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DISCRETE C^*-COACTIONS AND C^*-ALGEBRAIC BUNDLES

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Abstract

Discrete C^*-coactions are shown to be equivalent to discrete C^*-algebraic bundles. Simplicity, primeness, liminality, postliminality, and nuclearity are related to the fixed point algebra and the cocrossed product. Ergodic, and more generally homogeneous, C^*-coactions are characterized.

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Introduction

Coactions of an abelian locally compact group G on a C^*-algebra A correspond to actions of the Pontryagin dual group $\hat{G}$ on A. This works by rewriting the $\hat{G}$-action as a $C_0(\hat{G})$-comodule, which Fourier transforms to give a $C^*(G)$-comodule. This can be done even when G is nonabelian, and we think of coactions of G as corresponding to 'actions' of the non-existent dual group. When G is discrete and abelian, $\hat{G}$ is compact, so even in the nonabelian case we expect discrete coactions to behave like compact actions. Even better, G-coactions give rise to modules over the Fourier algebra $A(G)$, which is abelian, so discrete coactions should in fact behave like compact abelian actions.

In Section 1 we present the fundamental terminology and results for discrete C^*-coactions, taking advantage of the simplifications resulting from discreteness of the group.

In Section 2 we present a handful of results relating certain properties, for example, simplicity, primeness, liminality, postliminality, and nuclearity, of the fixed point algebra and the cocrossed product of a coaction of a discrete group G on a C^*-algebra A. When the coaction is saturated, the fixed point algebra is strongly Morita equivalent

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to the cocrossed product. We also relate the above properties to the corresponding properties of A. In the case of nuclearity, we get the best result for amenable G: if any one of the three algebras is nuclear, then all are. The results of this section are analogous to, and in some cases generalize, results concerning compact abelian group actions appearing in [11], [12], [16], and [18]. Also, there is a little overlap with results independently discovered by Ng [15].

In Section 3 we give a careful treatment which verifies what has been suspected by the cognoscenti, namely that discrete C^*-coactions are equivalent to discrete C^*-algebraic bundles. It would be interesting to find a connection with some recent work of Exel [5] concerning circle actions (that is, integer coactions).

In Section 4 we apply the results of Section 3 to the case of homogeneous discrete C^*-algebraic bundles, which are also roughly equivalent to discrete twisted actions in the sense of Green [8] or Busby and Smith [4], although we work exclusively with Green's setup. We call the corresponding coactions (unsurprisingly) homogeneous, and we use the connection with twisted actions, together with our duality theorem for twisted crossed products [20] to conclude that in this case the cocrossed product is the tensor product of the fixed point algebra and the compacts.

In Section 5 we specialize further to ergodic discrete coactions. Here life is made simpler by restricting attention to effective coactions, that is, those having 'full spectrum.' The corresponding discrete C^*-algebraic bundles are the so-called cocycle bundles of [7], and are classified by $H^2(G, \mathbf{T})$. We obtain a partial generalization of the theory of ergodic compact abelian actions, as found in [16]. Some of our results can be viewed as a C^*-version of [10], and again there is partial overlap with independent work of Ng [15]. The main results here, when δ is an ergodic coaction of a discrete group G on A, characterize nuclearity of A and show that A has a unique δ-invariant (in an appropriate sense) state, which is a trace, and is faithful exactly when δ satisfies a technical condition I call normality (which is automatic when G is amenable).

We wish to thank the referee for suggestions which greatly improved the exposition.

1. Discrete cosystems

We use the conventions of [19], with the simplification that all our groups are discrete. Thus, a coaction of a discrete group G on a C^*-algebra A is a nondegenerate homomorphism $\delta : A \to A \otimes C^*(G)$ such that $(\delta \otimes \iota) \circ \delta = (\iota \otimes \delta_G) \circ \delta$ (where $\delta_G(s) = s \otimes s$ for $s \in G$, and G is embedded in $UM(C^*(G))$ in the usual way), and we call (A, G, δ) a *discrete cosystem*. Throughout, unadorned $\otimes$ for C^*-algebras will always mean the completion relative to the minimal C^*-tensor norm. A acquires a Banach representation of $B(G)$ by $\delta_f = (\iota \otimes f) \circ \delta$ for $f \in B(G)$. Since G is discrete,

δ is nondegenerate in the sense that $\overline{\delta_{A(G)}(A)} = A$ (see [1]).

For every nondegenerate homomorphism $\mu : C_0(G) \to A$, $\delta^\mu = \operatorname{Ad} \mu \otimes \iota(w_G)(\cdot \otimes 1)$ is a nondegenerate coaction of G on A, where w_G denotes the unitary element of $M(C_0(G) \otimes C^*(G))$ determined by the function $w_G(s) = s$. A covariant representation of (A, G, δ) in $M(B)$ is a pair (π, μ) of nondegenerate homomorphisms $\pi : A \to M(B)$ and $\mu : C_0(G) \to M(B)$ such that $(\pi \otimes \iota) \circ \delta = \delta^\mu \circ \pi$, that is, $\pi : A \to M(B)$ is equivariant for the coactions δ and δ^μ. Then $C^*(\pi, \mu) = \overline{\pi(A)\mu(C_0(G))}$ is a C^*-algebra, and (π, μ) may be regarded as a covariant representation in $M(C^*(\pi, \mu))$. A cocrossed product of (A, G, δ) is a triple $(A \times_\delta G, j_A, j_G)$, where (j_A, j_G) is a covariant representation, $A \times_\delta G = C^*(j_A, j_G)$, and every covariant representation (π, μ) factors through (j_A, j_G) in the sense that there is a unique homomorphism $\pi \times \mu$ of $A \times_\delta G$ such that $(\pi \times \mu) \circ j_A = \pi$ and $(\pi \times \mu) \circ j_G = \mu$.

Let λ denote the left regular representation of G and M the canonical representation of $C_0(G)$ by pointwise multiplication on $L^2(G)$. If π is a homomorphism of A, then $((\pi \otimes \lambda) \circ \delta, 1 \otimes M)$ is a covariant representation of (A, G, δ) in $M(\pi(A) \otimes \mathcal{K}(L^2(G)))$, and $((\pi \otimes \lambda) \circ \delta) \times (1 \otimes M)$ is faithful if π is.

We call (A, G, δ) *normal* if j_A is faithful. For example, $(C^*(G), G, \delta_G)$ is normal if and only if G is amenable. For any covariant representation (π, μ) of (A, G, δ), the cosystem $(\pi(A), G, \delta^\mu)$ is normal. In particular, the cosystem $(j_A(A), G, \delta^{j_G})$ is normal, and we denote it by (A^n, G, δ^n), and refer to it as the *normalization* of (A, G, δ). There is a bijection between the covariant representations of (A, G, δ) and (A^n, G, δ^n) which associates to every covariant representation (ρ, μ) of (A^n, G, δ^n) the covariant representation $(\rho \circ j_A, \mu)$ of (A, G, δ). It follows from this that if π is a homomorphism of A with $\ker \pi \subset \ker j_A$, then the homomorphism $((\pi \otimes \lambda) \circ \delta) \times (1 \otimes M)$ of $A \times_\delta G$ is faithful.

For $s \in G$ define $\chi_s \in A(G)$ by $\chi_s(t) = 1$ if $t = s$ and 0 otherwise. Then $\{\chi_s\}$ is a mutually orthogonal set of projections summing strictly to 1 in $M(C_0(G))$.

DEFINITION 1.1. For $s \in G$ define $\delta_s = \delta_{\chi_s}$ and $A_s = \delta_s(A)$.

DEFINITION 1.2. The *fixed point algebra* of (A, G, δ) is $A^\delta = A_e$ (where e here denotes the identity element of G).

LEMMA 1.3. *For $s, t \in G$,*

(a) *δ_s is a projection of norm one of A onto A_s, and in particular δ_e is a conditional expectation of A onto A^δ.*
(b) *$A_s = \{a \in A \mid \delta_f(a) = f(s)a$ for all $f \in B(G)\} = \{a \in A \mid \delta(s) = a \otimes s\}$.*
(c) *$A_s A_t \subset A_{st}$ and $A_s^* = A_{s^{-1}}$.*

PROOF. (a) δ_s is a contractive projection of A onto A_s, hence has norm one.

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For (b), note that for $f \in B(G)$,

$$\delta_f \delta_s = \delta_{f\chi_s} = \delta_{f(s)\chi_s} = f(s)\delta_s,$$

so A_s is contained in the first set on the right. Since slicing by $B(G)$ separates $\delta(A)$, the first set on the right is contained in the set below it. To complete the circle of inclusions, note that δ_s acts as the identity on the second set on the right, so this set is contained in A_s.

(c) is now obvious.

LEMMA 1.4. *The conditional expectation δ_e of A onto A^δ is faithful if and only if (A, G, δ) is normal.*

PROOF. It suffices to show that for a positive element a of A, $\delta_e(a) = 0$ if and only if $j_A(a) = 0$. If $j_A(a) = 0$, then

$$0 = (\iota \otimes \chi_e) \circ (\iota \otimes \lambda) \circ \delta(a) = (\iota \otimes \chi_e) \circ \delta(a) = \delta_e(a).$$

Conversely, arguing by contradiction, suppose that $\delta_e(a) = 0$ but $j_A(a) \neq 0$. Then $(\iota \otimes \lambda) \circ \delta(a)$ is a nonzero positive element of $A \otimes C_r^*(G)$, so there is a state ω of A such that $(\omega \otimes \lambda) \circ \delta(a)$ is a nonzero positive element of $C_r^*(G)$. Since χ_e is a faithful state of $C_r^*(G)$,

$$\begin{aligned} 0 \neq \chi_e \circ (\omega \otimes \lambda) \circ \delta(a) &= \omega \circ (\iota \otimes \chi_e \circ \lambda) \circ \delta(a) = \omega \circ (\iota \otimes \chi_e) \circ \delta(a) \\ &= \omega \circ \delta_e(a) = \omega(0) = 0, \end{aligned}$$

a contradiction. Therefore, we must have $j_A(a) = 0$.

LEMMA 1.5. $\overline{\sum_{s\in G} A_s} = A$ *(where $\sum A_s$ denotes the set of finitely nonzero sums).*

PROOF. This follows immediately from nondegeneracy of δ, since $\{\chi_s \mid s \in G\}$ generates $A(G)$ and $A_s = \delta_s(A)$.

To avoid triviality we assume that $A \neq \{0\}$.

COROLLARY 1.6. *A^δ contains a bounded approximate identity for A.*

PROOF. First note that $A^\delta \neq \{0\}$, for by Lemma 1.5 some A_s contains a nonzero element a, and then $0 \neq a^*a \in A^\delta$. If $\{a_i\}$ is a bounded approximate identity for A^δ, then for all $s \in G$ and $b \in A_s$,

$$\|a_ib - b\|^2 = \|(a_ib - b)(a_ib - b)^*\| = \|a_ibb^*a_i - a_ibb^* - bb^*a_i + bb^*\| \to 0.$$

By Lemma 1.5, $\{a_i\}$ is a bounded approximate identity for A.

2. Fixed point algebras and cocrossed products

Let (A, G, δ) be a discrete cosystem. We want to explore the relation between A^δ and $A \times_\delta G$. To facilitate the computations, we introduce some temporary simplification of the notation. Let (B, G, ϵ) be the normalization of (A, G, δ), so that $B = j_A(A)$ is a nondegenerate C^*-subalgebra of $M(A \times_\delta G)$ and $\epsilon = \delta^{j_G}$. Next let

$$w = (j_G \otimes \iota)(w_G) \qquad \text{and} \qquad q_s = j_G(\chi_s),\ s \in G.$$

Then $\{q_s\}_{s\in G}$ is a mutually orthogonal set of projections summing strictly to 1 in $M(A \times_\delta G)$, and we have

$$\overline{\sum_s Bq_s} = A\times_\delta G; \qquad q_s = (\iota\otimes\chi_s)(w), \quad s \in G; \qquad \epsilon(b) = \operatorname{Ad} w(b\otimes 1), \quad b \in B.$$

The dual action $\hat{\delta}$ of G on $A \times_\delta G$ is determined by

$$\hat{\delta}_s(bq_t) = bq_{ts^{-1}}, \quad s, t \in G, b \in B.$$

LEMMA 2.1. *If $b \in B$ and $bq_s = 0$ for some $s \in G$, then $b = 0$.*

PROOF. For all $t \in G$ we have $0 = \hat{\delta}_t(bq_s) = bq_{st^{-1}}$, so $b = \sum_r bq_r = 0$ (where the sum is taken in the strict topology of $M(A \times_\delta G)$).

LEMMA 2.2. *For $s, t \in G$ and $b \in B_s$, $bq_t = q_{st}b$.*

PROOF.

$$\begin{aligned} bq_t &= b(\iota \otimes \chi_t)(w) = (\iota \otimes \chi_t)((b \otimes 1)w) = (\iota \otimes \chi_{st})((b \otimes s)w) \\ &= (\iota \otimes \chi_{st})(\epsilon(b)w) = (\iota \otimes \chi_{st})(w(b \otimes 1)) = (\iota \otimes \chi_{st})(w)b = q_{st}b. \end{aligned}$$

COROLLARY 2.3. *For $s, t \in G$, $q_s(A \times_\delta G)q_t = q_s Bq_t = q_s B_{st^{-1}} = B_{st^{-1}}q_t = q_s B_{st^{-1}} q_t$.*

PROOF. We have

$$q_s(A \times_\delta G)q_t = q_s\overline{\sum_{r,u} B_r q_u} q_t = \overline{\sum_r q_s B_r q_t} = \overline{\sum_r q_s q_{rt} B_r} = q_s B_{st^{-1}},$$

and the rest follows immediately.

REMARK 2.4. In the above proof we used $A \times_\delta G = \overline{\sum_{s,t} B_s q_t}$, which is equivalent to part of [15, Proposition 2.4]. When G is abelian, this is equivalent to [11, Theorem 1]. Further, the injectivity of $b \mapsto bq_s$ implies that if $bq_s = cq_t$ for some $b, c \in B$ and $s, t \in G$, then $b = c$ and $s = t$, which in turn implies the other part of [15, Proposition 2.4].

COROLLARY 2.5. *For* $s \in G$,

(a) *the hereditary* C^**-subalgebra of* $A \times_\delta G$ *generated by* q_s *is* $B^\epsilon q_s$*, and is isomorphic to* A^δ*;*

(b) *the closed ideal of* $A \times_\delta G$ *generated by* q_s *is* $\overline{Bq_sB}$*, and is strongly Morita equivalent to* A^δ*.*

PROOF. (a) The first part is immediate from the above corollary, and the other part follows since $a \mapsto j_A(a)q_s : A^\delta \to B^\epsilon q_s$ is an isomorphism.

(b) The closed ideal of $A \times G$ generated by q_s is

$$\overline{(A \times_\delta G)q_s(A \times_\delta G)} = \overline{\Big(\sum_t Bq_t\Big)q_s\Big(\sum_r q_r B\Big)} = \overline{Bq_sB}.$$

The strong Morita equivalence follows from (a) and [22, Example 6.7].

Recall from [7] that a C^*-algebraic bundle E over G is called *saturated* if $\overline{E_sE_t} = E_{st}$ for all $s, t \in G$, and it is enough to require $\overline{E_sE_s^*} = E_e$ for all $s \in G$. This motivates the following:

DEFINITION 2.6. A discrete cosystem (A, G, δ) is *saturated* if $\overline{A_sA_s^*} = A^\delta$ for all $s \in G$.

COROLLARY 2.7. *Conditions* (a)–(c) *below are equivalent:*

(a) (A, G, δ) *is saturated;*

(b) (B, G, ϵ) *is saturated;*

(c) $\overline{Bq_eB} = A \times_\delta G$.

Moreover, each of the above conditions implies:

(d) $A \times_\delta G$ *is strongly Morita equivalent to* A^δ.

PROOF. (a) and (b) are equivalent since $j_A|A_s : A_s \to B_s$ is bijective for all $s \in G$. If (b) holds, then

$$\begin{aligned}\overline{Bq_eB} &= \overline{\sum_{s,t} B_sq_eB_{t^{-1}}} = \overline{\sum B_sB_{t^{-1}}q_t} \supset \overline{\sum B_sB_s^*B_{st^{-1}}q_t} = \overline{\sum B^\epsilon B_{st^{-1}}q_t} \\ &= \overline{\sum B_{st^{-1}}q_t}, \qquad \text{by Corollary 1.6} \\ &= \overline{\sum B_sq_t} = A \times_\delta G,\end{aligned}$$

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showing (c).

Conversely, if (c) holds, then for $s \in G$

$$\overline{B_s B_s^* q_s} = \overline{q_s B q_e B q_s} = q_s(A \times_\delta G)q_s = B^\epsilon q_s.$$

Since $b \mapsto bq_s$ is injective, $\overline{B_s B_s^*} = B^\epsilon$, giving (b).

Finally, the preceding corollary now shows that (c) implies (d).

REMARK 2.8. In the separable case, [11, Theorem 2] follows from Corollaries 2.4 and 2.7 and [3].

DEFINITION 2.9. We let $\operatorname{sp}\delta = \{s \in G \mid A_s \neq \{0\}\}$.

THEOREM 2.10. *If (A, G, δ) is a discrete cosystem, then $A \times_\delta G$ is simple if and only if A^δ is simple and* $\operatorname{sp}\delta = G$.

PROOF. If $A \times_\delta G$ is simple, then the nonzero ideal $\overline{Bq_eB}$ of $A \times_\delta G$ cannot be proper, so $\overline{Bq_eB} = A \times_\delta G$ is simple, hence so is the strongly Morita equivalent algebra A^δ. Also, $\overline{Bq_eB} = A \times_\delta G$ implies (A, G, δ) is saturated, so in particular $\operatorname{sp}\delta = G$.

Conversely, if A^δ is simple and $\operatorname{sp}\delta = G$, then for each $s \in G$ the nonzero ideal $\overline{A_s A_s^*}$ of A^δ must be all of A^δ, so (A, G, δ) is saturated. Hence, $\overline{Bq_eB} = A \times_\delta G$, which must be simple since the strongly Morita equivalent algebra A^δ is.

THEOREM 2.11. *If (A, G, δ) is a discrete cosystem, then $A \times_\delta G$ is prime if and only if A^δ is prime and* $\operatorname{sp}\delta = G$.

PROOF. If $A \times_\delta G$ is prime, then the hereditary subalgebra $B^\epsilon q_e$ is also prime, hence so is the isomorphic algebra A^δ. Also, for each $s \in G$, $q_s(A \times_\delta G)q_e \neq \{0\}$, so B_s, hence A_s, is nonzero, giving $\operatorname{sp}\delta = G$.

Conversely, if A^δ is prime and $\operatorname{sp}\delta = G$, then the strongly Morita equivalent algebra $\overline{Bq_eB}$ is also prime. We finish by showing that $\overline{Bq_eB}$ is essential as an ideal of $A \times_\delta G$. Let $0 \neq x \in A \times_\delta G$. For some $s \in G$, $q_s x \neq 0$, hence $0 \neq q_s x x^* q_s \in B^\epsilon q_s$. Also, $q_s \overline{Bq_eB} q_s = q_s \overline{B_s B_s^*} q_s$ is a nonzero ideal of $B^\epsilon q_s$. Since $B^\epsilon q_s \cong A^\delta$, $B^\epsilon q_s$ is prime, so

$$\{0\} \neq (q_s \overline{Bq_eB} q_s)(q_s x x^* q_s) \subset \overline{Bq_eB} x x^* q_s.$$

A fortiori, $\overline{Bq_eB}x \neq \{0\}$, so $\overline{Bq_eB}$ is essential.

REMARK 2.12. The motivation for the techniques involving $B_s q_t$ and $\overline{Bq_sB}$ comes from [18], where results for actions of nonabelian compact groups are obtained.

DEFINITION 2.13. We call a property $\mathscr{P}$ of C^*-algebras *ideal* if every C^*-algebra has a largest ideal with property $\mathscr{P}$, and if property $\mathscr{P}$ is inherited by closed ideals and preserved by strong Morita equivalence.

For example, liminality and postliminality are ideal properties. Since nuclearity is preserved under strong Morita equivalence [2] and is inherited by ideals, the following lemma shows that nuclearity is an ideal property. The result is probably known, but we could not find a reference, so we include a proof.

LEMMA 2.14. *Every C^*-algebra A has a largest nuclear ideal.*

PROOF. Let $I = \{\sum J \mid J \text{ is a nuclear ideal of } A\}^-$. Then I is an ideal containing all nuclear ideals, so it remains to show I is nuclear. Let π be a factor representation of I. It suffices to show that $\pi(I)''$ is an amenable von Neumann algebra. Since π is nonzero, $\pi|J$ is nonzero for some nuclear ideal J. Then $\pi(J)''$ is a nonzero amenable von Neumann algebra which is a weak* closed ideal of $\pi(I)''$. Since $\pi(I)''$ is a factor, we must have $\pi(I)'' = \pi(J)''$.

THEOREM 2.15. *If (A, G, δ) is a discrete cosystem and $\mathscr{P}$ is an ideal property (for example, liminality, postliminality, or nuclearity), then $A \times_\delta G$ has property $\mathscr{P}$ if and only if A^δ does.*

PROOF. If $A \times_\delta G$ has property $\mathscr{P}$, then so does the ideal $\overline{Bq_eB}$, hence so does the strongly Morita equivalent algebra A^δ.

Conversely, if A^δ has property $\mathscr{P}$, then so do the strongly Morita equivalent ideals $\overline{Bq_sB}$ $(s \in G)$ of $A \times_\delta G$. But then the largest ideal of $A \times_\delta G$ having property $\mathscr{P}$ contains $\overline{\sum_s \overline{Bq_sB}} = A \times_\delta G$.

REMARK 2.16. Ng [15, Theorem 3.6] proves the above result for liminality and postliminality, generalizing the corresponding result for compact abelian group actions [12, Corollary 4]. Of course, if A is liminal or postliminal, then so is the subalgebra A^δ, hence the cocrossed product $A \times_\delta G$. However, this is not reversible, for example, for discrete G, δ_G is a coaction on $C^*(G)$ with fixed point algebra $\mathbf{C}$, but $C^*(G)$ is often not postliminal. For nuclearity, we have a much better situation, which we present in the next result. The converse direction of the following result improves [15, Corollary 3.8], where A^δ is required to be postliminal. The proof below (of the converse direction) is modelled after Ng's, and the new aspect is that $A \times_\delta G$ is nuclear if A^δ is. The converse direction is also a significant generalization of [16, Lemma 6.2] for ergodic compact abelian actions.

COROLLARY 2.17. *Let (A, G, δ) be a discrete cosystem. If A is nuclear, so is A^δ, and conversely if G is amenable.*

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PROOF. If A is nuclear, then so is $A \times_\delta G$, by a recent result of Raeburn [21, Theorem 4.6], hence so is A^δ by Theorem 2.15. Conversely, assume that A^δ is nuclear and G is amenable. Then $A \times_\delta G$ is nuclear by Theorem 2.15. Since G is amenable, $(A \times_\delta G) \times_{\hat{\delta}} G$ is nuclear [8]. By Katayama's Duality Theorem [9, Theorem 8], $A \otimes \mathscr{K}(L^2(G))$ is nuclear, hence so is A.

PROPOSITION 2.18. *If* (π, μ) *is a covariant representation of a discrete cosystem* (A, G, δ), *then* $\ker \pi = \ker j_A$ *if and only if* $\pi | A^\delta$ *is faithful.*

PROOF. If $\ker \pi = \ker j_A$, then $\pi | A^\delta$ is faithful since $j_A | A^\delta$ is. Conversely, suppose $\pi | A^\delta$ is faithful. Since (π, μ) factors through (j_A, j_G), $\ker \pi \supset \ker j_A$. For the opposite inclusion, let $a \in \ker \pi$, and without loss of generality assume that a is positive. Then $0 = \delta_e^\mu \circ \pi(a) = \pi \circ \delta_e(a)$. Since $\pi | A^\delta$ is faithful, $\delta_e(a) = 0$. Arguing by contradiction, suppose $a \notin \ker j_A$. Then $(\iota \otimes \lambda) \circ \delta(a)$ is a nonzero positive element of $A \otimes C_r^*(G)$, so there is a state ω of A such that $(\omega \otimes \lambda) \circ \delta(a)$ is a nonzero positive element of $C_r^*(G)$. Since χ_e is a faithful state of $C_r^*(G)$,

$$\begin{aligned} 0 \neq \chi_e \circ (\omega \otimes \lambda) \circ \delta(a) &= \omega \circ (\iota \otimes \chi_e \circ \lambda) \circ \delta(a) = \omega \circ (\iota \otimes \chi_e) \circ \delta(a) \\ &= \omega \circ \delta_e(a) = \omega(0) = 0, \end{aligned}$$

a contradiction. Therefore, we must have $a \in \ker j_A$.

COROLLARY 2.19. *Let* (A, G, δ) *be a discrete cosystem, and let* π *be a nondegenerate homomorphism of* A. *Then* $\ker \pi = \ker j_A$ *if and only if* $\pi | A^\delta$ *is faithful and there is a normal coaction* ϵ *of* G *on* $\pi(A)$ *such that* $\epsilon \circ \pi = (\pi \otimes \iota) \circ \delta$.

PROOF. By the above proposition it suffices to note that such an ϵ exists if and only if there is a covariant representation (ρ, μ) of (A, G, δ) such that $\ker \pi = \ker \rho$.

We will use the strong Morita equivalence between A^δ and $\overline{Bq_eB}$ to induce nondegenerate homomorphisms of A^δ to nondegenerate homomorphisms of $\overline{Bq_eB}$, $A \times_\delta G$, A, and $C_0(G)$. Since Rieffel induction uses (Hilbert space) representations, we will need an unambiguous way of converting nondegenerate homomorphisms into representations. For a nondegenerate homomorphism ϕ of A^δ we first compose with the universal representation of $\phi(A^\delta)$ to get a representation of A^δ. We next induce this representation of A^δ to a representation $\mathrm{Ind}^{\overline{Bq_eB}} \phi$ of $\overline{Bq_eB}$ using the strong Morita equivalence. We regard this latter representation as the nondegenerate homomorphism of $\overline{Bq_eB}$ induced by the nondegenerate homomorphism ϕ of A^δ. Next, we extend $\mathrm{Ind}^{\overline{Bq_eB}} \phi$ to a nondegenerate homomorphism $\mathrm{Ind}^{A \times_\delta G} \phi$ of $A \times_\delta G$ by composing with the natural nondegenerate homomorphism of $A \times_\delta G$ to $M(\overline{Bq_eB})$. Finally, we compose $\operatorname{Ind}^{A\times_\delta G}\phi$ with j_A and j_G to get nondegenerate homomorphisms $\operatorname{Ind}^A\phi$ and $\operatorname{Ind}^G\phi$ of A and $C_0(G)$, respectively. Then $(\operatorname{Ind}^A\phi, \operatorname{Ind}^G\phi)$ is a covariant representation of (A, G, δ), and $\operatorname{Ind}^{A\times_\delta G}\phi = \operatorname{Ind}^A\phi \times \operatorname{Ind}^G\phi$.

LEMMA 2.20. *Let (A, G, δ) be a discrete cosystem, and let ϕ be a faithful nondegenerate homomorphism of A^δ. Then* $\ker \operatorname{Ind}^A\phi = \ker j_A$.

PROOF. Since $(\operatorname{Ind}^A\phi, \operatorname{Ind}^G\phi)$ is a covariant representation of (A, G, δ), it suffices by Proposition 2.18 to show that $(\operatorname{Ind}^A\phi)|A^\delta$ is faithful. By strong Morita equivalence, $\operatorname{Ind}^{\overline{Bq_eB}}\phi$ is faithful because ϕ is. Since no nonzero element of B^ϵ annihilates $\overline{Bq_eB}$, and since $j_A|A^\delta$ is faithful, we conclude that $(\operatorname{Ind}^A\phi)|A^\delta$ must be faithful as well.

3. Cosystems and bundles

Let E be a C^*-algebraic bundle [7] over the discrete group G. We abuse notation by identifying E with its image in $C^*(E)$, taking care to note that the zero elements of the various fibers E_s lose their individuality when they are identified with the zero of $C^*(E)$. We let $M(E)$ denote the multiplier bundle, with fibers $M_s(E)$, and we identify $M_e(E)$ with $M(E_e)$.

DEFINITION 3.1. If E' is a C^*-algebraic bundle over another discrete group G', we call a map $\phi : E \to M(E')$ a *morphism* if there is a (necessarily unique) map $\gamma : G \to G'$ (necessarily a group homomorphism) such that

(a) $\phi|E_s : E_s \to M_{\gamma(s)}(E')$ is linear for all $s \in G$;
(b) $\phi(ab) = \phi(a)\phi(b)$ and $\phi(a^*) = \phi(a)^*$ for all $a, b \in E$;
(c) $\phi|E_e : E_e \to M(E'_e)$ is nondegenerate.

If ϕ is a bijection of E with E', then ϕ^{-1} is also a morphism, and we say ϕ is an *isomorphism.* If $G = G'$ and $\gamma = \iota$, we say ϕ is a *G-bundle morphism.*

DEFINITION 3.2. If E is a C^*-algebraic bundle over G and A is a C^*-algebra, a *representation of E in $M(A)$* is a map $\pi : E \to M(A)$ such that

(a) $\pi|E_s$ is linear for all $s \in G$;
(b) $\pi(ab) = \pi(a)\pi(b)$ and $\pi(a^*) = \pi(a)^*$ for all $a, b \in E$;
(c) $\pi|E_e : E_e \to M(A)$ is nondegenerate.

Every representation π of E in $M(A)$ corresponds to a unique nondegenerate homomorphism, still denoted by π, from $C^*(E)$ to $M(A)$. If $\pi(E) \subset A$, we say π is a representation in A.

In the context of reduced coactions, the following is [13, Example 2.3 (6)].

PROPOSITION 3.3. *There is a unique coaction δ_E of G on $C^*(E)$ such that*

$$\delta_E(a) = a \otimes s, \quad a \in E_s. \tag{3.1}$$

PROOF. (3.1) defines a representation of E in $C^*(E) \otimes C^*(G)$, hence a nondegenerate homomorphism $\delta_E : C^*(E) \to C^*(E) \otimes C^*(G)$. It suffices to check the coaction identity on the elements of E, where it is obvious. We verify injectivity of δ_E (as a homomorphism of $C^*(E)$) by showing every representation π of E factors through δ_E. Let 1_G denote the trivial one-dimensional representation of G. Then we can consider the homomorphism $\pi \otimes 1_G : C^*(E) \otimes C^*(G) \to \pi(C^*(E))$, and we have $(\pi \otimes 1_G) \circ \delta_E = \pi$.

LEMMA 3.4. *$C^*(E)_s = E_s$ for all $s \in G$.*

PROOF. We have

$$C^*(E)_s = (\delta_E)_s(C^*(E)) = (\delta_E)_s\left(\overline{\sum_t E_t}\right) = \overline{\sum_t (\delta_E)_s(E_t)} = E_s.$$

DEFINITION 3.5. Let E be C^*-algebraic bundle over a discrete group G, and let ι denote the identity homomorphism of E_e. Then we can form the covariant representation $(\mathrm{Ind}^{C^*(E)} \iota, \mathrm{Ind}^G \iota)$ of the cosystem $(C^*(E), G, \delta_E)$. We let

$$\lambda_E = \mathrm{Ind}^{C^*(E)}; \qquad C^*_r(E) = \lambda_E(C^*(E)); \qquad \mu_E = \mathrm{Ind}^G \iota; \qquad \delta^n_E = \delta^{\mu_E}.$$

Then $(C^*_r(E), G, \delta^n_E)$ is a normal cosystem which is isomorphic to the normalization of $(C^*(E), G, \delta_E)$.

REMARK 3.6. λ_E deserves to be called the *regular representation* of E and $C^*_r(E)$ the *reduced C^*-algebra* of E. Fell and Doran [7] do not define the reduced C^*-algebra of a C^*-algebraic bundle, although they define what they call 'generalized regular representations' of E, which are formed by inducing cyclic representations from E_e to E when E has enough unitary multipliers. Our induction of nondegenerate homomorphisms from E_e to $C^*(E)$ is consistent with Fell and Doran's induction of representations from E_e to E.

COROLLARY 3.7. *If π is a representation of E, then* $\ker \pi = \ker \lambda_E$ *(as ideals of $C^*(E)$) if and only if $\pi | E_e$ is faithful and there is a normal coaction δ of G on $\pi(A)$ such that $\delta \circ \pi = (\pi \otimes \iota) \circ \delta_E$.*

PROOF. Since $\ker \lambda_E = \ker j_{C^*(E)}$ and $E_e = C^*(E)^{\delta_E}$, the result follows from Corollary 2.19.

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The following result shows that for a discrete group G, normal coactions of G and C^*-algebraic bundles over G are essentially equivalent concepts:

THEOREM 3.8. *Let (A, G, δ) be a discrete normal cosystem. Then there are a C^*-algebraic bundle E over G, unique up to G-bundle isomorphism, and an isomorphism $\phi : C_r^*(E) \to A$ such that $\delta \circ \phi = (\phi \otimes \iota) \circ \delta_E^n$.*

PROOF. Let $E = \bigcup_s A_s \times \{s\}$. Then E becomes a C^*-algebraic bundle over G with $E_s = A_s \times \{s\}$, $(a, s)(b, t) = (ab, st)$, and $(a, s)^* = (a^*, s^{-1})$. Define $\pi : E \to A$ by $\pi(a, s) = a$. Since $\overline{\sum_s A_s} = A$, π is a representation with $\pi(C^*(E)) = A$. Moreover, $\delta \circ \pi = (\pi \otimes \iota) \circ \delta_E$. Since $\pi | E_e$ is faithful, Corollary 3.7 shows that π factors through an isomorphism $\phi : C_r^*(E) \to A$ with $\delta \circ \phi = (\phi \otimes \iota) \circ \delta_E^n$.

For the uniqueness, suppose E' is a another C^*-algebraic bundle over G and ϕ' is an isomorphism of $C_r^*(E')$ to A such that $\delta \circ \phi' = (\phi' \otimes \iota) \circ \delta_{E'}^n$. Then one checks that

$$a \mapsto (\phi'(\lambda_{E'}(a)), s), \qquad a \in E'_s,$$

defines an isomorphism of E' onto E.

COROLLARY 3.9. *Let A be a C^*-algebra, and let G be a discrete group. Then A is isomorphic to $C_r^*(E)$ for some C^*-algebraic bundle E over G if and only if there is a normal coaction of G on A.*

4. Homogeneous coactions

A discrete bundle $E = (E, P, G)$ is called *homogeneous* [6] if $UM_s(E)$ is nonempty for each $s \in G$. In [7], Fell and Doran use the phrase 'has enough unitary multipliers', but when G is discrete this is equivalent to homogeneity. This motivates the following:

DEFINITION 4.1. A discrete cosystem (A, G, δ) is *homogeneous* if A_s contains a unitary for each $s \in G$.

We will need the 'twisted covariant systems' of Green [8], which we use in the following form: let α be an action of a discrete group H on a C^*-algebra A, and let N be a normal subgroup of H. A *twist* for α over N is a representation τ of N in $M(A)$ such that

$$\alpha_n(a) = \operatorname{Ad}\tau(n)(a), \qquad n \in N, a \in A;$$
$$\alpha_h \circ \tau(n) = \tau \circ \operatorname{Ad} h(n), \qquad h \in H, n \in N,$$

and (A, H, N, α, τ) is a *discrete twisted system*. An alternate version of twisted systems is defined by Busby and Smith [4] (see also [14] and [17]). For discrete groups, Green twisted systems and Busby-Smith twisted systems are essentially the same concept, that is, for every twisted system of one type there is a twisted system of the other type with essentially the same representations and isomorphic twisted crossed products.

DEFINITION 4.2. A *covariant representation* of (A, H, N, α, τ) in $M(B)$ is a pair (ϕ, u), where $\phi : A \to M(B)$ is a nondegenerate homomorphism and u is a representation of H in $M(B)$, such that

$$\operatorname{Ad} u(h) \circ \phi = \phi \circ \alpha_h, \qquad h \in H; \qquad \pi \circ \tau = u|N.$$

Then $C^*(\phi, u) = \overline{\phi(A)u(H)}$ is a C^*-algebra, and (ϕ, u) may also be regarded as a covariant representation in $M(C^*(\phi, u))$. A *twisted crossed product* of (A, H, N, α, τ) is a triple $(A \times_{\alpha,\tau} (H, N), j_A, j_H)$, where (j_A, j_H) is a covariant representation, $A \times_{\alpha,\tau} (H, N) = C^*(j_A, j_H)$, and every covariant representation (ϕ, u) factors through (j_A, j_H) in the sense that there is a unique homomorphism $\phi \times_N u$ of $A \times_{\alpha,\tau} (H, N)$ such that $(\phi \times_N u) \circ j_A = \phi$ and $(\phi \times_N u) \circ j_H = u$. Warning: if we ignore the twist τ, we have a system (A, H, α), and covariant representations of (A, H, N, α, τ) are in particular covariant representations of (A, H, α). We should reserve the notation $\phi \times u$ to denote the corresponding nondegenerate homomorphism of the ordinary crossed product $A \times_\alpha H$.

We use a slight modification of the procedure of Fell [6] to relate discrete twisted systems and discrete homogeneous C^*-algebraic bundles. To each homogeneous C^*-algebraic bundle E over a discrete group G we associate the twisted system $(E_e, UM(E), UM_e(E), \operatorname{Ad}, \iota)$, where $UM(E)$ and $UM_e(E)$ are here given the discrete (!) topology, $\operatorname{Ad} h(a) = hah^{-1}$ for $h \in UM(E)$ and $a \in E_e$, and $\iota : UM_e(E) \to UM(E_e)$ is the natural map arising from the identification of $M_e(E)$ with $M(E_e)$. On the other hand, if (A, H, N, α, τ) is a discrete twisted system, let E be the orbit space of the Cartesian product $A \times H$ under the diagonal N-action $(a, h)\cdot n = (a\tau(n), n^{-1}h)$, and let $[a, h]$ denote the N-orbit of (a, h). Then E becomes a C^*-algebraic bundle over H/N, with

$$E_{hN} = \{[a, hn] \mid a \in A, n \in N\},$$

$[a, h][b, k] = [a\alpha_h(b), hk]$, and $[a, h]^* = [\alpha_{h^{-1}}(a^*), h^{-1}]$.

PROPOSITION 4.3. [6] *Let E be a homogeneous C^*-algebraic bundle over a discrete group G. As above, let (A, H, N, α, τ) be the twisted system associated to E, and E'*

the C^*-algebraic bundle over H/N associated to (A, H, N, α, τ). Then there is an isomorphism $\phi : E \to E'$ defined by

$$\phi(a) = [ah^*, h], \qquad a \in E_s, h \in UM_s(E).$$

COROLLARY 4.4. *Let (A, H, N, α, τ) be a discrete twisted system, and let E be the associated C^*-algebraic bundle. Then there is an isomorphism $\theta : A \times_{\alpha,\tau} (H, N) \to C^*(E)$ defined by*

$$\theta(j_A(a) j_H(h)) = [a, h], \qquad a \in A, h \in H. \tag{4.1}$$

PROOF. Evidently, the maps $a \mapsto \pi([a, e])$ of A and $h \mapsto ([1, h])$ of H define a representation of (A, H, N, α, τ), so (4.1) defines a nondegenerate homomorphism $\theta : A \times_{\alpha,\tau} (H, N) \to C^*(E)$. On the other hand, it is easy to check that the map $[a, h] \mapsto j_A(a) j_H(h)$ is well-defined and gives a representation of E in $A \times_{\alpha,\tau} (H, N)$. The corresponding homomorphism of $C^*(E)$ is an inverse of θ.

COROLLARY 4.5. *For any discrete twisted system (A, H, N, α, τ) there is a unique coaction $\hat{\alpha}$ of H/N on $A \times_{\alpha,\tau} (H, N)$ such that*

$$\hat{\alpha}(j_A(a) j_H(h)) = j_A(a) j_H(h) \otimes hN, \qquad a \in A, h \in H. \tag{4.2}$$

PROOF. Let E be the C^*-algebraic bundle over H/N associated to (A, H, N, α, τ), and let $\theta : A \times_{\alpha,\tau} (H, N) \to C^*(E)$ be the isomorphism of the preceding proposition. Then $\hat{\alpha} = (\theta^{-1} \otimes \iota) \circ \delta_E \circ \theta$ is a coaction of H/N on $A \times_{\alpha,\tau} (H, N)$ satisfying (4.2). Of course, (4.2) uniquely determines $\hat{\alpha}$.

REMARK 4.6. The above coaction $\hat{\alpha}$ agrees with the *dual coaction* of [20].

COROLLARY 4.7. *Let (A, H, N, α, τ) be a discrete twisted system. Then for all $h \in H$,*

$$(A \times_{\alpha,\tau} (H, N))_{hN} = j_A(A) j_H(h).$$

In particular, $(A \times_{\alpha,\tau} (H, N))^{\hat{\alpha}} = j_A(A)$.

PROOF. Let E be the associated C^*-algebraic bundle over H/N, and let $\theta : A \times_{\alpha,\tau} (H, N) \to C^*(E)$ be the above isomorphism. Since θ is equivariant for the coactions $\hat{\alpha}$ and δ_E, θ takes $(A \times_{\alpha,\tau} (H, N))_{hN}$ to $C^*(E)_{hN}$. Since the latter is identified with E_{hN}, the result follows immediately from the definitions of E and θ.

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PROPOSITION 4.8. *Let (A, H, N, α, τ) be a discrete twisted system, with associated C^*-algebraic bundle E, and let $\theta : A \times_{\alpha,\tau} (H, N) \to C^*(E)$ be the above isomorphism. Then $\lambda_E \circ \theta$ has the same kernel as the regular representation of $A \times_{\alpha,\tau} (H, N)$. Consequently, θ projects to an isomorphism θ_r from the reduced twisted crossed product $A \times_{\alpha,\tau,r} (H, N)$ to the reduced C^*-algebra $C^*_r(E)$, and there is a canonical normal coaction $\hat{\alpha}^n$ of H/N on $A \times_{\alpha,\tau,r} (H, N)$ such that the regular representation is equivariant.*

PROOF. Since $\ker \lambda_E \circ \theta = \ker j_{A \times_{\alpha,\tau}(H,N)}$, and since the regular representation of $A \times_{\alpha,\tau} (H, N)$ is induced in the sense of [8] by any faithful representation of A, by Lemma 2.20 it suffices to note that Green's induction uses essentially the same strong Morita equivalence as we do in our induction of nondegenerate homomorphisms.

THEOREM 4.9. *For a discrete cosystem (A, G, δ), the following are equivalent:*

(a) *(A, G, δ) is normal and homogeneous;*

(b) *there are a homogeneous C^*-algebraic bundle E over G, unique up to isomorphism, and an equivariant isomorphism of A with $C^*_r(E)$;*

(c) *there are a discrete twisted system (B, H, N, α, τ), with $H/N \cong G$, and an equivariant isomorphism of A with $B \times_{\alpha,\tau,r} (H, N)$. Moreover, in this case B can be taken to be A^δ.*

PROOF. The equivalence of (a) and (b) follows from Theorem 3.8 together with the fact that a discrete cosystem (A, G, δ) is homogeneous if and only if its associated C^*-algebraic bundle is. The equivalence of (b) and (c) follows from the above and our results relating (A, H, N, α, τ), its associated C^*-algebraic bundle E, and the normal coactions $\hat{\alpha}^n$ and δ^n_E.

COROLLARY 4.10. *If (A, G, δ) is a discrete homogeneous cosystem, then $A \times_\delta G \cong A^\delta \otimes \mathcal{K}(L^2(G))$.*

PROOF. This follows from the above theorem and the duality theorem for twisted crossed products [20, Theorem 3.6].

5. Ergodic cosystems

The following definition is the analogue for coactions of ergodicity for actions, and is taken from [10] and [15].

DEFINITION 5.1. A discrete cosystem (A, G, δ) is *ergodic* if $A^\delta = \mathbf{C}$.

PROPOSITION 5.2. *If* (A, G, δ) *is an ergodic discrete cosystem, then*

(a) *for all* $s \in \operatorname{sp}\delta$, A_s *is the linear span of a unitary;*

(b) $\operatorname{sp}\delta$ *is a subgroup of* G*;*

(c) δ *may be regarded as an ergodic coaction of* $\operatorname{sp}\delta$ *on* A*, and then* δ *is homogeneous.*

PROOF. Straightforward.

EXAMPLE 5.3. If (A, G, δ) is a discrete cosystem which is not ergodic, then condition (b) above need not hold. For example, let δ be the unique coaction of $\mathbf{Z}_4 = \{0, 1, 2, 3\}$ on $A = M_4(\mathbf{C})$ with

$$A^\delta = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix},$$

that is, δ corresponds to the action of $\hat{\mathbf{Z}}_4 \cong \mathbf{Z}_4$ on A generated by $\operatorname{Ad}\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$. Then $\operatorname{sp}\delta = \{0, 1, 3\}$, which is not a subgroup of $\mathbf{Z}_4$.

Observe that for a discrete cosystem (A, G, δ), $\operatorname{sp}\delta$ is the hull of the ideal $\{f \in A(G) \mid \delta_f = 0\}$. This motivates the following:

DEFINITION 5.4. A discrete cosystem (A, G, δ) is *effective* if $\operatorname{sp}\delta = G$.

PROPOSITION 5.5. *If* (A, G, δ) *is an effective ergodic discrete cosystem, then* $A \times_\delta G \cong \mathcal{K}(L^2(G))$.

PROOF. This immediate from Proposition 5.2 and Corollary 4.10.

The property for discrete C^*-algebraic bundles which corresponds to ergodicity for effective discrete cosystems is:

DEFINITION 5.6. [7] A C^*-algebraic bundle E over a discrete group G is a *cocycle bundle* if E_s is one dimensional for all $s \in G$.

PROPOSITION 5.7. *A discrete cosystem* (A, G, δ) *is effective and ergodic if and only if the associated* C^**-algebraic bundle is a cocycle bundle.*

PROOF. Straightforward.

REMARK 5.8. For a discrete group G, Theorem 3.8 tells us that normal coactions of G are classified (up to isomorphism) by C^*-algebraic bundles over G. Hence, by the above proposition, effective ergodic normal coactions of G are classified by cocycle bundles over G, which in turn are classified by $H^2(G, \mathbf{T})$ [7, VIII.4.9]. However, we cannot phrase this in terms of symplectic bicharacters as in [16] unless G is abelian.

THEOREM 5.9. *If* (A, G, δ) *is an ergodic discrete cosystem, then* A *is nuclear if and only if* $\operatorname{sp}\delta$ *is amenable.*

PROOF. Without loss of generality, let $\operatorname{sp}\delta = G$. If G is amenable, then by Corollary 2.17 A is nuclear since $A^\delta = \mathbf{C}$ is. On the other hand, if A is nuclear, then $\overline{j_A(A)}^{w*}$ (where $^{-w*}$ denotes weak* closure in $(A \times_\delta G)^{**}$) is an injective von Neumann algebra. Moreover (using self-explanatory terminology and notation) the cosystem $(j_A(A), G, \delta^{j_G})$ extends to give an effective ergodic W^*-cosystem $(\overline{j_A(A)}^{w*}, G, \bar{\delta})$, so by [10, Proposition 6.2] G must be amenable.

DEFINITION 5.10. If (A, G, δ) is a discrete cosystem, a state ω of A is called δ-*invariant* if

$$(\omega \otimes \iota) \circ \delta = \omega. \tag{5.1}$$

The following was discovered independently by Ng [15]:

THEOREM 5.11. [15, Propositions 4.5 and 4.6] *If* (A, G, δ) *is an ergodic discrete cosystem, then there is a unique* δ*-invariant state* ω *of* A. *Moreover,* ω *is a trace, and is faithful if and only if* (A, G, δ) *is normal.*

PROOF. Since this is largely proved in [15], we only give an outline of the argument to show how our above results can be used. By ergodicity and Lemma 1.4, $\omega(a) = \delta_e(a)$ defines a state of A which is faithful if and only if (A, G, δ) is normal.

Since our definition of δ-invariance is slightly different from Ng's 'averagely δ-invariant', we give the verification of this property. It suffices to consider a finitely nonzero sum $a = \sum_s a_s$ with $a_s \in A_s$ for all $s \in G$:

$$\begin{aligned}(\omega \otimes \iota) \circ \delta(a) &= \sum_s (\omega \otimes \iota) \circ \delta(a_s) = \sum_s (\omega \otimes \iota)(a_s \otimes s) \\ &= \sum_s \omega(a_s)s = \sum_s \delta_e(a_s)s = \delta_e(a) = \omega(a).\end{aligned}$$

By Proposition 5.2(c), A_s and A_s^* commute for all $s \in G$, which implies the trace property for ω.

Finally, any δ-invariant state of A is of the form $\omega' \circ \delta_e$ for some state ω' of $A^\delta = \mathbf{C}$, and so must be unique.

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