BULL. AUSTRAL. MATH. SOC. 20F99
VOL. 10 (1974), 277-279. (20F05, 20G99)

Some remarks on coherent soluble groups

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An example of Wehrfritz is pointed out to show that GL(4, Q) is not coherent. This answers a question of Serre. It is shown that finitely generated soluble coherent subgroups of GL(2, Q) need not be polycyclic, in sharp contrast to the fact that all soluble subgroups of GL(n, Z) are polycyclic and so automatically coherent.

A group G is said to be *coherent* if every finitely generated subgroup of G is finitely presented. Thus free groups, nilpotent groups, and polycyclic groups are coherent. In [2], Scott shows that the fundamental group of a three-manifold is coherent, and concludes that SL(2, R) is coherent, where R is the ring of integers of an imaginary quadratic field. This fact led Serre [3] to ask, amongst other questions, whether perhaps GL(n, Q) is coherent.

That this is not generally true is shown by the subgroup of GL(4, Q) generated by the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This is example 4.22 of Wehrfritz [4], and it is a metabelian linear group with entries in the subring $Z[\frac{1}{2}]$ of Q that is not finitely presented. As such it contrasts sharply with the theorem of Mal'cev (see [1,

Received 26 November 1973.

pp. 81-82]) stating that all soluble subgroups of $\mathrm{GL}(n, Z)$ are polycyclic, and so certainly coherent.

In view of this one might be tempted to conjecture that finitely generated coherent soluble subgroups of $\mathrm{GL}(n, Q)$ are polycyclic. One of the simplest known types of finitely presented soluble group shows that the answer is again in the negative:

THEOREM. *The subgroup* K *of* $\mathrm{GL}(2, Q)$ *generated by the matrices* $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ *and* $\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$ *is metabelian and coherent but not polycyclic.*

Proof. The group K is isomorphic with the group G given by the presentation $\langle a, b : a^b = a^2 \rangle$ (see Wehrfritz [4, Example 11.15]). We shall work with G. The derived group is $G' = \left\langle a, a^{b^{-1}}, a^{b^{-2}}, \ldots \right\rangle$, which is abelian and not finitely generated since it is isomorphic with the additive group of dyadic rationals. Thus G is not polycyclic. Let H be a finitely generated subgroup of G, which we may assume is not a subgroup of G' since finitely generated subgroups of G' are cyclic. Thus $HG' \neq G'$, so that G/HG' is finite since G/G' is infinite cyclic. This means that HG' is a finitely generated metabelian group; so $H \cap G'$ is the normal closure in HG' of finitely many elements, since $H \cap G'$ is normal in HG' and finitely generated metabelian groups satisfy max-n (see [1, Theorem 5.34]). Since G' is locally cyclic, it follows that $H \cap G'$ is the normal closure in HG' of a single element h. But $H/(H \cap G') \cong HG'/G'$, and so $H = \langle H \cap G', c \rangle$, where $c = gb^m$ for some integer $m > 0$ and some g in G'. Since G' is abelian, all these facts together show that $H = \langle h, c \rangle$. Furthermore, $h^c = h^{2^m}$ since G' is abelian, so that H is a homomorphic image of the group H^* with presentation $\left\langle x, y : x^y = x^{2^m} \right\rangle$. In fact these groups are isomorphic, but we do not need this information to prove the theorem. By the theorem on finitely generated metabelian groups mentioned above, H^* satisfies max-n, so that every homomorphic image of it is finitely presented; in particular, H is finitely presented; and G is coherent.

https://doi.org/10.1017/S0004972700040892 Published online by Cambridge University Press

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