{\mathfrak{t}}^*_{\geq 0}\times\mathfrak{t}^*_{\geq 0}$$

is a closed convex locally polyhedral set.

We consider now the action of $\tilde{K}\times K$ on the affine variety $\tilde{K}_{\mathbb{C}}\times\mathfrak{q}$. The set of highest weights of $\tilde{K}_{\mathbb{C}}\times\mathfrak{q}$ is the semigroup $\Pi^{\mathbb{Z}}(\tilde{K}_{\mathbb{C}}\times\mathfrak{q}) \subset \tilde{\wedge}^*_+ \times \wedge^*_+$ consisting of all dominant weights $(\tilde{\lambda},\lambda)$ such that the irreducible $\tilde{K}\times K$-representation $V^{\tilde{K}}_{\tilde{\lambda}}\otimes V^K_{\lambda}$ occurs in the coordinate ring $\mathbb{C}[\tilde{K}_{\mathbb{C}}\times\mathfrak{q}]$. We denote by $\Pi^{\mathbb{Q}}(\tilde{K}_{\mathbb{C}}\times\mathfrak{q})$ the $\mathbb{Q}$-convex cone generated by the semigroup $\Pi^{\mathbb{Z}}(\tilde{K}_{\mathbb{C}}\times\mathfrak{q})$: $(\tilde{\xi},\xi)\in\Pi^{\mathbb{Q}}(\tilde{K}_{\mathbb{C}}\times\mathfrak{q})$ if and only if $\exists N\geq 1$, $N(\tilde{\xi},\xi)\in\Pi^{\mathbb{Z}}(\tilde{K}_{\mathbb{C}}\times\mathfrak{q})$.

The following important fact is classical (see Theorem 4.9 in [35]).

Proposition 4.4. *The Kirwan polyhedron* $\Delta(\boldsymbol{T}^*\tilde{K}\times\mathfrak{q})$ *is equal to the closure of the* $\mathbb{Q}$*-convex cone* $\Pi^{\mathbb{Q}}(\tilde{K}_{\mathbb{C}}\times\mathfrak{q})$.

A direct application of the Peter–Weyl theorem gives the following characterization:

$$(\tilde{\lambda},\lambda)\in\Pi^{\mathbb{Z}}(\tilde{K}_{\mathbb{C}}\times\mathfrak{q})\Longleftrightarrow\left[V_{\tilde{\lambda}}^{\tilde{K}}|_K\otimes V_{\lambda}^{K}\otimes\mathrm{Sym}(\mathfrak{q})\right]^K\neq 0 \qquad (16)$$
$$\Longleftrightarrow\left[V_{\lambda^*}^{K}:V_{\tilde{\lambda}}^{\tilde{K}}|_K\otimes\mathrm{Sym}(\mathfrak{q})\right]\neq 0$$
$$\Longleftrightarrow(\tilde{\lambda},\lambda^*)\in\Pi^{\mathbb{Z}}_{\mathfrak{q}}(\tilde{K},K).$$

4.3. Proof of Theorem B

Consider the semigroup $\Pi^{\mathbb{Z}}_{\mathfrak{q}}(\tilde{K},K)$ of $\tilde{\wedge}^*_+\times\wedge^*_+$ (see Definition 1.3) and the $\mathbb{Q}$-convex cone $\Pi^{\mathbb{Q}}_{\mathfrak{q}}(\tilde{K},K):=\{(\tilde{\xi},\xi)\in\tilde{\mathfrak{t}}^*_{\geq 0}\times\mathfrak{t}^*_{\geq 0}\;\;\exists N\geq 1, N(\tilde{\xi},\xi)\in\Pi^{\mathbb{Z}}_{\mathfrak{q}}(\tilde{K},K)\}$.

The Jakobsen–Vergne theorem says that $\Pi^{\mathbb{Z}}_{\mathrm{hol}}(\tilde{G},G)=\Pi^{\mathbb{Z}}_{\mathfrak{q}}(\tilde{K},K)\bigcap\widehat{\tilde{G}}_{\mathrm{hol}}\times\widehat{G}_{\mathrm{hol}}$. Hence, the convex cone $\Pi^{\mathbb{Q}}_{\mathrm{hol}}(\tilde{G},G)$ is equal to $\Pi^{\mathbb{Q}}_{\mathfrak{q}}(\tilde{K},K)\cap\tilde{\mathcal{C}}_{\mathrm{hol}}\times\mathcal{C}_{\mathrm{hol}}$. Thanks to equation (16), we know that $(\tilde{\xi},\xi)\in\Pi^{\mathbb{Q}}_{\mathfrak{q}}(\tilde{K},K)$ if and only if $(\tilde{\xi},\xi^*)\in\Pi^{\mathbb{Q}}(\tilde{K}_{\mathbb{C}}\times\mathfrak{q})$. The density results obtained in Proposition 4.4 and Corollary 4.3 gives finally Theorem B.

4.4. Proof of Theorem C

We denote by $\bar{\mathfrak{q}}$ the K-vector space $\mathfrak{q}$ equipped with the opposite symplectic form $-\Omega_{\mathfrak{q}}$ and opposite complex structure $-\mathrm{ad}(z)$. The moment map relative to the K-action on $\bar{\mathfrak{q}}$ is denoted by $\Phi_{\bar{\mathfrak{q}}}=-\Phi_{\mathfrak{q}}$.

Lemma 4.5. *Any element $(\tilde{\xi},\xi)\in\tilde{\mathfrak{t}}^*_{\geq 0}\times\mathfrak{t}^*_{\geq 0}$ satisfies the equivalence*

$$(\tilde{\xi},\xi^*)\in\Delta(\boldsymbol{T}^*\tilde{K}\times\mathfrak{q})\Longleftrightarrow\xi\in\Delta_K(\tilde{K}\tilde{\xi}\times\bar{\mathfrak{q}}).$$

Proof. Thanks to equation (15), we see immediatly that $\exists(\tilde{a},\tilde{\eta},Y)\in\;\mathbf{T}^*\tilde{K}\times\mathfrak{q}$ such that $(\tilde{\xi},\xi^*)=\Phi(\tilde{a},\tilde{\eta},Y)$ if and only if $\exists(\tilde{b},Z)\in\tilde{K}\times\mathfrak{q}$ such that $\xi=\pi_{\mathfrak{k},\tilde{\mathfrak{k}}}(\tilde{b}\tilde{\xi})+\Phi_{\bar{\mathfrak{q}}}(Z)$. □

At this stage, we know that $\Delta_G(\tilde{G}\tilde{\mu})=\Delta_K(\tilde{K}\tilde{\mu}\times\bar{\mathfrak{q}})\cap\mathcal{C}_{\mathrm{hol}}$. Hence, Theorem C will follow from the next result.

Proposition 4.6. *For any $\tilde{\mu}\in\tilde{\mathcal{C}}_{\mathrm{hol}}$, the Kirwan polyhedron $\Delta_K(\tilde{K}\tilde{\mu}\times\bar{\mathfrak{q}})$ is contained in $\mathcal{C}_{\mathrm{hol}}$.*

Proof. By definition $\mathcal{C}_{\mathrm{hol}}=\mathcal{C}^0_{G/K}\cap\mathfrak{t}^*_{\geq 0}$, so we have to prove that $\pi_{\mathfrak{k},\tilde{\mathfrak{k}}}(\tilde{K}\tilde{\mu})+\mathrm{Image}(\Phi_{\bar{\mathfrak{q}}})$ is contained in $\mathcal{C}^0_{G/K}$. By definition $\tilde{K}\tilde{\mu}\subset\mathcal{C}^0_{\tilde{G}/\tilde{K}}$, and then $\pi_{\mathfrak{k},\tilde{\mathfrak{k}}}(\tilde{K}\tilde{\mu})\subset\mathcal{C}^0_{G/K}$. Since $\mathcal{C}^0_{G/K}+\mathcal{C}_{G/K}\subset\mathcal{C}^0_{G/K}$, it is sufficient to check that $\mathrm{Image}(\Phi_{\bar{\mathfrak{q}}})\subset\mathcal{C}_{G/K}$. Let $\Phi_{\tilde{\mathfrak{p}}}$ be the moment map relative to the action of $\tilde{K}$ on $(\tilde{\mathfrak{p}},\Omega_{\tilde{\mathfrak{p}}})$. As $\mathrm{Image}(\Phi_{\bar{\mathfrak{q}}})\subset\pi_{\mathfrak{k},\tilde{\mathfrak{k}}}\left(\mathrm{Image}(-\Phi_{\tilde{\mathfrak{p}}})\right)$, the following lemma will terminate the proof of Proposition 4.6. □

Lemma 4.7. *The image of the moment map $-\Phi_{\tilde{\mathfrak{p}}}$ is contained in $\mathcal{C}_{\tilde{G}/\tilde{K}}$.*

Proof. Let $z^*\in\tilde{\mathfrak{t}}^*$ such that $\langle z^*,\tilde{X}\rangle=-\tilde{\kappa}(z,\tilde{X})$, $\forall\tilde{X}\in\tilde{\mathfrak{g}}$. Consider the coadjoint orbit $\tilde{\mathcal{O}}=\tilde{G}z^*$ equipped with its canonical symplectic structure $\Omega_{\tilde{\mathcal{O}}}$: The symplectic vector space $\mathbf{T}_{z^*}\tilde{\mathcal{O}}$ is canonically isomorphic to $(\tilde{\mathfrak{p}},-\Omega_{\tilde{\mathfrak{p}}})$. In [26], McDuff proved that $(\tilde{\mathcal{O}},\Omega_{\tilde{\mathcal{O}}})$ is diffeomorphic, as a $\tilde{K}$-symplectic manifold, to the symplectic vector space $(\tilde{\mathfrak{p}},-\Omega_{\tilde{\mathfrak{p}}})$

(see [6, 8] for a generalization of this fact). McDuff's results show in particular that $\mathrm{Image}(-\Phi_{\tilde{\mathfrak{p}}}) = \pi_{\tilde{\mathfrak{g}},\tilde{\mathfrak{k}}}(\tilde{\mathcal{O}})$. Our proof is completed if we check that $\pi_{\tilde{\mathfrak{g}},\tilde{\mathfrak{k}}}(\tilde{\mathcal{O}}) \subset \mathcal{C}_{\tilde{G}/\tilde{K}}$: In other words, if $\langle \pi_{\tilde{\mathfrak{g}},\tilde{\mathfrak{k}}}(\tilde{g}_0 z^*), \tilde{g}_1 z\rangle \geq 0$, $\forall \tilde{g}_0, \tilde{g}_1 \in \tilde{G}$. But

$$2\langle \pi_{\tilde{\mathfrak{g}},\tilde{\mathfrak{k}}}(\tilde{g}_0 z^*), \tilde{g}_1 z\rangle = \langle \tilde{g}_0 z^*, \tilde{g}_1 z + \Theta(\tilde{g}_1) z\rangle$$
$$= -\tilde{\kappa}(z, \tilde{g}_0^{-1}\tilde{g}_1 z) - \tilde{\kappa}(z, \tilde{g}_0^{-1}\Theta(\tilde{g}_1) z).$$

With equation (7) in hand, it is not difficult to see that $-\tilde{\kappa}(z, \tilde{g} z) \geq 0$ for every $\tilde{g} \in \tilde{G}$. We thus verified that $\pi_{\tilde{\mathfrak{g}},\tilde{\mathfrak{k}}}(\tilde{\mathcal{O}}) \subset \mathcal{C}_{\tilde{G}/\tilde{K}}$. □

5. Inequalities characterizing the cones $\Delta_{\mathrm{hol}}(\tilde{G}, G)$

We come back to the framework of §4.2. We consider the Kähler Hamiltonian $\tilde{K} \times K$-manifold $\mathbf{T}^*\tilde{K} \times \mathfrak{q}$. The moment map, $\Phi : \mathbf{T}^*\tilde{K} \times \mathfrak{q} \to \tilde{\mathfrak{k}}^* \oplus \mathfrak{k}^*$, relative to the $\tilde{K} \times K$-action, is defined by equation (15).

In this section, we adapt to our case the result of §6 of [32] concerning the parametrization of the facets of Kirwan polyhedrons in terms of Ressayre's data.

5.1. Admissible elements

We choose maximal tori $\tilde{T} \subset \tilde{K}$ and $T \subset K$ such that $T \subset \tilde{T}$. Let $\mathfrak{R}_o$ and $\mathfrak{R}$ be, respectively, the set of roots for the action of T on $(\tilde{\mathfrak{g}}/\mathfrak{g}) \otimes \mathbb{C}$ and $\mathfrak{g} \otimes \mathbb{C}$. Let $\tilde{\mathfrak{R}}$ be the set of roots for the action of $\tilde{T}$ on $\tilde{\mathfrak{g}} \otimes \mathbb{C}$. Let $\mathfrak{R}^+ \subset \mathfrak{R}$ and $\tilde{\mathfrak{R}}^+ \subset \tilde{\mathfrak{R}}$ be the systems of positive roots defined in equation (6). Let $W, \tilde{W}$ be the Weyl groups of (T, K) and $(\tilde{T}, \tilde{K})$. Let $w_o \in W$ be the longest element.

We start by introducing the notion of admissible elements. The group $\hom(U(1), T)$ admits a natural identification with the lattice $\wedge := \frac{1}{2\pi}\ker(\exp : \mathfrak{t} \to T)$. A vector $\gamma \in \mathfrak{t}$ is called rational if it belongs to the $\mathbb{Q}$-vector space $\mathfrak{t}_{\mathbb{Q}}$ generated by $\wedge$.

We consider the $\tilde{K} \times K$-action on $N := \mathbf{T}^*\tilde{K} \times \mathfrak{q}$. We associate to any subset $\mathcal{X} \subset N$, the integer $\dim_{\tilde{K}\times K}(\mathcal{X})$ (see equation (5)).

Definition 5.1. A nonzero element $(\tilde{\gamma}, \gamma) \in \tilde{\mathfrak{t}} \times \mathfrak{t}$ is called *admissible* if the elements $\tilde{\gamma}$ and γ are rational and if $\dim_{\tilde{K}\times K}(N^{(\tilde{\gamma},\gamma)}) - \dim_{\tilde{K}\times K}(N) \in \{0, 1\}$.

If $\gamma \in \mathfrak{t}$, we denote by $\mathfrak{R}_o \cap \gamma^{\perp}$ the subsets of weight vanishing against γ. We start with the following lemma whose proof is left to the reader (see §6.1.1 of [32]).

Lemma 5.2.

1. *$N^{(\tilde{\gamma},\gamma)} \neq \emptyset$ if and only if $\tilde{\gamma} \in \tilde{W}\gamma$.*
2. $\dim_{\tilde{K}\times K}(N) = \dim_T(\tilde{\mathfrak{g}}/\mathfrak{g}) = \dim(\mathfrak{t}) - \dim(\mathrm{Vect}(\mathfrak{R}_o))$.
3. *For any $\tilde{w} \in \tilde{W}$,* $\dim_{\tilde{K}\times K}(N^{(\tilde{w}\gamma,\gamma)}) = \dim_T(\tilde{\mathfrak{g}}^{\gamma}/\mathfrak{g}^{\gamma}) = \dim(\mathfrak{t}) - \dim(\mathrm{Vect}(\mathfrak{R}_o \cap \gamma^{\perp}))$.

The next result is a direct consequence of the previous lemma.

Lemma 5.3. *The admissible elements relative to the* $\tilde{K}\times K$*-action on* $\boldsymbol{T}^*\tilde{K}\times\mathfrak{q}$ *are of the form* $(\tilde{w}\gamma,\gamma)$*, where* $\tilde{w}\in\tilde{W}$ *and* γ *is a nonzero rational element satisfying* $\mathrm{Vect}(\mathfrak{R}_o)\cap\gamma^\perp=\mathrm{Vect}(\mathfrak{R}_o\cap\gamma^\perp)$.

5.2. Ressayre's data

Definition 5.4.

1. Consider the linear action $\rho: G\to \mathrm{GL}_{\mathbb{C}}(V)$ of a compact Lie group on a complex vector space V. For any $(\eta,a)\in\mathfrak{g}\times\mathbb{R}$, we define the vector subspace $V^{\eta=a}=\{v\in V, d\rho(\eta)v=iav\}$. Thus, for any $\eta\in\mathfrak{g}$, we have the decomposition $V=V^{\eta>0}\oplus V^{\eta=0}\oplus V^{\eta<0}$, where $V^{\eta>0}=\sum_{a>0}V^{\eta=a}$, and $V^{\eta<0}=\sum_{a<0}V^{\eta=a}$.
2. The real number $\mathrm{Tr}_\eta(V^{\eta>0})$ is defined as the sum $\sum_{a>0}a\dim(V^{\eta=a})$.

We consider an admissible element $(\tilde{w}\gamma,\gamma)$. The submanifold of $N\simeq\tilde{K}_{\mathbb{C}}\times\mathfrak{q}$ fixed by $(\tilde{w}\gamma,\gamma)$ is $N^{(\tilde{w}\gamma,\gamma)}=\tilde{w}\tilde{K}_{\mathbb{C}}^{\gamma}\times\mathfrak{q}^\gamma$. There is a canonical isomorphism between the manifold $N^{(\tilde{w}\gamma,\gamma)}$ equipped with the action of $\tilde{w}\tilde{K}^\gamma\tilde{w}^{-1}\times K^\gamma$ with the manifold $\tilde{K}_{\mathbb{C}}^{\gamma}\times\mathfrak{q}^\gamma$ equipped with the action of $\tilde{K}^\gamma\times K^\gamma$. The tangent bundle $(\mathbf{T}N|_{N^{(\tilde{w}\gamma,\gamma)}})^{(\tilde{w}\gamma,\gamma)>0}$ is isomorphic to $N^{\gamma_w}\times\tilde{\mathfrak{k}}_{\mathbb{C}}^{\gamma>0}\times\mathfrak{q}^{\gamma>0}$.

The choice of positive roots $\mathfrak{R}^+$ (resp. $\tilde{\mathfrak{R}}^+$) induces a decomposition $\mathfrak{k}_{\mathbb{C}}=\mathfrak{n}\oplus\mathfrak{t}_{\mathbb{C}}\oplus\overline{\mathfrak{n}}$ (resp. $\tilde{\mathfrak{k}}_{\mathbb{C}}=\tilde{\mathfrak{n}}\oplus\tilde{\mathfrak{t}}_{\mathbb{C}}\oplus\overline{\tilde{\mathfrak{n}}}$), where $\mathfrak{n}=\sum_{\alpha\in\mathfrak{R}^+}(\mathfrak{k}\otimes\mathbb{C})_\alpha$ (resp. $\tilde{\mathfrak{n}}=\sum_{\tilde{\alpha}\in\tilde{\mathfrak{R}}^+}(\tilde{\mathfrak{k}}\otimes\mathbb{C})_{\tilde{\alpha}}$). We consider the map

$$\rho^{\tilde{w},\gamma}:\tilde{K}_{\mathbb{C}}^{\gamma}\times\mathfrak{q}^\gamma\longrightarrow \hom\left(\tilde{\mathfrak{n}}^{\tilde{w}\gamma>0}\times\mathfrak{n}^{\gamma>0},\tilde{\mathfrak{k}}_{\mathbb{C}}^{\gamma>0}\times\mathfrak{q}^{\gamma>0}\right)$$

defined by the relation

$$\rho^{\tilde{w},\gamma}(\tilde{x},v):(\tilde{X},X)\longmapsto((\tilde{w}\tilde{x})^{-1}\tilde{X}-X\,;X\cdot v)$$

for any $(\tilde{x},v)\in\tilde{K}_{\mathbb{C}}^{\gamma}\times\mathfrak{q}^\gamma$.

Definition 5.5. $(\gamma,\tilde{w})\in\mathfrak{t}\times\tilde{W}$ is a Ressayre's datum if

1. $(\tilde{w}\gamma,\gamma)$ is admissible,
2. $\exists(\tilde{x},v)$ such that $\rho^{\tilde{w},\gamma}(\tilde{x},v)$ is bijective.

Remark 5.6. In [32], the Ressayre's data were called *regular infinitesimal B-Ressayre's pairs.*

Since the linear map $\rho^{\tilde{w},\gamma}(\tilde{x},v)$ commutes with the γ-actions, we obtain the following necessary conditions.

Lemma 5.7. *If* $(\gamma,\tilde{w})\in\mathfrak{t}\times\tilde{W}$ *is a Ressayre's datum, then*

- *Relation (A):* $\dim(\tilde{\mathfrak{n}}^{\tilde{w}\gamma>0})+\dim(\mathfrak{n}^{\gamma>0})=\dim(\tilde{\mathfrak{k}}_{\mathbb{C}}^{\gamma>0})+\dim(\mathfrak{q}^{\gamma>0})$.
- *Relation (B):* $\mathrm{Tr}_{\tilde{w}\gamma}(\tilde{\mathfrak{n}}^{\tilde{w}\gamma>0})+\mathrm{Tr}_\gamma(\mathfrak{n}^{\gamma>0})=\mathrm{Tr}_\gamma(\tilde{\mathfrak{k}}_{\mathbb{C}}^{\gamma>0})+\mathrm{Tr}_\gamma(\mathfrak{q}^{\gamma>0})$.

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Lemma 5.8. *Relation (B) is equivalent to*

$$\sum_{\substack{\alpha\in\mathfrak{R}^+\\ \langle\alpha,\gamma\rangle>0}} \langle\alpha,\gamma\rangle = \sum_{\substack{\tilde{\alpha}\in\tilde{\mathfrak{R}}^+\\ \langle\tilde{\alpha},\tilde{w}_0\tilde{w}\gamma\rangle>0}} \langle\tilde{\alpha},\tilde{w}_0\tilde{w}\gamma\rangle. \tag{17}$$

Proof. First, one sees that $\mathrm{Tr}_\gamma(\mathfrak{q}^{\gamma>0}) = \mathrm{Tr}_\gamma(\tilde{\mathfrak{p}}^{\gamma>0}) - \mathrm{Tr}_\gamma(\mathfrak{p}^{\gamma>0}) = \sum_{\substack{\tilde{\alpha}\in\tilde{\mathfrak{R}}_n^+\\ \langle\tilde{\alpha},\gamma\rangle>0}} \langle\tilde{\alpha},\gamma\rangle - \sum_{\substack{\alpha\in\mathfrak{R}_n^+\\ \langle\alpha,\gamma\rangle>0}} \langle\alpha,\gamma\rangle$, and $\mathrm{Tr}_\gamma(\tilde{\mathfrak{k}}_{\mathbb{C}}^{\gamma>0}) = \mathrm{Tr}_{\tilde{w}\gamma}(\tilde{\mathfrak{k}}_{\mathbb{C}}^{\tilde{w}\gamma>0}) = \mathrm{Tr}_{\tilde{w}\gamma}(\tilde{\mathfrak{n}}^{\tilde{w}\gamma>0}) + \sum_{\substack{\tilde{\alpha}\in\tilde{\mathfrak{R}}_c^+\\ \langle\tilde{\alpha},\tilde{w}_0\tilde{w}\gamma\rangle>0}} \langle\tilde{\alpha},\tilde{w}_0\tilde{w}\gamma\rangle$. Relation (B) is equivalent to

$$\mathrm{Tr}_\gamma(\mathfrak{n}^{\gamma>0}) + \sum_{\substack{\alpha\in\mathfrak{R}_n^+\\ \langle\alpha,\gamma\rangle>0}} \langle\alpha,\gamma\rangle = \sum_{\substack{\tilde{\alpha}\in\tilde{\mathfrak{R}}_n^+\\ \langle\tilde{\alpha},\gamma\rangle>0}} \langle\tilde{\alpha},\gamma\rangle + \sum_{\substack{\tilde{\alpha}\in\tilde{\mathfrak{R}}_c^+\\ \langle\tilde{\alpha},\tilde{w}_0\tilde{w}\gamma\rangle>0}} \langle\tilde{\alpha},\tilde{w}_0\tilde{w}\gamma\rangle. \tag{18}$$

Since $\tilde{\mathfrak{R}}_n^+$ is invariant under the action of the Weyl group $\tilde{W}$, the right-hand side of equation (18) is equal to $\sum_{\substack{\tilde{\alpha}\in\tilde{\mathfrak{R}}^+\\ \langle\tilde{\alpha},\tilde{w}_0\tilde{w}\gamma\rangle>0}} \langle\tilde{\alpha},\tilde{w}_0\tilde{w}\gamma\rangle$. Since the left-hand side of equation (18) is equal to $\sum_{\substack{\alpha\in\mathfrak{R}^+\\ \langle\alpha,\gamma\rangle>0}} \langle\alpha,\gamma\rangle$, the proof of the lemma is complete. □

5.3. Cohomological characterization of Ressayre's data

Let $\gamma\in\mathfrak{t}$ be a nonzero rational element. We denote by $B\subset K_{\mathbb{C}}$ and by $\tilde{B}\subset\tilde{K}_{\mathbb{C}}$ the Borel subgroups with Lie algebra $\mathfrak{b}=\mathfrak{t}_{\mathbb{C}}\oplus\mathfrak{n}$ and $\tilde{\mathfrak{b}}=\tilde{\mathfrak{t}}_{\mathbb{C}}\oplus\tilde{\mathfrak{n}}$. Consider the parabolic subgroup $P_\gamma\subset K_{\mathbb{C}}$ defined by

$$P_\gamma = \{g\in K_{\mathbb{C}}, \lim_{t\to\infty}\exp(-it\gamma)g\exp(it\gamma) \text{ exists}\}. \tag{19}$$

Similarly, one defines a parabolic subgroup $\tilde{P}_\gamma\subset\tilde{K}_{\mathbb{C}}$.

We work with the projective varieties $\mathcal{F}_\gamma := K_{\mathbb{C}}/P_\gamma$, $\tilde{\mathcal{F}}_\gamma := \tilde{K}_{\mathbb{C}}/\tilde{P}_\gamma$ and the canonical embedding $\iota:\mathcal{F}_\gamma\to\tilde{\mathcal{F}}_\gamma$. We associate to any $\tilde{w}\in\tilde{W}$, the Schubert cell

$$\tilde{\mathfrak{X}}^o_{\tilde{w},\gamma} := \tilde{B}[\tilde{w}]\subset\tilde{\mathcal{F}}_\gamma$$

and the Schubert variety $\tilde{\mathfrak{X}}_{\tilde{w},\gamma} := \overline{\tilde{\mathfrak{X}}^o_{\tilde{w},\gamma}}$. If $\tilde{W}^\gamma$ denotes the subgroup of $\tilde{W}$ that fixes γ, we see that the Schubert cell $\tilde{\mathfrak{X}}^o_{\tilde{w},\gamma}$ and the Schubert variety $\tilde{\mathfrak{X}}_{\tilde{w},\gamma}$ depend only of the class of $\tilde{w}$ in $\tilde{W}/\tilde{W}^\gamma$.

On the variety $\mathcal{F}_\gamma$, we consider the Schubert cell $\mathfrak{X}^o_\gamma := B[e]$ and the Schubert variety $\mathfrak{X}_\gamma := \overline{\mathfrak{X}^o_\gamma}$.

We consider the cohomology[1] ring $H^*(\tilde{\mathcal{F}}_\gamma,\mathbb{Z})$ of $\tilde{\mathcal{F}}_\gamma$. If Y is an irreducible closed subvariety of $\tilde{\mathcal{F}}_\gamma$, we denote by $[Y]\in H^{2n_Y}(\tilde{\mathcal{F}}_\gamma,\mathbb{Z})$ its cycle class in cohomology: Here $n_Y = \mathrm{codim}_{\mathbb{C}}(Y)$. Let $\iota^*: H^*(\tilde{\mathcal{F}}_\gamma,\mathbb{Z})\to H^*(\mathcal{F}_\gamma,\mathbb{Z})$ be the pull-back map in cohomology. Recall that the cohomology class $[pt]$ associated to a singleton $Y=\{pt\}\subset\mathcal{F}_\gamma$ is a basis of $H^{\max}(\mathcal{F}_\gamma,\mathbb{Z})$.

[1] Here, we use singular cohomology with integer coefficients.

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To an oriented real vector bundle $\mathcal{E} \to N$ of rank r, we can associate its Euler class $\mathrm{Eul}(\mathcal{E}) \in H^{2r}(N,\mathbb{Z})$. When $\mathcal{V} \to N$ is a complex vector bundle, then $\mathrm{Eul}(\mathcal{V}_\mathbb{R})$ corresponds to the top Chern class $c_p(\mathcal{V})$, where p is the complex rank of $\mathcal{V}$, and $\mathcal{V}_\mathbb{R}$ means $\mathcal{V}$ viewed as a real vector bundle oriented by its complex structure (see [5], §21).

The isomorphism $\mathfrak{q}^{\gamma>0} \simeq \mathfrak{q}/\mathfrak{q}^{\gamma\leq 0}$ shows that $\mathfrak{q}^{\gamma>0}$ can be viewed as a P_γ-module. Let $[\mathfrak{q}^{\gamma>0}] = K_\mathbb{C} \times_{P_\gamma} \mathfrak{q}^{\gamma>0}$ be the corresponding complex vector bundle on $\mathcal{F}_\gamma$. We denote simply by $\mathrm{Eul}(\mathfrak{q}^{\gamma>0})$ the Euler class $\mathrm{Eul}([\mathfrak{q}^{\gamma>0}]_\mathbb{R}) \in H^*(\mathcal{F}_\gamma,\mathbb{Z})$.

The following characterization of Ressayre's data was obtained in [32], §6. Recall that $\mathfrak{R}_o$ denotes the set of weights relative to the T-action on $(\tilde{\mathfrak{g}}/\mathfrak{g})\otimes\mathbb{C}$.

Proposition 5.9. *An element $(\gamma,\tilde{w}) \in \mathfrak{t}\times\tilde{W}$ is a Ressayre's datum if and only if the following conditions hold:*

- *γ is nonzero and rational.*
- $\mathrm{Vect}(\mathfrak{R}_o \cap \gamma^\perp) = \mathrm{Vect}(\mathfrak{R}_o)\cap\gamma^\perp$.
- $[\mathfrak{X}_\gamma]\cdot\iota^*([\tilde{\mathfrak{X}}_{\tilde{w},\gamma}])\cdot\mathrm{Eul}(\mathfrak{q}^{\gamma>0}) = k[pt]$, *$k\geq 1$ in* $H^*(\mathcal{F}_\gamma,\mathbb{Z})$.
- $\sum_{\substack{\alpha\in\mathfrak{R}+\\ \langle\alpha,\gamma\rangle>0}} \langle\alpha,\gamma\rangle = \sum_{\substack{\tilde{\alpha}\in\tilde{\mathfrak{R}}+\\ \langle\tilde{\alpha},\tilde{w}_0\tilde{w}\gamma\rangle>0}} \langle\tilde{\alpha},\tilde{w}_0\tilde{w}\gamma\rangle$.

5.4. Parametrization of the facets

We can finally describe the Kirwan polyhedron $\Delta(\mathbf{T}^*\tilde{K}\times\mathfrak{q})$ (see [32], §6).

Theorem 5.10. *An element $(\tilde{\xi},\xi) \in \tilde{\mathfrak{t}}^*_{\geq 0}\times\mathfrak{t}^*_{\geq 0}$ belongs to $\Delta(\boldsymbol{T}^*\tilde{K}\times\mathfrak{q})$ if and only if*

$$\langle\tilde{\xi},\tilde{w}\gamma\rangle + \langle\xi,\gamma\rangle \geq 0$$

for any Ressayre's datum $(\gamma,\tilde{w}) \in \mathfrak{t}\times\tilde{W}$.

Theorem 5.10 and Theorem B permit us to give the following description of the convex cone $\Delta_{\mathrm{hol}}(\tilde{G},G)$.

Theorem 5.11. *An element $(\tilde{\xi},\xi)$ belongs to $\Delta_{\mathrm{hol}}(\tilde{G},G)$ if and only if $(\tilde{\xi},\xi) \in \tilde{\mathcal{C}}_{\mathrm{hol}}\times\mathcal{C}_{\mathrm{hol}}$ and*

$$\langle\tilde{\xi},\tilde{w}\gamma\rangle \geq \langle\xi,w_0\gamma\rangle$$

for any $(\gamma,\tilde{w}) \in \mathfrak{t}\times\tilde{W}$ satisfying the following conditions:

- *γ is nonzero and rational.*
- $\mathrm{Vect}(\mathfrak{R}_o \cap \gamma^\perp) = \mathrm{Vect}(\mathfrak{R}_o)\cap\gamma^\perp$.
- $[\mathfrak{X}_\gamma]\cdot\iota^*([\tilde{\mathfrak{X}}_{\tilde{w},\gamma}])\cdot\mathrm{Eul}(\mathfrak{q}^{\gamma>0}) = k[pt]$, *$k\geq 1$ in* $H^*(\mathcal{F}_\gamma,\mathbb{Z})$.
- $\sum_{\substack{\alpha\in\mathfrak{R}+\\ \langle\alpha,\gamma\rangle>0}} \langle\alpha,\gamma\rangle = \sum_{\substack{\tilde{\alpha}\in\tilde{\mathfrak{R}}+\\ \langle\tilde{\alpha},\tilde{w}_0\tilde{w}\gamma\rangle>0}} \langle\tilde{\alpha},\tilde{w}_0\tilde{w}\gamma\rangle$.

6. Example: the holomorphic Horn cone $\mathrm{Horn}_{\mathrm{hol}}(p,q)$

Let $p\geq q\geq 1$. We consider the pseudo-unitary group $G = U(p,q) \subset GL_{p+q}(\mathbb{C})$ defined by the relation: $g\in U(p,q)$ if and only if $g\mathrm{Id}_{p,q}g^* = \mathrm{Id}_{p,q}$, where $\mathrm{Id}_{p,q}$ is the diagonal matrix $\mathrm{Diag}(\mathrm{Id}_p,-\mathrm{Id}_q)$.

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We work with the maximal compact subgroup $K = U(p) \times U(q) \subset G$. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is identified with the vector space $M_{p,q}$ of $p \times q$ matrices through the map

$$X \in M_{p,q} \longmapsto \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}.$$

We work with the element $z_{p,q} = \frac{i}{2}\mathrm{Id}_{p,q}$ which belongs to the center of $\mathfrak{k}$. The adjoint action of $z_{p,q}$ on $\mathfrak{p}$ corresponds to the standard complex structure on $M_{p,q}$.

The trace on $\mathfrak{gl}_{p+q}(\mathbb{C})$ defines an identification $\mathfrak{g} \simeq \mathfrak{g}^* = \hom(\mathfrak{g}, \mathbb{R})$: To $X \in \mathfrak{g}$ we associate $\xi_X \in \mathfrak{g}^*$ defined by $\langle \xi_X, Y \rangle = -\mathrm{Tr}(XY)$. Thus, the G-invariant cone $\mathcal{C}_{G/K}$ defined by $z_{p,q}$ can be viewed as the following cone of $\mathfrak{g}$:

$$\mathcal{C}(p,q) = \left\{ X \in \mathfrak{g},\ \mathrm{Im}\left(\mathrm{Tr}(gXg^{-1}\mathrm{Id}_{p,q})\right) \geq 0,\ \forall g \in U(p,q) \right\}.$$

Let $T \subset U(p) \times U(q)$ be the maximal torus formed by the diagonal matrices. The Lie algebra $\mathfrak{t}$ is identified with $\mathbb{R}^p \times \mathbb{R}^q$ through the map $\mathbf{d} : \mathbb{R}^p \times \mathbb{R}^q \to \mathfrak{u}(p) \times \mathfrak{u}(q)$: $\mathbf{d}_x = \mathrm{Diag}(ix_1, \cdots, ix_p, ix_{p+1}, \cdots, ix_{p+q})$. The Weyl chamber is

$$\mathfrak{t}_{\geq 0} = \{ x \in \mathbb{R}^p \times \mathbb{R}^q,\ x_1 \geq \cdots \geq x_p \text{ and } x_{p+1} \geq \cdots \geq x_{p+q} \}.$$

Proposition 2.2 tells us that the $U(p,q)$ adjoint orbits in the interior of $\mathcal{C}(p,q)$ are parametrized by the holomorphic chamber

$$\mathcal{C}_{p,q} = \{ x \in \mathbb{R}^p \times \mathbb{R}^q, x_1 \geq \cdots \geq x_p > x_{p+1} \geq \cdots \geq x_{p+q} \} \subset \mathfrak{t}_{\geq 0}.$$

Definition 6.1. The holomorphic Horn cone $\mathrm{Horn}_{\mathrm{hol}}(p,q) := \mathrm{Horn}^2_{\mathrm{hol}}(U(p,q))$ is defined by the relations

$$\mathrm{Horn}_{\mathrm{hol}}(p,q) = \left\{ (A,B,C) \in (\mathcal{C}_{p,q})^3,\ U(p,q)\mathbf{d}_C \subset U(p,q)\mathbf{d}_A + U(p,q)\mathbf{d}_B \right\}.$$

Let us detail the description given of $\mathrm{Horn}_{\mathrm{hol}}(p,q)$ by Theorem B. For any $n \geq 1$, we consider the semigroup $\wedge_n^+ = \{(\lambda_1 \geq \cdots \geq \lambda_n)\} \subset \mathbb{Z}^n$. If $\lambda = (\lambda', \lambda'') \in \wedge_p^+ \times \wedge_q^+$, then $V_\lambda := V_{\lambda'}^{U(p)} \otimes V_{\lambda''}^{U(q)}$ denotes the irreducible representation of $U(p) \times U(q)$ with highest weight λ. We denote by $\mathrm{Sym}(M_{p,q})$ the symmetric algebra of $M_{p,q}$.

Definition 6.2.

1. $\mathrm{Horn}^{\mathbb{Z}}(p,q)$ is the semigroup of $(\wedge_p^+ \times \wedge_q^+)^3$ defined by the conditions:

$$(\lambda, \mu, \nu) \in \mathrm{Horn}^{\mathbb{Z}}(p,q) \Longleftrightarrow [V_\nu : V_\lambda \otimes V_\mu \otimes \mathrm{Sym}(M_{p,q})] \neq 0.$$

2. $\mathrm{Horn}(p,q)$ is the convex cone of $(\mathfrak{t}_{\geq 0})^3$ defined as the closure of $\mathbb{Q}^{>0} \cdot \mathrm{Horn}^{\mathbb{Z}}(p,q)$.

Theorem B asserts that

$$\mathrm{Horn}_{\mathrm{hol}}(p,q) = \mathrm{Horn}(p,q) \bigcap (\mathcal{C}_{p,q})^3. \tag{20}$$

In another article [33], we obtained a recursive description of the cones $\mathrm{Horn}(p,q)$. This allows us to give the following description of the holomorphic Horn cone $\mathrm{Horn}_{\mathrm{hol}}(2,2)$.

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Example 6.3. An element $(A,B,C) \in (\mathbb{R}^4)^3$ belongs to $\mathrm{Horn}_{\mathrm{hol}}(2,2)$ if and only if the following conditions hold:

$$\boxed{\begin{array}{ccc} a_1 \geq a_2 & > & a_3 \geq a_4 \\ b_1 \geq b_2 & > & b_3 \geq b_4 \\ c_1 \geq c_2 & > & c_3 \geq c_4 \end{array}}$$

$$\boxed{a_1+a_2+a_3+a_4+b_1+b_2+b_3+b_4 = c_1+c_2+c_3+c_4}$$

$$\boxed{a_1+a_2+b_1+b_2 \leq c_1+c_2}$$

$$\boxed{\begin{array}{ccc} a_2+b_2 & \leq & c_2 \\ a_2+b_1 & \leq & c_1 \\ a_1+b_2 & \leq & c_1 \end{array}}$$

$$\boxed{\begin{array}{ccc} a_3+b_3 & \geq & c_3 \\ a_3+b_4 & \geq & c_4 \\ a_4+b_3 & \geq & c_4 \end{array}}$$

$$\boxed{\begin{array}{ccc} a_2+a_4+b_2+b_4 & \leq & c_1+c_4 \\ a_2+a_4+b_2+b_4 & \leq & c_2+c_3 \\ a_2+a_4+b_1+b_4 & \leq & c_1+c_3 \\ a_1+a_4+b_2+b_4 & \leq & c_1+c_3 \\ a_2+a_4+b_2+b_3 & \leq & c_1+c_3 \\ a_2+a_3+b_2+b_4 & \leq & c_1+c_3 \end{array}}$$

7. A conjectural symplectomorphism

Let $\tilde{\mu} \in \tilde{\mathcal{C}}_{\mathrm{hol}}$. In this section, we are interested in the geometry of the coadjoint orbit $\tilde{G}\tilde{\mu}$ viewed as a Hamiltonian G-manifold with proper moment map $\Phi_G^{\tilde{\mu}} : \tilde{G}\tilde{\mu} \to \mathfrak{g}^*$.

We start with a decomposition that we have already used. The pullback $Y_{\tilde{\mu}} = (\Phi_G^{\tilde{\mu}})^{-1}(\mathfrak{k}^*)$ is a symplectic submanifold of $\tilde{G}\tilde{\mu}$ which is stable under the K-action: Let $\Omega_{\tilde{\mu}}$ be the corresponding two form on $Y_{\tilde{\mu}}$. The action of K on $(Y_{\tilde{\mu}},\Omega_{\tilde{\mu}})$ is Hamiltonian, with a proper moment map $\Phi_K^{\tilde{\mu}} : Y_{\tilde{\mu}} \to \mathfrak{k}^*$ equal to the restriction of $\Phi_G^{\tilde{\mu}}$ to $Y_{\tilde{\mu}}$.

The map $[g,x] \mapsto gx$ defines a symplectomorphism

$$G \times_K Y_{\tilde{\mu}} \simeq \tilde{G}\tilde{\mu} \tag{21}$$

so that $\Phi_G^{\tilde{\mu}}([g,x]) = g \cdot \Phi_K^{\tilde{\mu}}(x)$ [31]. This allows us to see that the Kirwan polytope $\Delta_G(\tilde{G}\tilde{\mu})$ relative to the G-action on $\tilde{G}\tilde{\mu}$ is equal to the Kirwan polytope $\Delta_K(Y_{\tilde{\mu}})$ relative to the K-action on $Y_{\tilde{\mu}}$.

We consider the orthogonal decomposition $\tilde{\mathfrak{p}} = \mathfrak{p} \oplus \mathfrak{q}$. Mostow's decomposition theorem [27] says that the map $\psi : \mathfrak{p} \times \mathfrak{q} \times \tilde{K} \to \tilde{G}$, $(X, Y, \tilde{k}) \mapsto e^X e^Y \tilde{k}$ is a diffeomorphism. This leads to the following result.

Lemma 7.1. *We have the following G-equivariant diffeomorphisms:*

$$\begin{aligned} \psi_o : G \times_K \left(\mathfrak{q} \times \tilde{K}\right) &\longrightarrow \tilde{G} \\ \left[g; Y, \tilde{k}\right] &\longmapsto g e^Y \tilde{k}, \end{aligned}$$

$$\begin{aligned} \psi_{\tilde{\mu}} : G \times_K \left(\mathfrak{q} \times \tilde{K}\tilde{\mu}\right) &\longrightarrow \tilde{G}\tilde{\mu} \\ [g; Y, \xi] &\longmapsto g e^Y \xi. \end{aligned}$$

We obtain the following geometric information on the K-manifold $Y_{\tilde{\mu}}$.

Corollary 7.2. *There exists a K-equivariant diffeomorphism* $\mathfrak{q} \times \tilde{K}\tilde{\mu} \simeq Y_{\tilde{\mu}}$.

Proof. Thanks to the diffeomorphisms (21) and $\psi_{\tilde{\mu}}$, we know that the manifolds $G \times_K Y_{\tilde{\mu}}$ and $G \times_K (\mathfrak{q} \times \tilde{K}\tilde{\mu})$ admit a G-equivariant diffeomorphism. Our result follows from this. □

Let $\tilde{\kappa}$ be the Killing form on the Lie algebra $\tilde{\mathfrak{g}}$. We consider the $\tilde{K}$-invariant symplectic structures $\Omega_{\tilde{\mathfrak{p}}}$ on $\tilde{\mathfrak{p}}$, defined by the relation $\Omega_{\tilde{\mathfrak{p}}}(\tilde{Y}, \tilde{Y}') = \tilde{\kappa}(z, [\tilde{Y}, \tilde{Y}'])$, $\forall \tilde{Y}, \tilde{Y}' \in \tilde{\mathfrak{p}}$. We denote by $\Omega_{\mathfrak{q}}$ the restriction of $\Omega_{\tilde{\mathfrak{p}}}$ on the symplectic subspace $\mathfrak{q}$.

We consider the following symplectic structure $-\Omega_{\mathfrak{q}} \times \Omega_{\tilde{K}\tilde{\mu}}$ on $\mathfrak{q} \times \tilde{K}\tilde{\mu}$. Knowing that $\Delta_G(\tilde{G}\tilde{\mu}) = \Delta_K(Y_{\tilde{\mu}})$, the following conjectural result would give another proof of Theorem C.

Conjecture 7.3. There exists a K-equivariant symplectomorphism between $(Y_{\tilde{\mu}}, \Omega_{\tilde{\mu}})$ and $(\mathfrak{q} \times \tilde{K}\tilde{\mu}, -\Omega_{\mathfrak{q}} \times \Omega_{\tilde{K}\tilde{\mu}})$.

This conjecture generalizes some results obtained when $G = \tilde{K}$:

1. In [26], McDuff showed that $\tilde{G}\tilde{\mu} \simeq \tilde{G}/\tilde{K}$ admit a $\tilde{K}$-equivariant symplectomorphism with $(\tilde{\mathfrak{p}}, -\Omega_{\tilde{\mathfrak{p}}})$ when $\tilde{\mu}$ is a central element of $\tilde{\mathfrak{k}}^*$.
2. In [8], Deltour extended the result of McDuff by showing that $\tilde{G}\tilde{\mu}$ admits a $\tilde{K}$-equivariant symplectomorphism with $(\tilde{\mathfrak{p}} \times \tilde{K}\tilde{\mu}, -\Omega_{\tilde{\mathfrak{p}}} \times \Omega_{\tilde{K}\tilde{\mu}})$ for any $\tilde{\mu} \in \tilde{\mathcal{C}}_{\mathrm{hol}}$.

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