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The hulls of semiprime rings

Paul F. Conrad

Each semiprime ring admits a unique projectable, strongly projectable, laterally complete and orthocomplete hull. Almost all of the theory for X-hulls of lattice-ordered groups in Paul Conrad, "The hulls of representable l-groups and f-rings", *J. Austral. Math. Soc.* **16** (1973), 385-415, has a counterpart for semiprime rings. The proofs of these results will appear elsewhere. They come in a large part directly from the corresponding theory for lattice-ordered groups. There is also a feedback from the rings to the groups.

Let G be a semiprime ring and for $a, b \in G$ define $a \geq b$ if $agb = bgb$ for all $g \in G$. This is equivalent ot the fact that a agrees with b on the support of b in each representation of G as a subdirect product of prime rings. Thus $\geq$ is a partial order for G with smallest element 0 and for $a, b, x \in G$,

$$a \geq b \text{ implies } ax \geq bx , \quad xa \geq xb \text{ and } ab = ba .$$

We say that a is *disjoint* from b or that a is *orthogonal* to b if $aGb = 0$ (notation $a \perp b$). This is equivalent to the fact that a and b have disjoint support in each representation of G as a subdirect product of prime rings. Thus $a \perp b$ if and only if $b \perp a$, and in this case $ab = ba$. If X is a subset of G then

$$X' = \{g \in G \mid g \perp x \text{ for each } x \in X\}$$

is the *annihilator ideal* of X . Lambek [5] has shown that these ideals form a complete boolean algebra which we will denote by $P(G)$. G will be called

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a *P-ring* if $G = g'' \oplus g'$ for each $g \in G$ (projectable),

an *SP-ring* if $G = X'' \oplus X'$ for each subset X of G (strongly projectable),

an *L-ring* if each pairwise disjoint subset of G has a least upper bound (laterally complete).

an *O-ring* if G is both an L-ring and an SP-ring (orthocomplete).

An overring H is a *left essential extension* of G if this is the case when H is considered as a left G-module. We prove the following theorems for $X = P$, SP, L or O.

THEOREM A. *Let G be a semiprime ring and let H be a left essential extension of G that is an X-ring. Then the intersection of all the subrings of H that contain G and are X-rings is a minimal left essential extension of G that is an X-ring; called an X-hull of G.*

THEOREM B. *Each semiprime ring admits a unique X-hull G^X. G^X is semiprime and G^X is reduced (commutative) if and only if G is reduced (commutative). If G has an identity 1, then 1 is also the identity for G^X. Finally G^X is the minimal right essential extension of G that is an X-ring.*

If G is reduced then the proofs of these theorems are almost identical with the proofs of the corresponding theorems for lattice-ordered groups in [3]; one simply replaces $a \wedge b$ by ab. For semiprime rings the proofs in [3] can be adapted. We show that

$$G \subseteq G^P \subseteq G^{SP} \subseteq (G^{SP})^L = (G^P)^L = G^O$$

and $(G^L)^P = (G^L)^{SP} \subseteq G^O$, but here we need not have equality.

In order to prove Theorems A and B we show that if H is a left essential extension of the semiprime ring G then H is semiprime and there is a natural isomorphism of $P(H)$ onto $P(G)$. If H is laterally complete then G is an L-subring of H (that is, for each disjoint subset $\{g_\lambda \mid \lambda \in \Lambda\}$ of G for which $\vee_G\, g_\lambda$ exists, it follows that $\vee_H\, g_\lambda = \vee_G\, g_\lambda$).

If G is a boolean ring then so is G^X and $G^L = G^0$. Moreover G^0 is the Dedekind-MacNeille completion of G if and only if G has an identity. If G is regular then so is G^P, G^{SP}, and G^0. We show that the ring G^X is determined by the addition and the partial order.

THEOREM. *Suppose that* G *is a semiprime ring and consider the system* $(G^X, +, \geq)$. *Then there is a unique multiplication on* G^X *so that*

(a) G^X *is a semiprime ring,*

(b) G *is a subring of* G^X, *and*

(c) the multiplication on G^X *induces the given partial order* $\geq$.

Almost all of the theory for the X-hulls of lattice-ordered groups in [3] has a counterpart for semiprime rings. In particular, this is true for the annihilator preserving endomorphisms of G and for the theory of semiprime rings with a basis.

$P(G)$ is atomic if and only if G^0 is a product of prime rings. From this it is easy to derive necessary and sufficient conditions for a reduced ring to be a product of integral domains; in particular, those in the literature for commutative rings (see, for example, [7], Theorem 4.3).

Abian [1] proved that a commutative semiprime ring G is a product of fields if and only if G is hyperatomic and laterally complete. A student of mine, Otis Kenny, has shown that a reduced ring H is a product of division rings if and only if H is hyperatomic and laterally complete. Thus H^L is a product of division rings if and only if H is hyperatomic.

If G is a commutative semiprime ring with 1, then G^P is the Baer extension of G that was introduced by Kist [4] and G^{SP} is the Baer extension of G that was introduced by Mewborn [6]. Thus for an arbitrary semiprime ring G with 1 we have the unique Baer hulls G^P and G^{SP}.

In [8] Speed, using the technique developed in [2] (which is somewhat

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cruder than that used in [3]), constructed G^P and G^{SP} and some hulls in between for commutative semiprime rings with 1 . His description of these hulls is categorical, but somewhat complicated.

If G is a semiprime ring then the complete ring of left (right) quotients of G is an O-ring that contains G^0 .

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Department of Mathematics,
University of Kansas,
Lawrence,
Kansas, USA.