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THE GROWTH SERIES OF COMPACT HYPERBOLIC COXETER GROUPS WITH 4 AND 5 GENERATORS

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ABSTRACT. The growth series of compact hyperbolic Coxeter groups with 4 and 5 generators are explicitly calculated. The assertions of J. Cannon and Ph. Wagreich for the 4-generated groups, that the poles of the growth series lie on the unit circle, with the exception of a single real reciprocal pair of poles, are verified. We also verify that for the 5-generated groups, this phenomenon fails.

1. **Introduction.** A Coxeter group is a group having presentation

$$W = \langle S : s^2 = 1,\ (st)^{m(s,t)} = 1,\ s,t \in S,\ s \neq t\rangle,$$

where $m(s,t) = m(t,s) \geq 2$. If no relation occurs for a pair s, t then we make the convention that $m(s,t) = \infty$.

The presentation of every finitely generated Coxeter group can be represented as an undirected graph, whose vertices correspond to the generators S of W. There is an edge between the vertices corresponding to s and t ($\in S$) with label $m(s,t)$. If $m(s,t) = 3$ then the label is omitted; if $m(s,t) = 2$ then both label and edge are omitted. We will call such a graph a *Coxeter graph*.

The growth series $G_S(t)$ of a group G relative to a finite generating set S is defined as follows: for any $1 \neq g \in G$ we define the length $\ell(g)$ to be the minimum length of all words in $S \cup S^{-1}$ representing g. By default $\ell(1) = 0$. Then

$$G_S(t) := \sum_{g \in G} t^{\ell(g)}.$$

We will also be using *reciprocal polynomials*: if $p(t) = \sum_{i=0}^{n} a_i t^i$ is a polynomial of degree n, then $p(t)$ is a reciprocal polynomial if $a_n \neq 0$ and $a_{n-i} = a_i$, for $i = 0, \ldots, n$. Note that this implies that $p(0) \neq 0$.

A Salem polynomial is a reciprocal polynomial which has the following properties:

1) it is monic, with integer coefficients, and
2) it has only one zero (of multiplicity 1) outside the unit circle.

(If λ is a zero of a Salem polynomial outside the unit circle, then $\frac{1}{\lambda}$ is also a zero and λ is real.)

It will also be convenient to define $[n] := 1 + t + \cdots + t^{n-1}$, ($n \in \mathbf{N}$).

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2. **Preliminary Results.** With regard to Coxeter groups, there is a rather nice recursive formula to obtain their growth series. If $W = \langle S : R\rangle$ and W_T is the subgroup of W generated by $T \subseteq S$, with growth series $W_T(t)$, then $W(t)$ $\big(\equiv W_S(t)\big)$ can be obtained from the following formula

$$\frac{L(W) + (-1)^{|S|+1}}{W(t)} = \sum_{T\subset S} \frac{(-1)^{|T|}}{W_T(t)}, \tag{2.1}$$

where $L(W) = 0$ if W is infinite, or $L(W) = t^\lambda$ where λ is the length of the element in W of maximal length if W is finite (see [4, Chap. IV, Section 1, Exercise 26]).

To help us to use the above relation, the growth series of (finite) Coxeter groups with 1, 2, and 3 generators are given below.

The growth series of a 1-generator Coxeter group $W = \langle s : s2 = 1\rangle$ is trivially $W(t) = 1 + t = [2]$.

The growth series of the (finite) 2-generator Coxeter group (see Figure 1, $p \geq 2$) is (see [1, Proposition 3.1]).

$$W(t) = [2][p]. \tag{2.2}$$

The growth series of the (finite) 3-generator Coxeter group (see Figure 1, $p, q, r \geq 2$, and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$) is

$$W(t) = \frac{[2][p][q][r](t^\lambda + 1)}{[2][p][q][r] - 3[p][q][r] + [p][q] + [p][r] + [q][r]}, \tag{2.3}$$

where λ, as before, is the length of the element in W of maximal length (see similar formula in [1, Proposition 3.1]).

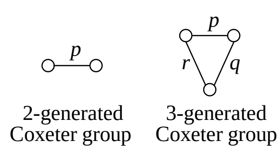

FIGURE 1.

Now let $p(t)$ be a reciprocal polynomial of degree $2m$ (if $p(t)$ has an odd degree, then $(t + 1)$ can be factored out to leave a reciprocal polynomial of even degree). If r is a zero of $p(t)$, then we see that $\frac{1}{r}$ is also a zero of $p(t)$. This implies that $p(t)$ can be written as

$$p(t) = \prod_{i=1}^{m} (t^2 - r_i t + 1), \tag{2.4}$$

where each r_i is of the form $r + \frac{1}{r}$, for r a zero of $p(t)$.

LEMMA 2.1. *If $a(s)$ is a polynomial, with indeterminate s, defined as follows*

$$a(s) = p\Big(\frac{s + \sqrt{s^2 - 4}}{2}\Big) \times \Big(\frac{s + \sqrt{s^2 - 4}}{2}\Big)^{-m}, \tag{2.5}$$

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with $p(t)$ as above, then the zeros of $a(s)$ are precisely the r_i, $i = 1, \ldots, m$.

PROOF.

$$\begin{aligned}
a(s) &= p\Big(\frac{s+\sqrt{s^2-4}}{2}\Big) \times \Big(\frac{s+\sqrt{s^2-4}}{2}\Big)^{-m} \\
&= \Big(\frac{s+\sqrt{s^2-4}}{2}\Big)^{-m} \prod_{i=1}^{m} \Big\{ \Big(\frac{s+\sqrt{s^2-4}}{2}\Big)^2 - r_i \Big(\frac{s+\sqrt{s^2-4}}{2}\Big) + 1 \Big\} \\
&= \Big(\frac{s+\sqrt{s^2-4}}{2}\Big)^{-m} \prod_{i=1}^{m} \Big\{ \Big(\frac{s+\sqrt{s^2-4}}{2}\Big)(s - r_i) \Big\} \\
&= \prod_{i=1}^{m} (s - r_i)
\end{aligned}$$

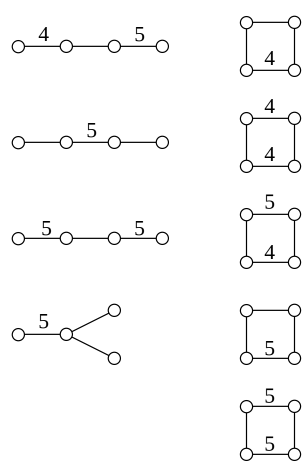
■

For a reciprocal polynomial $p(t)$ of even degree, we shall call $a(t)$ its *associated polynomial*.

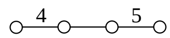
In order to show that the poles of the growth series lie on the unit circle, with the exception of a single real reciprocal pair, we shall show that, apart from a few simple factors, the denominator is a polynomial of even degree that is reciprocal, monic and integer. The following theorem then tells us that such a polynomial is Salem, as required.

THEOREM 2.2 ([1, THEOREM 2.12]). *Suppose $p(t)$ is a monic, integer, reciprocal polynomial of even degree with $p(t) = \prod_{i=1}^{m}(t^2 - r_i t + 1)$. Then p is Salem if and only if at least $m-1$ of the numbers r_i are real and lie in the real interval $[-2, 2]$. If $m-1$ of the r_i are real, then so is the m^{th}.*

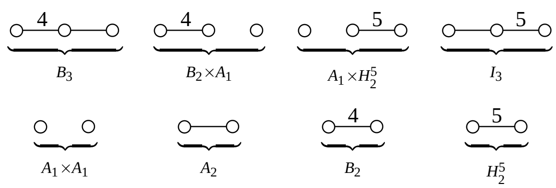
3. **Four Generators.** The complete list of the connected Coxeter graphs for the compact hyperbolic Coxeter groups having four generators is given below (see [4, p. 141]).



(To verify the assertions concerning the poles of the growth series, we shall only consider the first case in detail, as the other cases are similar. In both these cases and those of the five generator groups, the use of the mathematics package *Maple* is gratefully acknowledged.)



There are four subgroups with 3 generators and four non-isomorphic subgroups with 2 generators. Their corresponding growth series, calculated using 2.2 and 2.3, are given below.



name of subgroup	growth series
B_3	$[2][3][4](1+t^3)$
$B_2 \times A_1$	$[2][2][4]$
$A_1 \times H_2^5$	$[2][2][5]$
I_3	$[2][3][5](1+t^3)(1+t^5)$
$A_1 \times A_1$	$[2][2]$
A_2	$[2][3]$
B_2	$[2][4]$
H_2^5	$[2][5]$

Now by 2.1 we have that

$$-\frac{1}{W(t)} = \frac{-1}{B_3(t)} + \frac{-1}{B_2 \times A_1(t)} + \frac{-1}{A_1 \times H_2^5(t)} + \frac{-1}{I_3(t)} + \frac{3}{A_1 \times A_1(t)} + \frac{1}{A_2(t)} + \frac{1}{B_2(t)} + \frac{1}{H_2^5(t)} + \frac{-4}{1+t} + 1.$$

This yields the growth series

$$W(t) = \frac{-[4][5](t^3+1)(t^5+1)}{(t-1)(t^6+t^3+1)(t^8-t^7+t^6-2t^5+t^4-2t^3+t^2-t+1)}.$$

We now need to check that the denominator has all its zeros on the unit circle, apart from a pair of real reciprocal zeros. Trivially the zero of $(t-1)$ lies on the unit circle. Now if we let $s = t^3$, then $(t^6+t^3+1) = s^2+s+1$, whose zeros are given by $\frac{-1\pm 3i}{2}$, with $\left|\frac{-1\pm\sqrt{3}i}{2}\right| = 1$. So we see that the zeros of (t^6+t^3+1) all lie on the unit circle. For

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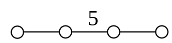
the reciprocal polynomial of degree 8, the associated polynomial, along with intervals indicating the positions of its zeros, is given below.

Associated Polynomial	$t^4 - t^3 - 3t^2 + t + 1$					
Value of t	-1.5	-1	0	1	2.05	2.2
Sign	+	–	+	–	–	+

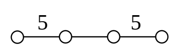
Since all the r_i but one lies in the interval $[-2, 2]$, we see that the polynomial of order 8 is Salem, yielding the required result.

The growth series for the other 4-generator compact hyperbolic Coxeter groups are given below. The denominator is reciprocal, apart from a factor of $(t-1)$. In one case $(t^6 + t^3 + 1)$ again occurs as a factor, and in the last case $(t^2 + 1)$ occurs, whose zeros are trivially on the unit circle. Intervals containing precisely one zero for the appropriate associated polynomial in the denominator are also given. (We have also, in most cases, worked out explicitly the values of the zeros, but we omit these to save space.)



$$W(t) = \frac{-[2][3](t^3+1)(t^5+1)}{(t-1)(t^{10} - t^9 - t^6 + t^5 - t^4 - t + 1)}$$

Associated Polynomial	$t^5 - t^4 - 5t^3 + 4t^2 + 4t - 1$						
Value of t	-1.974	$-.777$	$.216$	1.44	1.9	2.09	2.1
Sign	–	+	–	+	–	–	+



$$W(t) = \frac{-[2][5](t^3+1)(t^5+1)}{(t-1)(t^{12} - t^{11} - t^8 - t^6 - t^4 - t + 1)}$$

Associated Polynomial	$t^6 - t^5 - 6t^4 + 5t^3 + 8t^2 - 5t - 1$							
Value of t	-1.99	-1.3	$-.17$	$.7$	1.4	1.9	2.1	2.2
Sign	+	–	+	–	+	–	–	+

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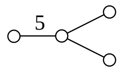
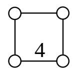


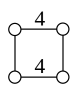
$$W(t) = \frac{-(t^2+1)(t+1)(t^3+1)(t^5+1)}{(t-1)(t^{10}-2t^9+2t^8-2t^7+t^6-t^5+t^4-2t^3+2t^2-2t+1)}$$

Associated Polynomial	$t^5 - 2t^4 - 3t^3 + 6t^2 - 1$						
Value of t	-1.71	$-.39$	$.4$	1.4	1.9	2.1	2.2
Sign	$-$	$+$	$-$	$+$	$-$	$-$	$+$



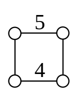
$$W(t) = \frac{-[2][4](t^3+1)}{(t-1)(t^6-t^5-t^4+t^3-t^2-t+1)}$$

Associated Polynomial	$t^3 - t^2 - 4t + 3$				
Value of t	-1.92	0	1.9	2.1	2.2
Sign	$-$	$+$	$-$	$-$	$+$



$$W(t) = \frac{-[2][4](t^3+1)}{(t-1)(t^6-t^5-t^4-t^2-t+1)}$$

Associated Polynomial	$t^3 - t^2 - 4t + 2$				
Value of t	-1.9	0	$.5$	2.1	2.5
Sign	$-$	$+$	$-$	$-$	$+$



$$W(t) = \frac{-[4](t^3+1)(t^5+1)}{(t-1)(t^4-2t^3+t^2-2t+1)(t^6+t^3+1)}$$

(The polynomial $(t^6 + t^3 + 1)$ has been dealt with above.) The associated polynomial is $t^2 - 2t - 1$, which has zeros at $1 \pm \sqrt{2}$.

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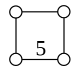
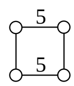


$$W(t) = \frac{-[4](t^3+1)(t^5+1)}{(t-1)(t^{10}-2t^9+t^8-2t^6+2t^5-2t^4+t^2-2t+1)}$$

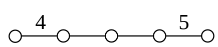
Associated Polynomial	$t^5-2t^4-4t^3+8t^2-2$						
Value of t	-1.99	$-.5$	$.6$	1.4	1.9	2.1	2.3
Sign	$-$	$+$	$-$	$+$	$-$	$-$	$+$



$$W(t) = \frac{-[2](t^3+1)(t^5+1)}{(t-1)(t^2+1)(t^6-2t^5-t^4+3t^3-t^2-2t+1)}$$

Associated Polynomial	t^3-2t^2-4t+7				
Value of t	-1.99	0	1.5	2.1	2.5
Sign	$-$	$+$	$-$	$-$	$+$

4. **Five Generators.** The poles of the growth series do not satisfy the statement "they all lie on the unit circle, apart from one pair of real reciprocal poles". The growth series are given with their numerators and denominators separated. (The list of the connected Coxeter graphs for the 5-generated hyperbolic Coxeter groups is taken from [4, p. 141].)



Numerator of $W(t) =$

$$[3]^2[5]^2(t^3+1)^2(t^5+1)^2(t^6+1)(t^{10}+1)(t^{16}-t^{14}+t^{10}-t^8+t^6-t^2+1)(t^4+1),$$

Denominator of $W(t) =$

$$\begin{aligned} &t^{64}-t^{63}-t^{61}+2t^{60}-2t^{59}+t^{58}-3t^{57}+3t^{56}-3t^{55}+3t^{54}-5t^{53}+5t^{52}-5t^{51} \\ &\quad +6t^{50}-7t^{49}+8t^{48}-8t^{47}+9t^{46}-9t^{45}+11t^{44}-11t^{43}+12t^{42}-11t^{41}+14t^{40} \\ &\quad -13t^{39}+14t^{38}-13t^{37}+16t^{36}-14t^{35}+15t^{34}-14t^{33}+17t^{32}-14t^{31}+15t^{30} \\ &\quad -14t^{29}+16t^{28}-13t^{27}+14t^{26}-13t^{25}+14t^{24}-11t^{23}+12t^{22}-11t^{21}+11t^{20} \\ &\quad -9t^{19}+9t^{18}-8t^{17}+8t^{16}-7t^{15}+6t^{14}-5t^{13}+5t^{12}-5t^{11}+3t^{10}-3t^9+3t^8 \\ &\quad -3t^7+t^6-2t^5+2t^4-t^3-t+1. \end{aligned}$$

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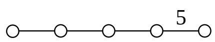
Associated Polynomial =

$$\begin{aligned} &t^{32} - t^{31} - 32t^{30} + 30t^{29} + 466t^{28} - 407t^{27} - 4087t^{26} + 3301t^{25} + 24077t^{24} \\ &\quad - 17826t^{23} - 100602t^{22} + 67549t^{21} + 306847t^{20} - 184375t^{19} - 692606t^{18} \\ &\quad + 366109t^{17} + 1160184t^{16} - 527411t^{15} - 1432805t^{14} + 542851t^{13} + 1283388t^{12} \\ &\quad - 387078t^{11} - 810414t^{10} + 181015t^{9} + 344745t^{8} - 50232t^{7} - 91842t^{6} \\ &\quad + 6685t^{5} + 13538t^{4} - 210t^{3} - 876t^{2} + 17. \end{aligned}$$

The associated polynomial has two zeros greater than 2, so that the denominator has (at least) four (reciprocal) zeros off the unit circle, because when $t = 2.01$ and 2.2, the associated polynomial is positive, but when $t = 2.1$, it is negative.



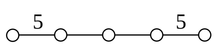
Numerator of $W(t)$ =

$$-[3]^2[5]^2(t^3+1)^2(t^5+1)^2(t^6+1)(t^{10}+1)(t^{16} - t^{14} + t^{10} - t^8 + t^6 - t^2 + 1),$$

Denominator of $W(t)$ =

$$\begin{aligned} &t^{59} + t^{57} + t^{56} + t^{55} + t^{54} + 2t^{53} + t^{52} + 3t^{51} + t^{50} + 3t^{49} + 2t^{48} + 3t^{47} + 2t^{46} \\ &\quad + 4t^{45} + 2t^{44} + 5t^{43} + 2t^{42} + 5t^{41} + 2t^{40} + 5t^{39} + 3t^{38} + 5t^{37} + 2t^{36} + 6t^{35} \\ &\quad + 2t^{34} + 6t^{33} + 2t^{32} + 5t^{31} + 2t^{30} + 5t^{29} + 2t^{28} + 5t^{27} + t^{26} + 5t^{25} + t^{24} \\ &\quad + 4t^{23} + t^{22} + 3t^{21} + t^{20} + 3t^{19} + 3t^{17} + 2t^{15} + t^{13} + t^{11} + t^{9} - t^{8} + t^{7} + t - 1. \end{aligned}$$

Therefore the denominator is not reciprocal.



Numerator of $W(t)$ =

$$[3]^2[5]^2(t^3+1)^2(t^5+1)^2(t^6+1)(t^{10}+1)(t^{16} - t^{14} + t^{10} - t^8 + t^6 - t^2 + 1),$$

Denominator of $W(t)$ =

$$\begin{aligned} &t^{60} - t^{59} - t^{57} - t^{53} - t^{51} + 2t^{50} - 2t^{49} + 2t^{48} - 2t^{47} + 2t^{46} + 2t^{44} - 2t^{43} + 2t^{42} \\ &\quad - 2t^{41} + 6t^{40} - 3t^{39} + 4t^{38} - 3t^{37} + 4t^{36} + 4t^{34} - 3t^{33} + 4t^{32} - 3t^{31} + 8t^{30} \\ &\quad - 3t^{29} + 4t^{28} - 3t^{27} + 4t^{26} + 4t^{24} - 3t^{23} + 4t^{22} - 3t^{21} + 6t^{20} - 2t^{19} + 2t^{18} \\ &\quad - 2t^{17} + 2t^{16} + 2t^{14} - 2t^{13} + 2t^{12} - 2t^{11} + 2t^{10} - t^{9} - t^{7} - t^{3} - t + 1. \end{aligned}$$

Associated Polynomial =

$$\begin{aligned} &t^{30} - t^{29} - 30t^{28} + 28t^{27} + 405t^{26} - 350t^{25} - 3250t^{24} + 2575t^{23} + 17250t^{22} \\ &\quad - 12375t^{21} - 63754t^{20} + 40753t^{19} + 168207t^{18} - 93804t^{17} - 319464t^{16} \\ &\quad + 151030t^{15} + 434690t^{14} - 167260t^{13} - 416260t^{12} + 122575t^{11} + 271132t^{10} \\ &\quad - 55177t^{9} - 113372t^{8} + 13141t^{7} + 27634t^{6} - 1165t^{5} - 3380t^{4} + 30t^{3} + 195t^{2} - 2. \end{aligned}$$

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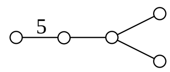
The associated polynomial has two zeros greater than 2, so that the denominator has (at least) four (reciprocal) zeros off the unit circle, because when $t = 2.01$ and 2.1, the associated polynomial is negative, but when $t = 2.02$, it is positive.



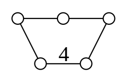
Numerator of $W(t) =$

$$-[3]^2[5]^2(t^3+1)^2(t^5+1)^2(t^6+1)(t^{10}+1)(t^{16}-t^{14}+t^{10}-t^8+t^6-t^2+1),$$

Denominator of $W(t) =$

$$\begin{aligned} &t^{59}+2t^{58}+t^{57}+2t^{56}+2t^{55}+4t^{54}+3t^{53}+4t^{52}+4t^{51}+7t^{50}+5t^{49}+6t^{48} \\ &\quad +6t^{47}+8t^{46}+8t^{45}+8t^{44}+9t^{43}+10t^{42}+10t^{41}+9t^{40}+11t^{39}+10t^{38}+12t^{37} \\ &\quad +9t^{36}+13t^{35}+10t^{34}+13t^{33}+9t^{32}+13t^{31}+8t^{30}+13t^{29}+7t^{28}+13t^{27} \\ &\quad +7t^{26}+12t^{25}+5t^{24}+11t^{23}+5t^{22}+10t^{21}+2t^{20}+9t^{19}+3t^{18}+8t^{17}+t^{16} \\ &\quad +6t^{15}+t^{14}+5t^{13}-t^{12}+4t^{11}+3t^9-t^8+2t^7+t^5-t^4+t^3+t-1. \end{aligned}$$

Therefore the denominator is not reciprocal.



Numerator of $W(t) =$

$$-[2][3][4]^2(t^3+1)(t^4+1)(t^8+1),$$

Denominator of $W(t) =$

$$\begin{aligned} &t^{24}+t^{23}+t^{22}+2t^{21}+t^{20}+t^{19}-t^{16}+t^{15}+t^{14}+t^{13} \\ &\quad +t^{12}+t^{11}+t^{10}+t^9-3t^8-t^5+t^4+t^2+t-1. \end{aligned}$$

Therefore the denominator is not reciprocal.

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