COMPOSITIO MATHEMATICA

A study of the representations supported by the orbit closure of the determinant

Shrawan Kumar

Compositio Math. **151** (2015), 292–312.

doi:10.1112/S0010437X14007660

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY 150 YEARS

Compositio Math. 151 (2015) 292–312
doi:10.1112/S0010437X14007660

A study of the representations supported by the orbit closure of the determinant

Shrawan Kumar

Abstract

We show the existence of a large family of representations supported by the orbit closure of the determinant. However, the validity of our result is based on the validity of the celebrated 'Latin square conjecture' due to Alon and Tarsi or, more precisely, on the validity of an equivalent 'column Latin square conjecture' due to Huang and Rota.

1. Introduction

Let $\mathfrak{v}$ be a complex vector space of dimension m and let $E := \operatorname{End}\mathfrak{v}$. Consider $\mathscr{D} \in Q := S^m(E)^*$, where $\mathscr{D}$ is the function taking the determinant of any $X \in \operatorname{End}\mathfrak{v}$. Fix a basis $\{v_1, \dots, v_m\}$ of $\mathfrak{v}$ and a positive integer $n < m$ and consider the function $\mathscr{P} \in Q$, defined by $\mathscr{P}(X) = x_{1,1}^{m-n}\operatorname{Perm}(X^o)$, X^o being the component of X in the right down $n \times n$ corner, where any element of $\operatorname{End}\mathfrak{v}$ is represented by an $m \times m$ matrix $X = (x_{i,j})_{1 \leqslant i,j \leqslant m}$ in the basis $\{v_i\}$ and Perm denotes the permanent. The group $G = \mathrm{GL}(E)$ canonically acts on Q. Let $\mathcal{X}$ (respectively $\mathcal{Y}$) be the G-orbit closure of $\mathscr{D}$ (respectively $\mathscr{P}$) inside Q. Then $\mathcal{X}$ and $\mathcal{Y}$ are closed (affine) subvarieties of Q which are stable under the standard homothety action of $\mathbb{C}^*$ on Q. Thus, their affine coordinate rings $\mathbb{C}[\mathcal{X}]$ and $\mathbb{C}[\mathcal{Y}]$ are nonnegatively graded G-algebras over the complex numbers $\mathbb{C}$. Clearly, $\mathscr{D} \odot \operatorname{End} E \subset \mathcal{X}$, where $\operatorname{End} E$ acts on Q on the right via $(q \odot A)(X) = q(A \cdot X)$, for $A \in \operatorname{End} E, q \in Q$ and $X \in E$.

For any positive integer n, let $\bar{m} = \bar{m}(n)$ be the smallest positive integer such that the permanent of any $n \times n$ matrix can be realized as a linear projection of the determinant of an $\bar{m} \times \bar{m}$ matrix. This is equivalent to saying that $\mathscr{P} \in \mathscr{D} \odot \operatorname{End} E$ for the pair $(\bar{m}, n)$. Then Valiant conjectured that the function $\bar{m}(n)$ grows faster than any polynomial in n (cf. [Val79]).

Similarly, let $m = m(n)$ be the smallest integer such that $\mathscr{P} \in \mathcal{X}$ (for the pair (m, n)). Clearly, $m(n) \leqslant \bar{m}(n)$. Now, Mulmuley and Sohoni strengthened Valiant's conjecture. They conjectured that, in fact, the function $m(n)$ grows faster than any polynomial in n (cf. [MS01, MS08] and the references therein). They further conjectured that if $\mathscr{P} \notin \mathcal{X}$, then there exists an irreducible G-module which occurs in $\mathbb{C}[\mathcal{Y}]$ but does not occur in $\mathbb{C}[\mathcal{X}]$. (Of course, if $\mathscr{P} \in \mathcal{X}$, then $\mathbb{C}[\mathcal{Y}]$ is a G-module quotient of $\mathbb{C}[\mathcal{X}]$.) This geometric complexity theory program initiated by Mulmuley and Sohoni provides a significant mathematical approach to solving Valiant's conjecture (in fact, the strengthened version of Valiant's conjecture proposed by them).

By [Kum13, Theorem 5.2], if an irreducible G-module $V_E(\lambda)$ (with highest weight λ) appears in $\mathbb{C}[\mathcal{Y}]$, then $V_E(\lambda)$ is a polynomial representation of G given by a partition

$$\lambda : (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{n^2+1} \geqslant 0 \geqslant \cdots \geqslant 0)$$

with last $m^2 - (n^2 + 1)$ zeros.

Received 19 October 2012, accepted in final form 1 June 2014, published online 24 October 2014.
2010 Mathematics Subject Classification 14L30, 14L24, 14R20, 20G05, 22E46, 68W30 (primary).
Keywords: determinant orbit closure, Latin squares, Alon–Tarsi conjecture, geometric complexity theory, Valiant's conjecture, representations.

From now on (in this Introduction), we assume that m is even. Our principal result in this paper (Corollary 6.2) asserts that for any partition $\mu : (\mu_1 \geqslant \cdots \geqslant \mu_m \geqslant 0 \geqslant \cdots \geqslant 0)$ with last $m^2 - m$ zeros, the irreducible G-module $V_E(m\mu)$ appears in $\mathbb{C}[\mathcal{X}]$ with nonzero multiplicity, provided the column Latin (m, m)-square conjecture is valid (cf. Conjecture 4.3). In particular, if $m \geqslant n^2 + 1$, for any irreducible representation $V_E(\lambda)$ appearing in $\mathbb{C}[\mathcal{Y}]$, $V_E(m\lambda)$ appears in $\mathbb{C}[\mathcal{X}]$ (again assuming the validity of the column Latin (m, m)-square conjecture). Thus, finding an irreducible representation in $\mathbb{C}[\mathcal{Y}]$ which does not occur in $\mathbb{C}[\mathcal{X}]$ (on which the success of the Mulmuley–Sohoni program relies) for $m \geqslant n^2 + 1$ is not so easy. As a consequence of our Corollary 6.2, we deduce that the symmetric Kronecker coefficient $sk_{m\overline{\lambda}, d\delta_m, d\delta_m} > 0$ for any partition $\overline{\lambda} : (\overline{\lambda}_1 \geqslant \overline{\lambda}_2 \geqslant \cdots \geqslant \overline{\lambda}_m \geqslant 0)$ of d, where δ_m is the partition $\delta_m : (1 \geqslant 1 \geqslant \cdots \geqslant 1)$ (m factors) (cf. Corollary 6.5).

This paper is organized as follows. By a result of Howe (cf. Corollary 2.4), for any fundamental weight δ_i $(1 \leqslant i \leqslant m^2 = \dim E)$ of $\mathrm{GL}(E)$, the irreducible $\mathrm{GL}(E)$-module $V_E(d\delta_i)$, for $0 < d < m$, does not occur in $S^\bullet(S^m(E))$, whereas $V_E(m\delta_i)$ occurs with multiplicity one in $S^\bullet(S^m(E))$. In fact, it occurs in $S^i(S^m(E))$. In § 2 we give an explicit construction of the highest weight vector $P_i = \gamma_{m,i}$ in this unique copy of $V_E(m\delta_i)$ in $S^i(S^m(E))$ (cf. Definition 2.5).

In § 3, for any $1 \leqslant i \leqslant m$, we calculate $\gamma_{m,i}$ on a certain subset $\theta(M(m, i))$ of $\mathcal{X}$ given by a morphism $\theta : M(m, i) \to \mathcal{X}$, where $M(m, i)$ denotes the set of $m \times i$ matrices. The induced map θ^* on the level of affine coordinate rings is identified with a certain very explicit map φ. The main result of this section is Proposition 3.4, which asserts that $\gamma_{m,i}$ restricted to the image $\theta(M(m, i))$ is nonzero if and only if the $\mathrm{GL}(V_m)$-submodule U_i generated by $v_o^{\otimes i} \in S^i(S^m(V_m^*))$ intersects the isotypic component $\mathcal{I}_{m\delta_i}$ of $S^i(S^m(V_m^*))$ corresponding to the irreducible $\mathrm{GL}(V_m)$-module $V_m(m\delta_i)^*$ nontrivially, where the element v_o is defined by the identity (9).

In § 4, we turn to Latin squares (more generally Latin rectangles) and state the *column Latin square conjecture* due to Huang and Rota. As shown by Huang and Rota, their conjecture is equivalent to the celebrated *Latin square conjecture* due to Alon and Tarsi. We recall that the Latin square conjecture is known to be true for $p - 1$ as well as $p + 1$, for any odd prime p; in particular, it is true for any even integer up to 24 (cf. Remark 4.5).

Section 5 is devoted to proving that the validity of the column Latin square conjecture implies that the isotypic component $\mathcal{I}_{m\delta_i}$ of $S^i(S^m(V_m^*))$ corresponding to the irreducible $\mathrm{GL}(V_m)$-module $V_m(m\delta_i)^*$ intersects the $\mathrm{GL}(V_m)$-submodule U_i generated by $v_o^{\otimes i}$ nontrivially (cf. Theorem 5.6). In fact, for $i = m$, we show that the latter assertion is equivalent to the column Latin square conjecture.

This sets the stage for the proof of our main theorem (cf. Theorem 6.1), which asserts that the irreducible module $V_E(m\delta_i)$ occurs in $\mathbb{C}[\mathcal{X}]$ with multiplicity one for any $1 \leqslant i \leqslant m$ if the column Latin square conjecture is true for $m \times m$ Latin squares. This is shown by proving that P_i does not vanish identically on $\mathcal{X}$. As an immediate corollary (cf. Corollary 6.2), we deduce that for any partition $\lambda : \lambda_1 \geqslant \cdots \geqslant \lambda_m \geqslant 0, V_E(m\lambda)$ occurs in $\mathbb{C}[\mathcal{X}]$ (if the column Latin square conjecture is true for $m \times m$ Latin squares).

Finally, in Remark 6.6(b), we observe that $V_E(m\delta_i)$ (for any $1 \leqslant i \leqslant m$) occurs in $\mathbb{C}[\overline{\mathrm{GL}(E) \cdot \mathfrak{P}}]$ with multiplicity one, where $\mathfrak{P}$ is the function $E \to \mathbb{C}$ taking any matrix $A \in E = \operatorname{End} \mathfrak{v}$ to its permanent. (Of course, as mentioned above, $V_E(d\delta_i)$, for any $0 < d < m$ and $1 \leqslant i \leqslant m^2$, does not occur in $S^\bullet(S^m(E))$, and hence it does not occur in $\mathbb{C}[\overline{\mathrm{GL}(E) \cdot \mathfrak{P}}]$ or in $\mathbb{C}[\mathcal{X}]$.)

For any vector space W over the complex numbers, in this paper, we view $S^k(W)$ as the subspace of $\otimes^k W$ consisting of symmetric tensors.

2. An explicit realization of multiples of fundamental $\mathrm{GL}(E)$-representations in $S^\bullet(S^\bullet(E))$

Let E be a finite-dimensional complex vector space with basis $\{e_1, \dots, e_\ell\}$. Let δ_i, $1 \leqslant i \leqslant \ell$, be the ith fundamental weight of $\mathrm{GL}(E) = \mathrm{GL}(\ell)$. This corresponds to the partition $1 \geqslant 1 \geqslant \cdots \geqslant 1$ (i factors).

Lemma 2.1. *For any positive integers d, j and m, the multiplicity of the irreducible $\mathrm{GL}(E)$-module $V_E(d\delta_i)$ (with highest weight $d\delta_i$) in $S^j(S^m(E))$ is the same as the multiplicity of the irreducible $\mathrm{GL}(E_i)$-module $V_{E_i}(d\delta_i)$ in $S^j(S^m(E_i))$, where E_i is the subspace of E spanned by $\{e_1, \dots, e_i\}$.*

In fact, the highest weight vectors in $S^j(S^m(E))$ for the irreducible $\mathrm{GL}(E)$-module $V_E(d\delta_i)$ coincide with the highest weight vectors in $S^j(S^m(E_i))$ for the irreducible $\mathrm{GL}(E_i)$-module $V_{E_i}(d\delta_i)$.

Proof. Let B_E be the standard Borel subgroup of $\mathrm{GL}(E)$ consisting of all the invertible upper triangular matrices (with respect to the basis $\{e_1, \dots, e_\ell\}$). Let $v \in S^j(S^m(E))$ be a B_E-eigenvector of weight $d\delta_i$. Then clearly $v \in S^j(S^m(E_i))$ and v is a B_{E_i}-eigenvector of weight $d\delta_i$. Conversely, let $v' \in S^j(S^m(E_i))$ be a B_{E_i}-eigenvector of weight $d\delta_i$. Then the line $\mathbb{C}v'$ is clearly stable under B_E. Moreover, the vector v' is a weight vector of weight $d\delta_i$ with respect to the standard maximal torus T_E (consisting of invertible diagonal matrices) of $\mathrm{GL}(E)$. This proves the lemma. $\square$

Corollary 2.2. *With the notation as above, the multiplicity $\mu_E(d\delta_i)$ of $V_E(d\delta_i)$ in $S^j(S^m(E))$ is equal to the dimension of the invariant space $[S^j(S^m(E_i))]^{\mathrm{SL}(E_i)}$ if $di = jm$. If $di \neq jm$, then $\mu_E(d\delta_i) = 0$.*

We recall the following result from [How87, Proposition 4.3].

Proposition 2.3. *Let E be a vector space of dimension ℓ as above. For positive integers j, m, we have*

(a) $[S^j(S^m(E))]^{\mathrm{SL}(E)} = (0)$, *if* $0 < j < \ell$;

(b) $[S^\ell(S^m(E))]^{\mathrm{SL}(E)} \simeq \begin{cases} (0) & \text{if } m \text{ is odd,} \\ \mathbb{C} & \text{if } m \text{ is even.} \end{cases}$

Combining Corollary 2.2 with Proposition 2.3, together with the action of the center of $\mathrm{GL}(E)$, we get the following result.

Corollary 2.4. *Let E be a vector space of dimension ℓ as above. Let m be a positive even integer and let $1 \leqslant i \leqslant \ell$. Let d be the smallest positive integer such that $V_E(d\delta_i)$ occurs in $S^\bullet(S^m(E))$ as a $\mathrm{GL}(E)$-submodule. Then $d = m$. Moreover, $V_E(m\delta_i)$ occurs in $S^\bullet(S^m(E))$ with multiplicity 1 and it occurs in $S^i(S^m(E))$.*

From now on in this section, let m be an even positive integer.

We first give an explicit construction of the invariant $[S^i(S^m(E_i))]^{\mathrm{SL}(E_i)}$ for any $1 \leqslant i \leqslant \ell$. Recall from Proposition 2.3 that it is one-dimensional.

Definition 2.5 (An explicit construction of $[S^i(S^m(E_i))]^{\mathrm{SL}(E_i)}$). Recall that E_i has a basis $\{e_1, \dots, e_i\}$. Let $M(i,i)$ be the space of $i \times i$ matrices over $\mathbb{C}$. Define a linear isomorphism

$$\theta : (\otimes^2 E_i)^* \xrightarrow{\sim} M(i,i), \quad \theta(f) = (\theta(f)_{p,q})_{1 \leqslant p,q \leqslant i},$$

where $\theta(f)_{p,q} = f(e_p \otimes e_q)$, for any $f \in (\otimes^2 E_i)^*$.

Let $\mathrm{GL}(E_i)$ act on $M(i,i)$ via

$$g \cdot A = (g^{-1})^t A g^{-1} \quad \text{for } g \in \mathrm{GL}(E_i) \text{ and } A \in M(i,i).$$

Then, θ is $\mathrm{GL}(E_i)$-equivariant. Now, define the map (setting $m' = m/2$)

$$\theta^{\otimes m'} : (\otimes^m E_i)^* \xrightarrow{\sim} \otimes^{m'}(M(i,i))$$

by identifying

$$(\otimes^m E_i)^* \simeq ((\otimes^2 E_i)^*) \otimes \cdots \otimes ((\otimes^2 E_i)^*) \quad (m' \text{ factors})$$

and setting

$$\theta^{\otimes m'}(f_1 \otimes \cdots \otimes f_{m'}) = \theta(f_1) \otimes \cdots \otimes \theta(f_{m'}) \quad \text{for } f_k \in (\otimes^2 E_i)^*.$$

For any finite-dimensional vector space W and nonnegative integer k, let $\mathcal{P}^k(W)$ be the space of homogeneous polynomials of degree k on W, i.e.,

$$\mathcal{P}^k(W) = \{f : W \to \mathbb{C} \text{ polynomial such that } f(\lambda w) = \lambda^k f(w)\ \forall w \in W \text{ and } \lambda \in \mathbb{C}\}.$$

Then there is a linear isomorphism (cf. [GW09, Proposition B.2.4])

$$\xi : S^k(W)^* \xrightarrow{\sim} \mathcal{P}^k(W)$$

defined by $\xi(\theta)(w) = \theta(w^{\otimes k})$, for $\theta \in S^k(W)^*$ and $w \in W$. If an algebraic group G acts linearly on W, then ξ is G-equivariant.

Define the linear map $\bar{\xi} : \mathcal{P}^k(W) \to (\otimes^k W)^*$ by

$$(\bar{\xi}(f))(w_1 \otimes \cdots \otimes w_k) = \frac{1}{k!} \quad (\text{the coefficient of } t_1 \ldots t_k \text{ in } f(t_1 w_1 + \cdots + t_k w_k)),$$

for $f \in \mathcal{P}^k(W)$ and $w_1, \ldots, w_k \in W$. Then the inverse map

$$\xi^{-1} : \mathcal{P}^k(W) \to S^k(W)^*$$

is given by the composition of $\bar{\xi}$ with the restriction map $(\otimes^k W)^* \to S^k(W)^*$.

Consider the linear projection obtained via the symmetrization

$$\pi : \otimes^m E_i \to S^m(E_i), \quad w_1 \otimes \cdots \otimes w_m \mapsto \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(m)},$$

where $\mathfrak{S}_m$ is the permutation group on the symbols $[m] := \{1, \ldots, m\}$. Thus, we have $\mathrm{GL}(E_i)$-equivariant linear maps

$$S^m(E_i)^* \overset{\pi^*}{\hookrightarrow} (\otimes^m E_i)^* \xrightarrow[\sim]{\theta^{\otimes m'}} \otimes^{m'}(M(i,i)).$$

This gives rise to a linear $\mathrm{GL}(E_i)$-equivariant map

$$S^i(S^m(E_i)^*) \to S^i(\otimes^{m'}(M(i,i))). \tag{1}$$

Now consider the map

$$\det{}^{\otimes m'} : M(i,i) \to \mathbb{C}, \quad A \mapsto (\det A)^{m'}.$$

It is clearly a homogeneous polynomial of degree im', which is $\mathrm{SL}(E_i)$-invariant. Thus, via the above isomorphism ξ^{-1}, we get an $\mathrm{SL}(E_i)$-invariant linear map

$$\widehat{\det^{\otimes m'}} : S^{im'}(M(i,i)) \to \mathbb{C}. \tag{2}$$

Of course, we have a canonical $\mathrm{GL}(E_i)$-equivariant projection

$$S^i(\otimes^{m'}(M(i,i))) \to S^{im'}(M(i,i)), \tag{3}$$

obtained via the inclusion

$$S^i(\otimes^{m'}(M(i,i))) \subset \otimes^i(\otimes^{m'}(M(i,i))) \simeq \otimes^{im'}(M(i,i))$$

followed by the symmetrization $\otimes^{im'}(M(i,i)) \to S^{im'}(M(i,i))$.

Composing the maps $(2) \circ (3) \circ (1)$, we get an $\mathrm{SL}(E_i)$-invariant linear map

$$\gamma_{m,i} : S^i(S^m(E_i)^*) \to \mathbb{C}.$$

For any vector space W, we have a canonical $\mathrm{GL}(W)$-equivariant identification

$$S^i(W^*) \simeq S^i(W)^* \tag{4}$$

via $S^i(W^*) \subset \otimes^i(W^*) \simeq (\otimes^i W)^* \to S^i(W)^*$, where the last map is the restriction map. Thus, $\gamma_{m,i}$ can be thought of as an element of $[S^i(S^m(E_i))]^{\mathrm{SL}(E_i)}$.

Lemma 2.6.

$$\gamma_{m,i}\left(\left(\sum_{j=1}^{i}(e_j^*)^{\otimes m}\right)^{\otimes i}\right) \neq 0,$$

where $\{e_1^*, \ldots, e_i^*\}$ is the basis of E_i^* dual to the basis $\{e_1, \ldots, e_i\}$ of E_i.

Proof. Let $E_{j,k} \in M(i,i)$ be the matrix with all entries 0, except (j,k) which is 1. By the definition,

$$\begin{aligned}
\gamma_{m,i}\left(\left(\sum_{j=1}^{i}(e_j^*)^{\otimes m}\right)^{\otimes i}\right) &= \gamma_{m,i}\left(\sum_{1\leqslant j_1,\ldots,j_i\leqslant i}((e_{j_1}^*)^{\otimes m}\otimes\cdots\otimes(e_{j_i}^*)^{\otimes m})\right)\\
&= \sum_{1\leqslant j_1,\ldots,j_i\leqslant i}\widehat{\det^{\otimes m'}}(E_{j_1,j_1}^{\otimes m'}\otimes\cdots\otimes E_{j_i,j_i}^{\otimes m'})\\
&= \sum_{1\leqslant j_1,\ldots,j_i\leqslant i}\frac{1}{(im')!}\quad \text{the coefficient of } (t_1\ldots t_{im'}) \text{ in}\\
&\qquad [\det((t_1+\cdots+t_{m'})E_{j_1,j_1}+(t_{m'+1}+\cdots+t_{2\,m'})E_{j_2,j_2}\\
&\qquad +\cdots+(t_{(i-1)m'+1}+\cdots+t_{im'})E_{j_i,j_i})]^{m'}\\
&= \sum_{\sigma\in\mathfrak{S}_i}\frac{1}{(im')!}\quad \text{the coefficient of } (t_1t_2\ldots t_{im'}) \text{ in}\\
&\qquad [(t_1+\cdots+t_{m'})^{m'}\ldots(t_{(i-1)m'+1}+\cdots+t_{im'})^{m'}]\\
&= \frac{i!}{(im')!}(m'!)^i \neq 0.
\end{aligned}$$

This proves the lemma. $\square$

We record this in the following.

Lemma 2.7. *The element $\gamma_{m,i}$ is the unique (up to a scalar multiple) nonzero element of $[S^i(S^m(E_i))]^{\mathrm{SL}(E_i)}$.*

3. Calculation of $\gamma_{m,i}$ on the determinant orbit closure

We continue to assume that m is even and $m' = m/2$.

Let $\mathfrak{v}$ be a complex vector space of dimension m and let $E := \operatorname{End}\mathfrak{v} = \mathfrak{v} \otimes \mathfrak{v}^*$. Fix a basis $\{v_1, \ldots, v_m\}$ of $\mathfrak{v}$ and let $\{v_1^*, \ldots, v_m^*\}$ be the dual basis of $\mathfrak{v}^*$. Take the basis $\{v_i \otimes v_j^*\}_{1\leqslant i,j\leqslant m}$ of E and order the basis elements as $\{e_1, e_2, \ldots, e_{m^2}\}$ satisfying

$$e_1 = v_1 \otimes v_1^*, \quad e_2 = v_2 \otimes v_2^*, \ldots, e_m = v_m \otimes v_m^*.$$

Fix $1 \leqslant i \leqslant m$ and consider the subspace E_i of E spanned by $\{e_1, \ldots, e_i\}$. Take $A \in \operatorname{End} E$ of the form

$$Ae_j = \sum_{p=1}^{m} a_p^j e_p, \quad 1 \leqslant j \leqslant i. \tag{5}$$

In the sequel, we will only consider $A \in \operatorname{End} E$ of the above form and the values of Ae_j for $j > i$ will be irrelevant for us. Thus, we can (and will) think of $A = (a_p^j)_{1\leqslant p\leqslant m, 1\leqslant j\leqslant i}$ as an $m \times i$ matrix.

Define a right action of the semigroup $\operatorname{End} E$ on $Q := \mathcal{P}^m(E) \simeq S^m(E)^*$ (cf. Definition 2.5 for the last identification under ξ) via

$$(f \odot A)(e) = f(Ae) \quad \text{for } f \in Q, A \in \operatorname{End} E \text{ and } e \in E. \tag{6}$$

Take $f = \mathscr{D} \odot A \in Q$, where A is of the form (5) and $\mathscr{D} \in \mathcal{P}^m(E)$ is the function taking the determinant of any $X \in E$. For $1 \leqslant l_1, \ldots, l_m \leqslant i$, let $A^{l_1,\ldots,l_m}$ denote the $m \times m$ matrix with the first column $\begin{bmatrix} a_1^{l_1} \\ \vdots \\ a_m^{l_1} \end{bmatrix}$, etc. For integers $d_1, \ldots, d_i \geqslant 0$ with $d_1 + \cdots + d_i = m$, $\{1^{d_1}, 2^{d_2}, \ldots, i^{d_i}\}$ means the collection

$$\Big\{\underbrace{1, \ldots, 1}_{d_1 \text{ times}}; \underbrace{2, \ldots, 2}_{d_2 \text{ times}}; \ldots; \underbrace{i, \ldots, i}_{d_i \text{ times}}\Big\},$$

and $A^{(d_1,\ldots,d_i)}$ means the $m \times m$ matrix with columns

$$\underbrace{A^1, \ldots, A^1}_{d_1 \text{ times}}; \underbrace{A^2, \ldots, A^2}_{d_2 \text{ times}}; \ldots; \underbrace{A^i, \ldots, A^i}_{d_i \text{ times}}, \quad \text{where } A^j \text{ is the column } \begin{bmatrix} a_1^j \\ \vdots \\ a_m^j \end{bmatrix}.$$

Lemma 3.1. *The image of $f_{|E_i}$ in $\otimes^{m'}(M(i,i))$ under $\theta^{\otimes m'} \circ \pi^*$ (cf. Definition 2.5) is given by*

$$\frac{1}{m!} \sum_{\substack{d_1+\cdots+d_i=m \\ d_j \geqslant 0}} \operatorname{Perm} A^{(d_1,\ldots,d_i)} \sum_{1\leqslant j_p,k_p\leqslant i} E_{j_1,k_1} \otimes \cdots \otimes E_{j_{m'},k_{m'}},$$

where the last summation runs over those ordered m-tuples $(j_1, k_1, \ldots, j_{m'}, k_{m'})$ such that the collection (without regard to the order)

$$\{j_1, k_1, \ldots, j_{m'}, k_{m'}\} = \{1^{d_1}, 2^{d_2}, \ldots, i^{d_i}\},$$

and Perm denotes the permanent of the matrix.

https://doi.org/10.1112/S0010437X14007660 Published online by Cambridge University Press

Proof. For any $1 \leqslant j_p, k_p \leqslant i$,

$$
\begin{aligned}
f(\pi((e_{j_1} \otimes e_{k_1}) \otimes \cdots \otimes (e_{j_{m'}} \otimes e_{k_{m'}})))
&= \mathscr{D}((Ae_{j_1} \otimes Ae_{k_1}) \otimes \cdots \otimes (Ae_{j_{m'}} \otimes Ae_{k_{m'}})) \\
&= \mathscr{D}\bigg(\sum_{1 \leqslant p_1, \ldots, p_m \leqslant m} (a^{j_1}_{p_1} e_{p_1} \otimes a^{k_1}_{p_2} e_{p_2}) \otimes \cdots \otimes (a^{j_{m'}}_{p_{m-1}} e_{p_{m-1}} \otimes a^{k_{m'}}_{p_m} e_{p_m})\bigg) \\
&= \sum_{1 \leqslant p_1, \ldots, p_m \leqslant m} (a^{j_1}_{p_1} a^{k_1}_{p_2}) \ldots (a^{j_{m'}}_{p_{m-1}} a^{k_{m'}}_{p_m}) \mathscr{D}(e_{p_1} \otimes e_{p_2} \otimes \cdots \otimes e_{p_{m-1}} \otimes e_{p_m}) \\
&= \sum_{\sigma \in \mathfrak{S}_m} (a^{j_1}_{\sigma(1)} a^{k_1}_{\sigma(2)}) \ldots (a^{j_{m'}}_{\sigma(m-1)} a^{k_{m'}}_{\sigma(m)}) \mathscr{D}(e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \cdots \otimes e_{\sigma(m-1)} \otimes e_{\sigma(m)}) \\
&= \frac{1}{m!} \operatorname{Perm}(A^{j_1, k_1, \ldots, j_{m'}, k_{m'}}). \qquad (7)
\end{aligned}
$$

Now the image of $f_{|E_i}$ in $\otimes^{m'}(M(i,i))$ under $\theta^{\otimes m'} \circ \pi^*$ is given by

$$
\begin{aligned}
&\sum_{1 \leqslant j_p, k_p \leqslant i} f((e_{j_1} \otimes e_{k_1}) \otimes \cdots \otimes (e_{j_{m'}} \otimes e_{k_{m'}})) E_{j_1,k_1} \otimes \cdots \otimes E_{j_{m'},k_{m'}} \\
&\quad = \frac{1}{m!} \sum_{1 \leqslant j_p, k_p \leqslant i} \operatorname{Perm}(A^{j_1,k_1,\ldots,j_{m'},k_{m'}}) E_{j_1,k_1} \otimes \cdots \otimes E_{j_{m'},k_{m'}}, \quad \text{by (7)} \\
&\quad = \frac{1}{m!} \sum_{\substack{d_1 + \cdots + d_i = m \\ d_j \geqslant 0}} \operatorname{Perm} A^{(d_1, \ldots, d_i)} \sum_{1 \leqslant j_p, k_p \leqslant i} E_{j_1,k_1} \otimes \cdots \otimes E_{j_{m'},k_{m'}},
\end{aligned}
$$

where the last summation runs over those ordered m-tuples $(j_1, k_1, \ldots, j_{m'}, k_{m'})$ such that the collection (without regard to the order)

$$\{j_1, k_1, \ldots, j_{m'}, k_{m'}\} = \{1^{d_1}, 2^{d_2}, \ldots, i^{d_i}\}.$$

This proves the lemma. □

On the vector space $M(m,i)$ of $m \times i$ matrices, $\mathrm{GL}(m) \times \mathrm{GL}(i)$ acts via

$$(g,h) \cdot X = gXh^{-1}, \quad \text{for } g \in \mathrm{GL}(m), h \in \mathrm{GL}(i), X \in M(m,i).$$

In particular, the permutation group $\mathfrak{S}_m$, thought of as the subgroup of permutation matrices in $\mathrm{GL}(m)$, acts on $M(m,i)$ and hence on any $\mathcal{P}^k(M(m,i))$. For any $\mathfrak{d} = (d_1, \ldots, d_i)$, $|\mathfrak{d}| := d_1 + \cdots + d_i = m$ and $d_j \geqslant 0$, set (for any A of the form (5))

$$a_{\mathfrak{d}}(A) = \operatorname{Perm} A^{(d_1, \ldots, d_i)}.$$

Then, clearly, for any $\mathfrak{d}$ as above,

$$a_{\mathfrak{d}} \in \mathcal{P}^m(M(m,i))^{-(\epsilon_1 + \cdots + \epsilon_m), \mathfrak{S}_m},$$

where the superscript '$-(\epsilon_1 + \cdots + \epsilon_m), \mathfrak{S}_m$' denotes the $\mathfrak{S}_m$-invariants of weight $-(\epsilon_1 + \cdots + \epsilon_m)$ with respect to the action of $\mathrm{GL}(m)$, i.e., the invertible diagonal matrices $(t_1, \ldots, t_m)$ act via $(t_1 \ldots t_m)^{-1}$. Recall from [GW09, Theorem 5.6.7] that, as $\mathrm{GL}(m) \times \mathrm{GL}(i)$-modules, for any $j \geqslant 0$,

$$\mathcal{P}^j(M(m,i)) \simeq \sum_{\substack{\mu : \mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_i \geqslant 0 \\ |\mu| = j}} V_m(\mu)^* \otimes E_i(\mu), \qquad (8)$$

https://doi.org/10.1112/S0010437X14007660 Published online by Cambridge University Press

where $|\mu| := \sum \mu_p$ and $V_m(\mu)$ (respectively $E_i(\mu)$) denotes the irreducible $\mathrm{GL}(m)$-module (respectively $\mathrm{GL}(i)$-module) corresponding to the partition μ.

Let $V_m := \mathbb{C}^m$ with the standard basis $\{v_1, \dots, v_m\}$. Define the elements

$$\overline{v}_o := v_1^* \otimes \cdots \otimes v_m^* \in \otimes^m(V_m^*); \quad v_o := \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \sigma \cdot \overline{v}_o \in S^m(V_m^*), \tag{9}$$

and

$$\mathfrak{v}_o := \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \sigma \cdot (v_1 \otimes \cdots \otimes v_m) \in S^m(V_m). \tag{10}$$

From (8), by a classical result due to Kostant [Kos76, Remark 4.1] (which asserts that for any irreducible $\mathrm{SL}(V_m)$-module $V_m(\lambda)$ corresponding to the partition λ with $|\lambda| = m$, its zero weight space is an irreducible representation W_λ of $\mathfrak{S}_m$ corresponding to the partition λ), we get

$$\begin{aligned} \mathcal{P}^m(M(m,i))^{-(\epsilon_1+\cdots+\epsilon_m),\mathfrak{S}_m} &\simeq (S^m(V_m)^{\epsilon_1+\cdots+\epsilon_m,\mathfrak{S}_m})^* \otimes S^m(E_i), \\ S^m(V_m)^{\epsilon_1+\cdots+\epsilon_m,\mathfrak{S}_m} &\simeq \mathbb{C}\mathfrak{v}_o. \end{aligned}$$

Thus,

$$\mathcal{P}^m(M(m,i))^{-(\epsilon_1+\cdots+\epsilon_m),\mathfrak{S}_m} \simeq S^m(E_i), \tag{11}$$

as $\mathrm{GL}(i)$-modules. It is easy to see that $\{a_{\mathfrak{d}}\}_{\mathfrak{d}=(d_1,\dots,d_i)}$ with $|\mathfrak{d}| = m$ are linearly independent (by taking, for example, $a_p^j = a_1^j$ for all $1 \leqslant p \leqslant m$). Hence, by the dimensional consideration, $\{a_{\mathfrak{d}}\}_{|\mathfrak{d}|=m}$ provides a basis of $S^m(E_i)$ under the above identification (11). The $\mathrm{GL}(i)$-module isomorphism (11) clearly induces a $\mathrm{GL}(i)$-algebra homomorphism:

$$\varphi : S^\bullet(S^m(E_i)) \to \mathcal{P}^{m\bullet}(M(m,i))^{-\bullet(\epsilon_1+\cdots+\epsilon_m),\mathfrak{S}_m} \simeq \oplus (V_m(\mu)^{\bullet(\epsilon_1+\cdots+\epsilon_m),\mathfrak{S}_m})^* \otimes E_i(\mu), \tag{12}$$

where the above sum runs over $\mu : \mu_1 \geqslant \cdots \geqslant \mu_i \geqslant 0$, $|\mu| = m\bullet$; the last isomorphism follows by the identity (8).

We now give an alternative description of the map

$$\varphi : S^\bullet(S^m(E_i)) \to \mathcal{P}^{m\bullet}(M(m,i))^{-\bullet(\epsilon_1+\cdots+\epsilon_m),\mathfrak{S}_m}.$$

First of all, as $\mathrm{GL}(m) \times \mathrm{GL}(i)$-modules,

$$\mathcal{P}^{mj}(M(m,i)) \simeq \mathcal{P}^{mj}(V_m \otimes E_i^*) \simeq S^{mj}(V_m^* \otimes E_i), \tag{13}$$

where the last identification is obtained from the isomorphism ξ^{-1} of Definition 2.5 followed by the identification (4). Define the map

$$\overline{\varphi} : \otimes^m E_i \to \otimes^m(V_m^* \otimes E_i) = (\otimes^m(V_m^*)) \otimes (\otimes^m E_i), \quad \overline{\varphi}(v) = v_o \otimes v.$$

Clearly, the map $\overline{\varphi}$ is a $\mathrm{GL}(E_i)$-module map. Moreover, it restricts to the map $\overline{\varphi}_1$:

$$\begin{tikzcd} S^m(E_i) \arrow[r, "\overline{\varphi}_1"] \arrow[d, hook] & S^m(V_m^* \otimes E_i) \arrow[d, hook] \\ \otimes^m(E_i) \arrow[r, "\overline{\varphi}"] & \otimes^m(V_m^* \otimes E_i), \end{tikzcd}$$

where the vertical maps are the canonical inclusions.

https://doi.org/10.1112/S0010437X14007660 Published online by Cambridge University Press

It is easy to see that $\overline{\varphi}_1$ is a $\mathrm{GL}(E_i)$-module map with image inside $S^m(V_m^* \otimes E_i)^{-(\epsilon_1+\cdots+\epsilon_m),\mathfrak{S}_m}$. Thus, from the irreducibility of $S^m E_i$ as a $\mathrm{GL}(E_i)$-module, applying Schur's lemma, we can choose the identification (11) so that $\overline{\varphi}_1$ coincides with the map $\varphi_{|S^m(E_i)}$ under the identification (13).

The map $\overline{\varphi}_1 : S^m(E_i) \to S^m(V_m^* \otimes E_i)$ extends to an algebra homomorphism (still denoted by)

$$\overline{\varphi}_1 : S^\bullet(S^m(E_i)) \to S^\bullet(S^m(V_m^* \otimes E_i)).$$

The isomorphism (13) for $j = 1$,

$$\mathcal{P}^m(M(m,i)) \simeq S^m(V_m^* \otimes E_i), \tag{14}$$

induces an algebra homomorphism $\beta : S^\bullet(S^m(V_m^* \otimes E_i)) \to \mathcal{P}^{m\bullet}(M(m,i))$. Let $\overline{\varphi}'_1 : S^\bullet(S^m(E_i)) \to \mathcal{P}^{m\bullet}(M(m,i))$ be the $\mathrm{GL}(E_i)$-algebra homomorphism which is the composite $\beta \circ \overline{\varphi}_1$.

Since $\overline{\varphi}'_1$ coincides with φ on $S^m(E_i)$, and both φ and $\overline{\varphi}'_1$ are algebra homomorphisms, we get that

$$\overline{\varphi}'_1 = \varphi. \tag{15}$$

Consider the function (for $i \leqslant m$)

$$\theta : M(m,i) \to \mathcal{P}^m(E_i) \simeq S^m(E_i)^*, \quad A \mapsto (\mathscr{D} \odot A)_{|E_i}.$$

Explicitly,

$$\theta(A)\biggl(\sum_{j=1}^{i} \lambda_j e_j\biggr) = \prod_{p=1}^{m}\biggl(\sum_{j=1}^{i} \lambda_j a_p^j\biggr) \quad \text{for } A = (a_p^j)_{1 \leqslant p \leqslant m, 1 \leqslant j \leqslant i}.$$

Clearly, θ is a polynomial function of homogeneous degree m. Moreover, it is $\mathrm{GL}(E_i)$-equivariant:

$$\begin{aligned}\theta(A \cdot g^{-1}) &= (\mathscr{D} \odot (Ag^{-1}))_{|E_i}\\ &= g \cdot ((\mathscr{D} \odot A)_{|E_i})\\ &= g \cdot \theta(A).\end{aligned}$$

Of course, θ gives rise to a $\mathrm{GL}(E_i)$-algebra homomorphism

$$\theta^* : S^\bullet(S^m(E_i)) \to \mathcal{P}^{m\bullet}(M(m,i)).$$

Lemma 3.2. $\mathrm{Im}(\theta^*_{|S^m(E_i)}) \subset \mathcal{P}^m(M(m,i))^{-(\epsilon_1+\cdots+\epsilon_m),\mathfrak{S}_m}$.

Proof. Let $\mathbf{t}$ be the diagonal matrix $(t_1, \dots, t_m) \in \mathrm{GL}(m)$. For any $f \in S^m(E_i)$, $A \in M(m,i)$,

$$\begin{aligned}(\theta^* f)(\mathbf{t}^{-1}A) &= f((\mathscr{D} \odot \mathbf{t}^{-1}A)_{|E_i})\\ &= t_1^{-1} \dots t_m^{-1} f((\mathscr{D} \odot A)_{|E_i}).\end{aligned}$$

This shows that

$$\mathrm{Im}(\theta^*_{|S^m(E_i)}) \subset \mathcal{P}^m(M(m,i))^{-(\epsilon_1+\cdots+\epsilon_m)}.$$

We next show that for any $f \in S^m(E_i)$, $\theta^* f$ is $\mathfrak{S}_m$-invariant. Take $\sigma \in \mathfrak{S}_m$ (considered as a permutation matrix), then

$$\begin{aligned}(\theta^* f)(\sigma A) &= f((\mathscr{D} \odot \sigma A)_{|E_i})\\ &= f((\mathscr{D} \odot A)_{|E_i}).\end{aligned}$$

This proves the lemma. □

https://doi.org/10.1112/S0010437X14007660 Published online by Cambridge University Press

Lemma 3.3. *The function θ^* coincides (up to a nonzero scalar multiple in any degree) with the function $\varphi : S^\bullet(S^m(E_i)) \to \mathcal{P}^{m\bullet}(M(m,i))^{-\bullet(\epsilon_1+\cdots+\epsilon_m),\mathfrak{S}_m}$ defined earlier (cf. (12)).*

Proof. First of all the function θ is clearly nonzero. Further, both θ^* and φ are $\mathrm{GL}(i)$-algebra homomorphisms. Moreover, $S^m(E_i)$ and $\mathcal{P}^m(M(m,i))^{-(\epsilon_1+\cdots+\epsilon_m),\mathfrak{S}_m}$ are both irreducible $\mathrm{GL}(i)$-modules (use the isomorphism (11) for the latter). Combining these, the lemma follows using Schur's lemma. □

Now, $S^i(S^m(E_i))$ has a unique (up to a scalar multiple) $\mathrm{SL}(E_i)$-invariant (by Proposition 2.3). We want to determine if $\varphi_{|[S^i(S^m(E_i))]^{\mathrm{SL}(E_i)}} \neq 0$.

By the definition, $S^j(S^m(E_i)) = [\otimes^j(\otimes^m E_i)]^{H_j}$, where $H_j \subset \mathfrak{S}_{mj}$ is the subgroup $\mathfrak{S}_m^{\times j} \rtimes \mathfrak{S}_j$ acting on the right as

$$\begin{aligned}((v_1^1 \otimes \cdots \otimes v_m^1) \otimes \cdots \otimes (v_1^j \otimes \cdots \otimes v_m^j)) \cdot ((\sigma_1, \ldots, \sigma_j), \mu) \\ = (v^{\mu(1)}_{\sigma_{\mu(1)}(1)} \otimes \cdots \otimes v^{\mu(1)}_{\sigma_{\mu(1)}(m)}) \otimes \cdots \otimes (v^{\mu(j)}_{\sigma_{\mu(j)}(1)} \otimes \cdots \otimes v^{\mu(j)}_{\sigma_{\mu(j)}(m)}),\end{aligned}$$

for $\sigma_p \in \mathfrak{S}_m$ and $\mu \in \mathfrak{S}_j$.

Proposition 3.4. *The map $\varphi_{|[S^i(S^m(E_i))]^{\mathrm{SL}(E_i)}} \neq 0$ if and only if the $\mathrm{GL}(V_m)$-submodule U_i generated by $v_o^{\otimes i} \in S^i(S^m(V_m^*)) = [\otimes^i(\otimes^m V_m^*)]^{H_i}$ intersects the isotypic component $\mathcal{I}_{m\delta_i}$ of $S^i(S^m(V_m^*))$ corresponding to the irreducible $\mathrm{GL}(V_m)$-module $V_m(m\delta_i)^*$ nontrivially.*

Proof. Take $0 \neq v \in [S^i(S^m(E_i))]^{\mathrm{SL}(E_i)} = [\otimes^i(\otimes^m E_i)]^{H_i \times \mathrm{SL}(E_i)}$.

Recall that for any partition $\lambda : \lambda_1 \geqslant \cdots \geqslant \lambda_d > 0$, d is called the *height* $ht\,\lambda$ of λ. We set $|\lambda| := \sum \lambda_j$. Let W_λ be the corresponding irreducible $\mathfrak{S}_{|\lambda|}$-module and let $E_i(\lambda)$ be the corresponding irreducible $\mathrm{GL}(i)$-module for any $i \geqslant d$. By the Schur–Weyl duality (cf. [GW09, Thoerem 9.1.2]),

$$S^i(S^m(E_i)) \simeq \bigoplus_{\substack{ht\,\lambda \leqslant i \\ |\lambda| = mi}} [W_\lambda]^{H_i} \otimes E_i(\lambda). \tag{16}$$

Thus, we get

$$[S^i(S^m(E_i))]^{\mathrm{SL}(E_i)} \simeq [W_{m\delta_i}]^{H_i} \otimes E_i(m\delta_i). \tag{17}$$

In particular, $[W_{m\delta_i}]^{H_i}$ is one-dimensional. Also, consider the analogous decomposition,

$$S^i(S^m(V_m^*)) \simeq \bigoplus_{\substack{ht\,\mu \leqslant m \\ |\mu| = mi}} [W_\mu]^{H_i} \otimes V_m(\mu)^*, \tag{18}$$

and write

$$v_o^{\otimes i} = \sum_\mu v_\mu, \tag{19}$$

under the above decomposition.

Let $M \subset [\otimes^i(\otimes^m E_i)]^{\mathrm{SL}(E_i)}$ be the $\mathfrak{S}_{mi}$-submodule generated by v and, for any μ with $ht\,\mu \leqslant m$ and $|\mu| = mi$, let $M_\mu \subset \otimes^i(\otimes^m V_m^*)$ be the $\mathfrak{S}_{mi}$-submodule generated by v_μ. Then, by the

https://doi.org/10.1112/S0010437X14007660 Published online by Cambridge University Press

Schur–Weyl duality, $M \simeq W_{m\delta_i}$ and M_μ (if nonzero) is isotypic of type W_μ. By the definition,

$$\begin{aligned}\bar{\varphi}_1'(v) &= \frac{1}{im!}\sum_{\sigma\in\mathfrak{S}_{mi}} \sigma\cdot(v_o^{\otimes i}\otimes v) \in S^{mi}(V_m^*\otimes E_i) \subset (\otimes^i(\otimes^m V_m^*))\otimes(\otimes^i(\otimes^m E_i))\\ &= \frac{1}{im!}\sum_{ht\,\mu\leqslant m, |\mu|=mi}\;\sum_{\sigma\in\mathfrak{S}_{mi}} \sigma\cdot(v_\mu\otimes v). \end{aligned} \tag{20}$$

Now, W_μ being self-dual, we get that for $\mu \neq m\delta_i$,

$$\sum_{\sigma\in\mathfrak{S}_{mi}} \sigma\cdot(v_\mu\otimes v) = 0. \tag{21}$$

Moreover, if $v_{m\delta_i}\neq 0$, we claim that

$$\sum_{\sigma\in\mathfrak{S}_{mi}} \sigma\cdot(v_{m\delta_i}\otimes v) \neq 0. \tag{22}$$

By projecting to an irreducible component, we can assume that $M_{m\delta_i}\simeq W_{m\delta_i}$. Now, take a $\mathfrak{S}_{mi}$-invariant nondegenerate bilinear form $\alpha : M_{m\delta_i}\otimes M \to \mathbb{C}$. Since α is $\mathfrak{S}_{mi}$-invariant, $\alpha_{|M_{m\delta_i}^{H_i}\otimes M^{H_i}}$ remains nondegenerate. Since both of v and $v_{m\delta_i}$ are H_i-invariant, and $[W_{m\delta_i}]^{H_i}$ is one-dimensional as observed above, we get $\alpha(v_{m\delta_i}\otimes v)\neq 0$. Thus,

$$\begin{aligned}\alpha\Bigg(\sum_{\sigma\in\mathfrak{S}_{mi}} \sigma\cdot(v_{m\delta_i}\otimes v)\Bigg) &= \sum_{\sigma\in\mathfrak{S}_{mi}} \alpha(v_{m\delta_i}\otimes v)\\ &\neq 0.\end{aligned}$$

This proves (22). Now, as it is easy to see, $v_{m\delta_i}\neq 0$ if and only if U_i intersects $\mathcal{I}_{m\delta_i}$ nontrivially. Hence the proposition is proved by combining the identities (20)–(22) since $\varphi = \bar{\varphi}_1'$ (by the identity (15)). $\square$

4. Latin squares

Definition 4.1. Let $1\leqslant i\leqslant m$. By a *Latin (i,m)-rectangle* A, one means an $i\times m$ matrix

$$A = (a_p^q)_{\substack{1\leqslant p\leqslant i\\ 1\leqslant q\leqslant m}}$$

such that each row $A_p := \{a_p^1, \ldots, a_p^m\}$ is a permutation σ_p of $[m]$ (i.e. $\sigma_p(q) = a_p^q$) and each column $A^q := \{a_1^q, \ldots, a_i^q\}$ consists of distinct numbers. We define the *sign* $\epsilon(A^q)$ of A^q as follows:

$$\epsilon(A^q) := \text{sign of} \prod_{1\leqslant p<p'\leqslant i}(a_{p'}^q - a_p^q).$$

We call a Latin rectangle A *column-even* if $\epsilon_c(A) := \prod_{q=1}^m \epsilon(A^q)$ is $+1$ and *column-odd* otherwise.

Let $\mathcal{A}^q$ denote the set A^q without regard to order. Then we call the m-tuple $\mathcal{A} = (\mathcal{A}^1,\ldots,\mathcal{A}^m)$ the *pattern* of A. Let $L_{\mathcal{A}}$ denote the set of Latin (i,m)-rectangles A with pattern $\mathcal{A}$. Let $S(i,m)$ be the set of all patterns of size (i,m), where by a *pattern $\mathcal{A}$ of size* (i,m) (or an (i,m)-*pattern*) we mean a m-tuple $\mathcal{A} = (\mathcal{A}^1,\ldots,\mathcal{A}^m)$ of subsets of $[m]$, each of cardinality exactly i such that any integer $q\in[m]$ occurs in exactly i sets $\mathcal{A}^\bullet$.

For $\mathcal{A}\in S(i,m)$, let $L_{\mathcal{A}}^+$ (respectively $L_{\mathcal{A}}^-$) denote the subset of $L_{\mathcal{A}}$ consisting of column even (respectively odd) Latin rectangles.

https://doi.org/10.1112/S0010437X14007660 Published online by Cambridge University Press

We have the following simple lemma.

Lemma 4.2. *Fix any $1 \leqslant i \leqslant m$. Assume that there exists a pattern $\mathcal{A}$ of size (i, m) such that*

$$\sharp L^+_{\mathcal{A}} \neq \sharp L^-_{\mathcal{A}}.$$

Then, for any $1 \leqslant i' \leqslant i$, there exists a pattern $\mathcal{B}$ of size (i', m) such that

$$\sharp L^+_{\mathcal{B}} \neq \sharp L^-_{\mathcal{B}}.$$

Proof. It suffices to prove the lemma for $i' = i - 1$. Define the map $\varphi : L_{\mathcal{A}} \to \bigsqcup_{\mathcal{B} \in S(i-1,m)} L_{\mathcal{B}}$ by removing the last row of any Latin rectangle A in $L_{\mathcal{A}}$. The map φ is clearly injective. Moreover, the image of φ consists exactly of the union $\bigsqcup_{\mathcal{B} \in S_{\mathcal{A}}(i-1,m)} L_{\mathcal{B}}$, where

$$S_{\mathcal{A}}(i-1, m) := \left\{ \begin{matrix} (i-1, m)\text{-patterns } \mathcal{B} \text{ such that there exists } A \in L_{\mathcal{A}} \\ \text{with its top } (i-1) \text{ rows having pattern } \mathcal{B} \end{matrix} \right\}.$$

By the definition of $L^{\pm}_{\mathcal{A}}$, it is clear that for any $\mathcal{B} \in S_{\mathcal{A}}(i-1, m)$, there exists a sign $\epsilon(\mathcal{B}) \in \{\pm 1\}$ such that

$$\varphi^{-1}(L^{\pm}_{\mathcal{B}}) \subset L^{\pm\epsilon(\mathcal{B})}_{\mathcal{A}}. \tag{23}$$

Assume, if possible, that the lemma is false, that is,

$$\sharp L^+_{\mathcal{B}} = \sharp L^-_{\mathcal{B}} \quad \text{for every } (i-1, m)\text{-pattern } \mathcal{B}; \tag{24}$$

in particular, for any $\mathcal{B} \in S_{\mathcal{A}}(i-1, m)$.

Combining (23) and (24), we get (since φ is a bijection) $\sharp L^+_{\mathcal{A}} = \sharp L^-_{\mathcal{A}}$. This contradicts the assumption and hence proves the lemma. $\square$

We recall the following celebrated *column Latin (m, m)-square conjecture* due to Huang and Rota [HR94, Conjecture 3].

Conjecture 4.3. For any positive even integer m,

$$\sharp\mathrm{CELS}(m) \neq \sharp\mathrm{COLS}(m),$$

where $\mathrm{CELS}(m)$ (respectively $\mathrm{COLS}(m)$) denotes the set of column-even (respectively column-odd) Latin (m, m)-squares. (Observe that for Latin (m, m)-squares, there is a unique pattern: $([m], [m], \ldots, [m])$.)

Combining the above conjecture with Lemma 4.2, we get the following proposition.

Proposition 4.4. *Let m be a positive even integer. Assume that Conjecture 4.3 is true for m. Then, for any $1 \leqslant i \leqslant m$, there exists a pattern $\mathcal{A}$ of size (i, m) such that*

$$\sharp L^+_{\mathcal{A}} \neq \sharp L^-_{\mathcal{A}}.$$

Remark 4.5. As proved by Huang and Rota [HR94, § 3], their column Latin (m, m)-square conjecture is equivalent to the (full) Latin (m, m)-square conjecture given by Alon and Tarsi [AT92]. Now the (full) Latin (m, m)-square conjecture is known to be valid in the following cases:

(a) $m = p - 1$, for any odd prime p, due to Glynn [Gly10, Theorem 3.2];

(b) $m = p + 1$, for any odd prime p, due to Drisko [Dri97, Theorem 9].

We have the following very simple lemma.

LEMMA 4.6. *Let $\mathcal{A}$ (respectively $\mathcal{B}$) be a pattern of type (i,m) (respectively (i,m')) such that*

$$\sharp L^+_{\mathcal{A}} \neq \sharp L^-_{\mathcal{A}} \quad \text{and} \quad \sharp L^+_{\mathcal{B}} \neq \sharp L^-_{\mathcal{B}}.$$

Then,

$$\sharp L^+_{(\mathcal{A},\mathcal{B})} \neq \sharp L^-_{(\mathcal{A},\mathcal{B})},$$

where each entry in $\mathcal{B}$ is shifted by m.

Proof. Clearly, under the concatenation,

$$L_{\mathcal{A}} \times L_{\mathcal{B}} \xrightarrow{\sim} L_{(\mathcal{A},\mathcal{B})}.$$

Moreover, under the above bijection,

$$L^{\epsilon_1}_{\mathcal{A}} \times L^{\epsilon_2}_{\mathcal{B}} \to L^{\epsilon_1 \cdot \epsilon_2}_{(\mathcal{A},\mathcal{B})}$$

where $\epsilon_i = \pm 1$. From this the lemma follows. $\square$

5. Existence of a certain isotypic component in the module generated by $v_o^{\otimes i}$

Recall from § 3 that $V_m = \mathbb{C}^m$ has standard basis $\{v_1, \dots, v_m\}$. Recall from the identity (10) that

$$\mathfrak{v}_o := \frac{1}{m!} \sum_{\sigma_1 \in \mathfrak{S}_m} v_{\sigma_1(1)} \otimes \cdots \otimes v_{\sigma_1(m)} \in S^m(V_m),$$

so that, as elements of $S^i(S^m(V_m))$,

$$\mathfrak{v}_o^{\otimes i} = \frac{1}{(m!)^i} \sum_{\sigma = (\sigma_1, \dots, \sigma_i) \in \mathfrak{S}_m^i} \big(v_{\sigma_1(1)} \otimes \cdots \otimes v_{\sigma_1(m)}\big) \otimes \cdots \otimes \big(v_{\sigma_i(1)} \otimes \cdots \otimes v_{\sigma_i(m)}\big).$$

Let λ be a partition of k into at most m parts and let A be a tableau of shape λ. As in [GW09, Proposition 9.3.7], define

$$\begin{aligned} \text{Row}\, A &= \{\sigma \in \mathfrak{S}_k : \sigma \ \text{ preserves the rows of } A\}, \\ \text{Col}\, A &= \{\mu \in \mathfrak{S}_k : \mu \ \text{ preserves the columns of } A\}, \\ S(A) &= \Big(\sum_{\mu \in \text{Col}\, A} \epsilon(\mu)\mu\Big) \cdot \sum_{\sigma \in \text{Row}\, A} \sigma, \\ v_A &= v_{i_1} \otimes \cdots \otimes v_{i_k} \in \otimes^k(V_m), \end{aligned} \tag{25}$$

where $i_j = r$ if j occurs in the rth row of A. (Here $\epsilon(\mu)$ denotes the sign of μ.)

Example 5.1.

$$A = \begin{array}{|c|c|c|} \hline 1 & 5 & 9 \\ \hline 2 & 6 & 10 \\ \hline 3 & 7 \\ \cline{1-2} 4 & 8 \\ \cline{1-2} \end{array}$$

$$v_A = v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_1 \otimes v_2.$$

https://doi.org/10.1112/S0010437X14007660 Published online by Cambridge University Press

Consider the tableau $B_o = B_o(i,m)$ of shape $m \geqslant m \geqslant \cdots \geqslant m$ (i factors):

1	2	3	$\dots$	m
$m+1$	$m+2$	$m+3$	$\dots$	$2m$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(i-1)m+1$	$(i-1)m+2$	$(i-1)m+3$	$\dots$	im

PROPOSITION 5.2. *For any $1 \leqslant i \leqslant m$ and even m,*

$$\langle v_o^{\otimes i}, S(B_o)\cdot \mathfrak{v}_o^{\otimes i}\rangle = \left(\frac{1}{m!}\right)^i \sum_{\mathcal{A}\in S(i,m)} (\sharp L_{\mathcal{A}}^{+} - \sharp L_{\mathcal{A}}^{-})^2,$$

where $\langle\cdot,\cdot\rangle$ is the standard pairing between $\otimes^i(\otimes^m(V_m^*))$, $\otimes^i(\otimes^m V_m)$ and $v_o \in S^m(V_m^*) \subset \otimes^m(V_m^*)$ is defined by the identity (9).

Proof. First of all, by the definition of $S(B_o)$,

$$\begin{aligned} S(B_o)\cdot \mathfrak{v}_o^{\otimes i} &= (m!)^i \sum_{\mu\in \mathrm{Col}\ B_o} \epsilon(\mu)\,\mu\cdot \mathfrak{v}_o^{\otimes i} \\ &= \sum_{\substack{\sigma=(\sigma_1,\dots,\sigma_i)\in\mathfrak{S}_m^i \\ \mu=(\mu_1,\dots,\mu_m)\in\mathfrak{S}_i^m}} \epsilon(\mu)(v_{\sigma_{\mu_1(1)}(1)}\otimes\cdots\otimes v_{\sigma_{\mu_m(1)}(m)}) \\ &\quad \otimes (v_{\sigma_{\mu_1(2)}(1)}\otimes\cdots\otimes v_{\sigma_{\mu_m(2)}(m)})\otimes\cdots\otimes(v_{\sigma_{\mu_1(i)}(1)}\otimes\cdots\otimes v_{\sigma_{\mu_m(i)}(m)}), \end{aligned} \tag{26}$$

where $\epsilon(\mu) := \epsilon(\mu_1)\cdots\epsilon(\mu_m)$ and μ_j is embedded in $\mathfrak{S}_{mi}$ as permuting $\{j,\ j+m,\dots,j+(i-1)m\}$ only.

For any $i\times m$ matrix

$$A = (a_{p,q})_{\substack{1\leqslant p\leqslant i\\ 1\leqslant q\leqslant m}}$$

of integers $a_{p,q}\in[m]$, let

$$V_A := (v_{a_{1,1}}\otimes v_{a_{1,2}}\otimes\cdots\otimes v_{a_{1,m}})\otimes\cdots\otimes(v_{a_{i,1}}\otimes v_{a_{i,2}}\otimes\cdots\otimes v_{a_{i,m}}) \in \otimes^i(\otimes^m V_m).$$

Thus, we can rewrite the identity (26) as

$$S(B_o)\cdot\mathfrak{v}_o^{\otimes i} = \sum_{\substack{\sigma\in\mathfrak{S}_m^i\\ \mu\in\mathfrak{S}_i^m}} \epsilon(\mu)V_{A(\sigma,\mu)},$$

where $A(\sigma,\mu)$ is the $i\times m$ matrix

$$A(\sigma,\mu) = \begin{pmatrix} \sigma_{\mu_1(1)}(1) & \dots & \sigma_{\mu_m(1)}(m) \\ \vdots & & \vdots \\ \sigma_{\mu_1(i)}(1) & \dots & \sigma_{\mu_m(i)}(m)\end{pmatrix}.$$

We claim that

$$\langle v_o^{\otimes i}, S(B_o)\cdot\mathfrak{v}_o^{\otimes i}\rangle = \left\langle v_o^{\otimes i}, \sum_{(\sigma,\mu)\in R} \epsilon(\mu)V_{A(\sigma,\mu)}\right\rangle, \tag{27}$$

https://doi.org/10.1112/S0010437X14007660 Published online by Cambridge University Press

where the last summation runs over R consisting of those $\sigma = (\sigma_1, \ldots, \sigma_i) \in \mathfrak{S}_m^i$ and $\mu = (\mu_1, \ldots, \mu_m) \in \mathfrak{S}_i^m$ such that $A(\sigma, \mu)$ is a Latin (i, m)-rectangle.

Since v_o is, by definition, $(1/m!) \sum_{\sigma_1 \in \mathfrak{S}_m} v^*_{\sigma_1(1)} \otimes \cdots \otimes v^*_{\sigma_1(m)}$, unless each row of $A(\sigma, \mu)$ is a permutation of $[m]$, we have $\langle v_o^{\otimes i}, V_{A(\sigma,\mu)} \rangle = 0$. Further, assume that the entries in some column of $A(\sigma, \mu)$ are nondistinct, say

$$\sigma_{\mu_q(p)}(q) = \sigma_{\mu_q(p')}(q) \quad \text{for some } 1 \leqslant q \leqslant m \text{ and some } 1 \leqslant p \neq p' \leqslant i.$$

Let $\tau \in \mathfrak{S}_i$ be the transposition (p, p'). Then

$$V_{A(\sigma,\mu)} = V_{A(\sigma,\mu')},$$

where $\mu' := (\mu_1, \ldots, \mu_q \circ \tau, \ldots, \mu_m)$.

Hence,

$$\epsilon(\mu) V_{A(\sigma,\mu)} + \epsilon(\mu') V_{A(\sigma,\mu')} = 0.$$

This proves the identity (27).

Let $R' \subset \mathfrak{S}_m^i$ be the subset consisting of $\sigma = (\sigma_1, \ldots, \sigma_i)$ such that

$$A(\sigma) = \begin{pmatrix} \sigma_1(1) & \ldots & \sigma_1(m) \\ \vdots & & \vdots \\ \sigma_i(1) & \ldots & \sigma_i(m) \end{pmatrix}$$

is a Latin (i, m)-rectangle. For any $\sigma \in R'$, let $\widehat{\sigma}$ be the pattern $(\widehat{\sigma}^1, \ldots, \widehat{\sigma}^m)$, where

$$\widehat{\sigma}^q := \{\sigma_1(q), \ldots, \sigma_i(q)\}.$$

Define an equivalence relation on R' by $\sigma \sim \sigma'$ if the patterns $\widehat{\sigma} = \widehat{\sigma'}$. Denote the equivalence class containing σ by $[\sigma]$. Then the sum $\sum_{(\sigma,\mu) \in R} \epsilon(\mu) V_{A(\sigma,\mu)}$ can clearly be written as

$$\begin{aligned} \sum_{\sigma \in R'} \sum_{\substack{\mu \in \mathfrak{S}_i^m: \\ (\sigma,\mu) \in R}} \epsilon(\mu) V_{A(\sigma,\mu)} &= \sum_{\sigma \in R'} \epsilon_c(A(\sigma)) \sum_{B \in L_{\widehat{\sigma}}} \epsilon_c(B) V_B \\ &= \sum_{[\sigma] \in R'/\sim} \sum_{A \in L_{\widehat{\sigma}}} \epsilon_c(A) \sum_{B \in L_{\widehat{\sigma}}} \epsilon_c(B) V_B \\ &= \sum_{\mathcal{A} \in S(i,m)} \sum_{A \in L_{\mathcal{A}}} \epsilon_c(A) \sum_{B \in L_{\mathcal{A}}} \epsilon_c(B) V_B. \end{aligned}$$

Thus, by the identity (27),

$$\begin{aligned} \langle v_o^{\otimes i}, S(B_o) \cdot \mathfrak{v}_o^{\otimes i} \rangle &= \left(\frac{1}{m!}\right)^i \sum_{\mathcal{A} \in S(i,m)} \sum_{A \in L_{\mathcal{A}}} \epsilon_c(A) \sum_{B \in L_{\mathcal{A}}} \epsilon_c(B) \\ &= \left(\frac{1}{m!}\right)^i \sum_{\mathcal{A} \in S(i,m)} \left(\sum_{A \in L_{\mathcal{A}}} \epsilon_c(A) \right)^2 \\ &= \left(\frac{1}{m!}\right)^i \sum_{\mathcal{A} \in S(i,m)} (\sharp L^+_{\mathcal{A}} - \sharp L^-_{\mathcal{A}})^2. \end{aligned}$$

This proves the proposition. □

As an immediate consequence of the above proposition, we get the following corollary.

COROLLARY 5.3. *$\langle v_o^{\otimes i}, S(B_o) \cdot \mathfrak{v}_o^{\otimes i} \rangle \neq 0$ if and only if for some pattern $\mathcal{A} \in S(i,m)$, $\sharp L^+_{\mathcal{A}} \neq \sharp L^-_{\mathcal{A}}$.*

For any partition λ of k into at most m parts, let G^λ be the highest weight space in $\otimes^k(V_m)$ for $\mathrm{GL}(V_m)$ corresponding to the highest weight λ. Then we have the following lemma (cf. [GW09, Lemma 9.3.2]).

LEMMA 5.4. *Let A be a tableau of shape λ. Then, $S(A) \cdot v_A$ is a nonzero element of G^λ. Thus,*

$$G^\lambda = \sum_{\tau \in \mathfrak{S}_k} \mathbb{C}\tau \cdot (S(A) \cdot v_A).$$

We specialize the above lemma to $k = m^2$ and λ the partition:

$$m\delta_m : \quad \underbrace{m \geqslant m \geqslant \cdots \geqslant m}_{m \text{ factors}}.$$

In this case $V_m(m\delta_m)$ is a one-dimensional representation of $\mathrm{GL}(V_m)$.

Consider the tableau $B_o = B_o(m,m)$ (with $i = m$) given just above Proposition 5.2. Then

$$S(B_o) \cdot v_{B_o} = (m!)^m \sum_{\mu = (\mu_1, \ldots, \mu_m) \in \mathfrak{S}^m_{i=m}} \epsilon(\mu)(v_{\mu_1(1)} \otimes v_{\mu_2(1)} \otimes \cdots \otimes v_{\mu_m(1)}) \otimes (v_{\mu_1(2)} \otimes v_{\mu_2(2)} \otimes \cdots \otimes v_{\mu_m(2)}) \otimes \cdots \otimes (v_{\mu_1(m)} \otimes v_{\mu_2(m)} \otimes \cdots \otimes v_{\mu_m(m)}). \quad (28)$$

By the above lemma, the isotypic component G^λ of $\otimes^m(\otimes^m V_m)$ for the partition $\lambda = m\delta_m$ is the span of

$$\{\tau \cdot (S(B_o) \cdot v_{B_o}) : \tau \in \mathfrak{S}_{m^2}\}.$$

I thank J. Landsberg for the part (b) of the following proposition.

PROPOSITION 5.5. (a) *For CELS and COLS as defined in Conjecture 4.3,*

$$\langle v_o^{\otimes m}, S(B_o) \cdot v_{B_o} \rangle = \sharp\,\mathrm{CELS}(m) - \sharp\,\mathrm{COLS}(m).$$

(b) *For any $\tau \in \mathfrak{S}_{m^2}$,*

$$\langle v_o^{\otimes m}, \tau \cdot (S(B_o) \cdot v_{B_o}) \rangle = \alpha \langle v_o^{\otimes m}, S(B_o) \cdot v_{B_o} \rangle \quad \text{for some } \alpha \in \{0, \pm 1\}.$$

Proof. By the identity (28),

$$\langle v_o^{\otimes m}, S(B_o) \cdot v_{B_o} \rangle = \sum \epsilon(\mu),$$

where the summation runs over those $\mu = (\mu_1, \ldots, \mu_m) \in \mathfrak{S}^m_m$ such that

$$B(\mu) := \begin{pmatrix} \mu_1(1) & \mu_2(1) & \cdots & \mu_m(1) \\ \mu_1(2) & \mu_2(2) & \cdots & \mu_m(2) \\ \vdots & \vdots & & \vdots \\ \mu_1(m) & \mu_2(m) & \cdots & \mu_m(m) \end{pmatrix}$$

is a Latin square (i.e., each row and each column of the above matrix is a permutation of $[m]$), and

$$\epsilon(\mu) := \epsilon(\mu_1) \cdots \epsilon(\mu_m).$$

From this, part (a) follows.

(b) By the expression of $S(B_o)\cdot v_{B_o}$ as in the identity (28), clearly

$$\tau\cdot(S(B_o)\cdot v_{B_o}) = (m!)^m \sum_{\mu=(\mu_1,\ldots,\mu_m)\in\mathfrak{S}_m^m} \epsilon(\mu)(v_{\mu_{i_1^1}(j_1^1)}\otimes\cdots\otimes v_{\mu_{i_m^1}(j_m^1)}) \otimes\cdots\otimes(v_{\mu_{i_1^m}(j_1^m)}\otimes\cdots\otimes v_{\mu_{i_m^m}(j_m^m)}),$$

for some fixed i_q^p and $j_q^p\in[m]$ (depending only upon τ).

We claim that if for any $1\leqslant p\leqslant m$, $i_a^p = i_b^p =: q$ for some $a\neq b$, then $D_\tau = 0$, where $D_\tau := \langle v_o^{\otimes m}, \tau\cdot(S(B_o)\cdot v_{B_o})\rangle$. Observe that $j_a^p\neq j_b^p$ since the element

$$(v_{\mu_{i_1^1}(j_1^1)}\otimes\cdots\otimes v_{\mu_{i_m^1}(j_m^1)})\otimes\cdots\otimes(v_{\mu_{i_1^m}(j_1^m)}\otimes\cdots\otimes v_{\mu_{i_m^m}(j_m^m)})$$

is a permutation of

$$(v_{\mu_1(1)}\otimes v_{\mu_2(1)}\otimes\cdots\otimes v_{\mu_m(1)})\otimes(v_{\mu_1(2)}\otimes v_{\mu_2(2)}\otimes\cdots\otimes v_{\mu_m(2)})\otimes\cdots\otimes(v_{\mu_1(m)}\otimes v_{\mu_2(m)}\otimes\cdots\otimes v_{\mu_m(m)}).$$

Consider the element $\theta = (j_a^p, j_b^p)\in\mathfrak{S}_m$. Then, replacing μ_q by $\mu_q\theta$ in the above expression for $\tau\cdot(S(B_o)\cdot v_{B_o})$, we clearly get

$$D_\tau = \epsilon(\theta)D_\tau.$$

Thus, $D_\tau = 0$.

So let us assume that for any $1\leqslant p\leqslant m$, $i_a^p\neq i_b^p$ for $a\neq b$. Since $v_o^{\otimes m}$ is H_m-invariant (where H_m is defined above Proposition 3.4), to calculate D_τ, we can assume that

$$\tau\cdot(S(B_o)\cdot v_{B_o}) = (m!)^m\sum_{\mu\in\mathfrak{S}_m^m}\epsilon(\mu)(v_{\mu_1(j_1^1)}\otimes\cdots\otimes v_{\mu_m(j_m^1)})\otimes\cdots\otimes(v_{\mu_1(j_1^m)}\otimes\cdots\otimes v_{\mu_m(j_m^m)}),$$

where, for any $1\leqslant q\leqslant m$, $\{j_q^1,\ldots,j_q^m\}$ is a permutation σ_q of $[m]$. Now, replacing μ_q by $\mu_q\circ\sigma_q$, we get (setting $\sigma = (\sigma_1,\ldots,\sigma_m)$)

$$\begin{aligned}\tau\cdot(S(B_o)\cdot v_{B_o}) &= \epsilon(\sigma)(m!)^m\sum_{\mu\in\mathfrak{S}_m^m}\epsilon(\mu)(v_{\mu_1(1)}\otimes\cdots\otimes v_{\mu_m(1)})\otimes\cdots\otimes(v_{\mu_1(m)}\otimes\cdots\otimes v_{\mu_m(m)})\\ &= \epsilon(\sigma)\,S(B_o)\cdot v_{B_o}.\end{aligned}$$

This proves the proposition. □

Theorem 5.6. *Let m be an even positive integer and let $1\leqslant i\leqslant m$. If there exists a pattern $\mathcal{B}$ of size (i,m) such that*

$$\sharp L_{\mathcal{B}}^+ \neq \sharp L_{\mathcal{B}}^-, \tag{29}$$

then the $\mathrm{GL}(V_m)$-submodule U_i generated by $v_o^{\otimes i}\in S^i(S^m(V_m^*)) = [\otimes^i(\otimes^m V_m^*)]^{H_i}$ intersects the isotypic component $\mathcal{I}_{m\delta_i}$ of $S^i(S^m(V_m^*))$ corresponding to the irreducible $\mathrm{GL}(V_m)$-module $V_m(m\delta_i)^*$ nontrivially, where H_i is defined over Proposition 3.4.

In particular, if Conjecture 4.3 is true for m, then $U_i\cap\mathcal{I}_{m\delta_i}\neq(0)$, for all $1\leqslant i\leqslant m$.

For $i=m$, $U_m\cap\mathcal{I}_{m\delta_m}\neq(0)$ if and only if Conjecture 4.3 is true for m.

Proof. Let $y_o = v_{m\delta_i}$ be the component of $v_o^{\otimes i}$ in $\mathcal{I}_{m\delta_i}$ (cf. the identity (19)). Then, as observed in the proof of Proposition 3.4, $U_i\cap\mathcal{I}_{m\delta_i}\neq 0$ if and only if $y_o\neq 0$. By [GW09, Theorem 9.3.10],

https://doi.org/10.1112/S0010437X14007660 Published online by Cambridge University Press

$S(B_o)\cdot\mathfrak{v}_o^{\otimes i}$ belongs to an irreducible $\mathrm{GL}(V_m)$-submodule of $\otimes^i(\otimes^m V_m)$ of highest weight $m\delta_i$. Thus,

$$\begin{aligned}\langle y_o,\, S(B_o)\cdot\mathfrak{v}_o^{\otimes i}\rangle &= \langle v_o^{\otimes i},\, S(B_o)\cdot\mathfrak{v}_o^{\otimes i}\rangle\\ &= \left(\frac{1}{m!}\right)^i \sum_{\mathcal{A}\in S(i,m)} (\sharp L^+_{\mathcal{A}} - \sharp L^-_{\mathcal{A}})^2, \quad \text{by Proposition 5.2}\\ &\neq 0 \quad \text{by the assumption of the theorem.}\end{aligned}$$

Thus, $y_o\neq 0$, proving the first part of the theorem.

The second 'In particular' part of the theorem, of course, follows from Proposition 4.4.

For the last part, by Lemma 5.4, $S(B_o)\cdot v_{B_o}$ is a nonzero highest weight vector of $\otimes^m(\otimes^m V_m)$ with highest weight $m\delta_m$ and the isotypic component of $\otimes^m(\otimes^m V_m)$ corresponding to the highest weight $m\delta_m$ is given by $\sum_{\tau\in\mathfrak{S}_{m^2}}\mathbb{C}\,\tau\cdot(S(B_o)\cdot v_{B_o})$ (since $V_m(m\delta_m)$ is a one-dimensional representation).

Thus, $y_o\in\mathcal{I}_{m\delta_m}$ is nonzero if and only if

$$\langle v_o^{\otimes m},\, x\rangle = \langle y_o,\, x\rangle \neq 0,$$

for some $x\in\sum_{\tau\in\mathfrak{S}_{m^2}}\mathbb{C}\,\tau\cdot(S(B_o)\cdot v_{B_o})$. The above condition is equivalent to the nonvanishing of $\langle v_o^{\otimes m}, S(B_o)\cdot v_{B_o}\rangle$ by Proposition 5.5(b); which, in turn, is equivalent to the validity of Conjecture 4.3 by Proposition 5.5(a). This proves the theorem. □

Remark 5.7. It is quite possible that for any $1\leqslant i\leqslant m$, $U_i\cap\mathcal{I}_{m\delta_i}\neq 0$ if and only if (29) is satisfied for some pattern $\mathcal{B}$ of size (i,m).

6. Statement of the main theorem and its consequences

Let $\mathfrak{v}$ be a complex vector space of dimension m and let $E:=\mathfrak{v}\otimes\mathfrak{v}^* = \operatorname{End}\mathfrak{v}$, $Q:=\mathcal{P}^m(E)\simeq S^m(E)^*$ (under the isomorphism ξ of Definition 2.5). Consider $\mathscr{D}\in Q$, where $\mathscr{D}$ is the function taking determinant of any $A\in E=\operatorname{End}\mathfrak{v}$. The group $G=\mathrm{GL}(E)$ acts canonically on Q. Let $\mathcal{X}$ be the G-orbit closure of $\mathscr{D}$ inside Q.

Fix a basis $\{v_1,\ldots,v_m\}$ of $\mathfrak{v}$ and let $\{v_1^*,\ldots,v_m^*\}$ be the dual basis of $\mathfrak{v}^*$. Take the basis $\{v_i\otimes v_j^*\}_{1\leqslant i,j\leqslant m}$ of E and order the basis elements as $\{e_1,e_2,\ldots,e_{m^2}\}$ satisfying

$$e_1 = v_1\otimes v_1^*,\quad e_2 = v_2\otimes v_2^*,\ldots, e_m = v_m\otimes v_m^*.$$

Assume that m is even. Recall from Corollary 2.4 that for any $1\leqslant i\leqslant m^2$, the irreducible $\mathrm{GL}(E)$-module $V_E(m\delta_i)$ occurs in $S^i(S^m(E))$ with multiplicity one (and $V_E(m\delta_i)$ does not occur in any $S^j(S^m(E))$, for $j\neq i$). Let $P_i=\gamma_{m,i}\in S^i(S^m(E))$ be the highest weight vector of $V_E(m\delta_i)\subset S^i(S^m(E))$ (which is unique up to a nonzero scalar multiple) with respect to the standard Borel subgroup $B=B_E$ of G consisting of upper triangular invertible matrices, where $\mathrm{GL}(E)$ is identified with $\mathrm{GL}(m^2)$ with respect to the basis $\{e_1,\ldots,e_{m^2}\}$ of E given above. By Lemma 2.1, in fact $P_i\in[S^i(S^m(E_i))]^{\mathrm{SL}(E_i)}$, where E_i is the subspace of E spanned by $\{e_1,\ldots,e_i\}$.

Recall an explicit construction of P_i from Lemma 2.7. Since $P_i\in S^i(S^m(E))$, we can think of P_i as a homogeneous polynomial of degree i on the vector space $Q=S^m(E)^*$.

The following is our main result.

Theorem 6.1. *Assume, as above, that m is even. Assume further that Conjecture 4.3 is true for m. Then, with the above notation, for any $1 \leqslant i \leqslant m$, the polynomial P_i does* not *vanish identically on $\mathcal{X}$.*

In particular, the irreducible $\mathrm{GL}(E)$-module $V_E(m\delta_i)$ occurs with multiplicity one in the affine coordinate ring $\mathbb{C}[\mathcal{X}]$. Moreover, by Corollary 2.4, $V_E(d\delta_i)$, for any $d < m$ and any $1 \leqslant i \leqslant m^2$, does not occur in $S^\bullet(S^m(E))$; in particular, it does not occur in $\mathbb{C}[\mathcal{X}]$.

Proof. Recall the definition of the right action of the semigroup End E on $Q = \mathcal{P}^m(E)$ from the identity (6). Consider the map

$$\hat{\theta} : M(m,i) \to Q, \quad A \mapsto \mathscr{D} \odot \hat{A},$$

where $\hat{A} \in \operatorname{End} E$ is defined by

$$\hat{A}e_j = \sum_{p=1}^{m} a_p^j e_p \quad \text{for } 1 \leqslant j \leqslant i, \quad \text{and} \quad \hat{A}e_j = 0 \quad \text{for } j > i,$$

where $A = (a_p^j)_{1\leqslant p\leqslant m, 1\leqslant j\leqslant i}$. Clearly,

$$\operatorname{Im}\hat{\theta} \subset \mathcal{X}.$$

To prove that $P_i \in \mathcal{P}^i(Q) \simeq S^i(S^m(E))$ restricts to a nonzero function on $\mathcal{X}$, it suffices to show that P_i restricts to a nonzero function on $M(m,i)$ via the morphism $\hat{\theta}$. Since

$$P_i \in S^i(S^m(E_i)) \simeq S^i(S^m(E_i^*)^*) \simeq \mathcal{P}^i(\mathcal{P}^m(E_i)),$$

P_i is the pullback of a function $\bar{P}_i \in \mathcal{P}^i(\mathcal{P}^m(E_i))$ via the restriction map $r : \mathcal{P}^m(E) \to \mathcal{P}^m(E_i)$. Thus, it suffices to prove that $\bar{P}_i$ restricts to a nonzero function on $M(m,i)$ via $\theta : M(m, i) \to \mathcal{P}^m(E_i) \simeq S^m(E_i)^*$ defined as the composite $\theta = r \circ \hat{\theta}$. (Observe that this θ coincides with the map θ defined just before Lemma 3.2.) Now, by Lemma 3.3, the induced map

$$\theta^* : S^\bullet(S^m(E_i)) \to \mathcal{P}^{m\bullet}(M(m,i))$$

coincides with the map φ (up to a nonzero scalar multiple in any degree). Since $\bar{P}_i$ is the unique (up to a nonzero multiple) nonzero element of $[S^i(S^m(E_i))]^{\mathrm{SL}(E_i)}$ (by Proposition 2.3), it suffices to show that $\varphi_{|[S^i(S^m(E_i))]^{\mathrm{SL}(E_i)}} \neq 0$. This follows from Proposition 3.4 and Theorem 5.6. The 'In particular' part follows from Corollary 2.4, and hence the theorem is proved. □

Corollary 6.2. *With the notation and assumptions as in the last theorem (in particular, assuming the validity of Conjecture 4.3 for m), for any dominant integral weight λ for $\mathrm{GL}(E)$ of the form $\lambda = \sum_{i=1}^m n_i\delta_i$, $n_i \in \mathbb{Z}_+$, the irreducible $\mathrm{GL}(E)$-module $V_E(m\lambda)$ occurs in $\mathbb{C}[\mathcal{X}]$ with nonzero multiplicity.*

Proof. First of all, $\mathcal{X}$ being an irreducible variety, $\mathbb{C}[\mathcal{X}]$ is an integral domain. Take a B_E-eigenvector $\tilde{P}_i \in \mathbb{C}[\mathcal{X}]$ of weight $m\delta_i$ for any $1 \leqslant i \leqslant m$; which exists by the last theorem (assuming the validity of Conjecture 4.3). Now consider the function

$$\tilde{P}_\lambda = \prod_{i=1}^{m} \tilde{P}_i^{n_i} \in \mathbb{C}[\mathcal{X}].$$

Clearly, $\tilde{P}_\lambda$ is a nonzero B_E-eigenvector of weight $m\lambda$. This proves the corollary. □

https://doi.org/10.1112/S0010437X14007660 Published online by Cambridge University Press

Let $\mathcal{X}^o$ be the G-orbit $G \cdot \mathscr{D} \subset Q$. Then, by a classical result due to Frobenius (cf. [Kum13, Proposition 2.1 and Corollary 2.3]), the isotropy subgroup $G_{\mathscr{D}}$ of $\mathscr{D}$ is a reductive subgroup. In particular, by a result of Matsushima, $\mathcal{X}^o$ is an affine variety. Moreover, by Frobenius reciprocity, we get the following proposition.

PROPOSITION 6.3. *$\mathbb{C}[\mathcal{X}^o] \simeq \bigoplus_\lambda V_E(\lambda) \otimes [V_E(\lambda)^*]^{G_{\mathscr{D}}}$ as G-modules, where the summation runs over all the dominant integral weights λ of G (i.e., λ runs over $\sum_{i=1}^{m^2} n_i \delta_i$, $n_i \in \mathbb{Z}_+$ for all $1 \leqslant i < m^2$ and $n_{m^2} \in \mathbb{Z}$) and $[V_E(\lambda)^*]^{G_{\mathscr{D}}}$ denotes the subspace of $G_{\mathscr{D}}$-invariants in the dual space $V_E(\lambda)^*$. The action of G on the right is via its standard action on the first factor and it acts trivially on the second factor.*

In particular, the multiplicity of $V_E(\lambda)$ in $\mathbb{C}[\mathcal{X}^o]$ is the dimension of the invariant space $[V_E(\lambda)^*]^{G_{\mathscr{D}}}$.

It is easy to see that if $[V_E(\lambda)^*]^{G_{\mathscr{D}}} \neq 0$, then $|\lambda| := \sum_{i=1}^{m^2} i\, n_i \in m\mathbb{Z}$, where (as earlier) $\lambda = \sum_{i=1}^{m^2} n_i \delta_i$.

Applying [BLMW11, Proposition 5.2.1], we get that for any polynomial representation $V_E(\lambda)$ (i.e., $\lambda = \sum_{i=1}^{m^2} n_i \delta_i$ with all $n_i \in \mathbb{Z}_+$) with $|\lambda| = md$, $d \in \mathbb{Z}_+$,

$$\dim [V_E(\lambda)^*]^{G_{\mathscr{D}}} = sk_{\overline{\lambda}, d\delta_m, d\delta_m}, \tag{30}$$

where δ_m (as earlier) is the partition $\delta_m : (1 \geqslant 1 \geqslant \cdots \geqslant 1)$ (m factors), $\overline{\lambda}$ is the partition $(n_1 + \cdots + n_{m^2} \geqslant n_2 + \cdots + n_{m^2} \geqslant n_3 + \cdots + n_{m^2} \geqslant \cdots \geqslant n_{m^2} \geqslant 0)$ and $sk_{\overline{\lambda}, d\delta_m, d\delta_m}$ is the symmetric Kronecker coefficient (i.e., the multiplicity of the irreducible $\mathfrak{S}_{dm}$-module $W_{\overline{\lambda}}$ in the symmetric product $S^2(W_{d\delta_m})$, where, as earlier, $W_{\overline{\lambda}}$ denotes the irreducible $\mathfrak{S}_{dm}$-module corresponding to the partition $\overline{\lambda}$).

As a corollary of (30), and Proposition 6.3, we get the following result (since $\mathbb{C}[\mathcal{X}] \hookrightarrow \mathbb{C}[\mathcal{X}^o]$ as a G-module).

COROLLARY 6.4. *For any irreducible polynomial representation $V_E(\lambda)$ of G, such that $|\lambda| = dm$, for $d \in \mathbb{Z}_+$, the multiplicity $\mu(\lambda)$ of $V_E(\lambda)$ in $\mathbb{C}[\mathcal{X}]$ is bounded by:*

$$\mu(\lambda) \leqslant sk_{\overline{\lambda}, d\delta_m, d\delta_m}.$$

Observe that unless $V_E(\lambda)$ is a polynomial representation of G and $|\lambda| \in m\mathbb{Z}_+$, we have $\mu(\lambda) = 0$.

As an immediate consequence of Corollaries 6.2 and 6.4, we get the following result.

COROLLARY 6.5. *Let m be any positive even integer. Assume that Conjecture 4.3 is true for m. Then, for any partition $\overline{\lambda} : (\overline{\lambda}_1 \geqslant \overline{\lambda}_2 \geqslant \cdots \geqslant \overline{\lambda}_m \geqslant 0)$ (with at most m parts) of d (i.e., $|\overline{\lambda}| = d$), the symmetric Kronecker coefficient*

$$sk_{m\overline{\lambda}, d\delta_m, d\delta_m} > 0.$$

Remark 6.6. (a) Compare the above corollary with [BCI11, Theorem 1, § 3].

(b) The following generalization of Theorem 6.1 holds by exactly the same proof. Let $\mathscr{F} \in Q = S^m(E)^*$ be any (homogeneous) polynomial such that, writing $\mathscr{F}$ as a sum of monomials (in a basis of E^*), some monomial with no repeated factors occurs with nonzero coefficient. Assume further that Conjecture 4.3 is true for m, which is assumed to be even. Then, for any $1 \leqslant i \leqslant m$, the polynomial P_i does not vanish identically on the orbit $\mathrm{GL}(E) \cdot \mathscr{F}$.

In particular, this remark applies to $\mathscr{F} = \mathfrak{P}$, where $\mathfrak{P}$ is the function $E \to \mathbb{C}$ taking any matrix $A \in E := \operatorname{End} \mathfrak{v}$ to its permanent.

https://doi.org/10.1112/S0010437X14007660 Published online by Cambridge University Press

Thus, the irreducible $\mathrm{GL}(E)$-module $V_E(m\delta_i)$ occurs with multiplicity one in $\mathbb{C}[\overline{\mathrm{GL}(E)\cdot\mathfrak{P}}]$ for any $1 \leqslant i \leqslant m$ (assuming the validity of Conjecture 4.3 for m). Moreover, $V_E(d\delta_i)$, for any $d < m$ and $1 \leqslant i \leqslant m^2$, does not occur in $\mathbb{C}[\overline{\mathrm{GL}(E)\cdot\mathfrak{P}}]$ (cf. Corollary 2.4).

Acknowledgements

I am grateful to P. Bürgisser and C. Ikenmeyer for pointing out a gap in an earlier version of the paper which resulted in this greatly modified version. The gap did not alter the main Theorem 6.1, however unfortunately now the validity of the theorem is proved only under the hypothesis that the Latin square conjecture is valid.

I thank J. Landsberg for helpful conversations and correspondence; in particular, § 5 is greatly influenced by discussions with him and Proposition 5.5(b) is due to him. I also thank the referee for some helpful suggestions and Jeroen Zuiddam for pointing out some typos.

Partial support from the NSF grants is gratefully acknowledged.

References

AT92 N. Alon and M. Tarsi, *Colorings and orientations of graphs*, Combinatorica **12** (1992), 125–134.

BCI11 P. Bürgisser, M. Christandl and C. Ikenmeyer, *Nonvanishing of Kronecker coefficients for rectangular shapes*, Adv. Math. **227** (2011), 2082–2091.

BLMW11 P. Bürgisser, J. M. Landsberg, L. Manivel and J. Weyman, *An overview of mathematical issues arising in the geometric complexity theory approach to* $VP \neq VNP$, SIAM J. Comput. **40** (2011), 1179–1209.

Dri97 A. A. Drisko, *On the number of even and odd Latin squares of order* $p+1$, Adv. Math. **128** (1997), 20–35.

Gly10 D. Glynn, *The conjectures of Alon–Tarsi and Rota in dimension prime minus one*, SIAM J. Discrete Math. **24** (2010), 394–399.

GW09 R. Goodman and N. Wallach, *Symmetry, representations, and invariants*, Graduate Texts in Mathematics, vol. 255 (Springer, New York, 2009).

How87 R. Howe, $(\mathrm{GL}_n, \mathrm{GL}_m)$*-duality and symmetric plethysm*, Proc. Indian Acad. Sci. Math. Sci. **97** (1987), 85–109.

HR94 R. Huang and G.-C. Rota, *On the relations of various conjectures on Latin squares and straightening coefficients*, Discrete Math. **128** (1994), 225–236.

Kos76 K. Kostant, *On Macdonald's* η*-function formula, the Laplacian and generalized exponents*, Adv. Math. **20** (1976), 79–212.

Kum13 S. Kumar, *Geometry of orbits of permanents and determinants*, Comment. Math. Helv. **88** (2013), 759–788.

MS01 K. Mulmuley and M. Sohoni, *Geometric complexity theory I. An approach to the P vs. NP and related problems*, SIAM J. Comput. **31** (2001), 496–526.

MS08 K. Mulmuley and M. Sohoni, *Geometric complexity theory II: Towards explicit obstructions for embeddings among class varieties*, SIAM J. Comput. **38** (2008), 1175–1206.

Val79 L. G. Valiant, *Completeness classes in algebra*, in *Proceedings of the eleventh annual ACM symposium on theory of computing (Atlanta, GA, 1979)* (ACM, New York, 1979), 249–261.

Shrawan Kumar shrawan@email.unc.edu
Department of Mathematics, University of North Carolina,
Chapel Hill, NC 27599-3250, USA

https://doi.org/10.1112/S0010437X14007660 Published online by Cambridge University Press