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Abstract

Suppose that $f : X \to C$ is a general Jacobian elliptic surface over $\mathbb{C}$ of irregularity q and positive geometric genus h. Assume that $10h > 12(q-1)$, that $h > 0$ and let $\overline{\mathcal{E}\ell\ell}$ denote the stack of generalized elliptic curves. (1) The moduli stack $\mathcal{JE}$ of such surfaces is smooth at the point X and its tangent space T there is naturally a direct sum of lines $(v_a)_{a\in Z}$, where $Z \subset C$ is the ramification locus of the classifying morphism $\phi : C \to \overline{\mathcal{E}\ell\ell}$ that corresponds to $X \to C$. (2) For each $a \in Z$ the map $\overline{\nabla}_{v_a} : H^{2,0}(X) \to H^{1,1}_{\mathrm{prim}}(X)$ defined by the derivative per_* of the period map per is of rank one. Its image is a line $\mathbb{C}[\eta_a]$ and its kernel is $H^0(X, \Omega^2_X(-E_a))$, where $E_a = f^{-1}(a)$. (3) The classes $[\eta_a]$ form an orthogonal basis of $H^{1,1}_{\mathrm{prim}}(X)$ and $[\eta_a]$ is represented by a meromorphic 2-form η_a in $H^0(X, \Omega^2_X(2E_a))$ of the second kind. (4) We prove a local Schottky theorem; that is, we give a description of per_* in terms of a certain additional structure on the vector bundles that are involved. Assume further that $8h > 10(q-1)$ and that $h \geq q+3$. (5) Given the period point $per(X)$ of X that classifies the Hodge structure on the primitive cohomology $H^2_{\mathrm{prim}}(X)$ and the image of T under per_* we recover Z as a subset of $\mathbb{P}^{h-1}$ and then, by quadratic interpolation, the curve C. (6) We prove a generic Torelli theorem for these surfaces. Everything relies on the construction, via certain kinds of Schiffer variations of curves, of certain variations of X for which per_* can be calculated. (In an earlier version of this paper we used variations constructed by Fay. However, Schiffer variations are slightly more powerful.)

1. Introduction

Suppose that $\mathcal{M}$ is a moduli stack of smooth projective varieties over $\mathbb{C}$ and that $per : \mathcal{M} \to \mathcal{P} = D/\Gamma$ is a corresponding period map. The derivative of per is a homomorphism

$$per_* : T_{\mathcal{M}} \to per^* T_{\mathcal{P}}.$$

The *local Torelli problem* is that of describing the kernel of this homomorphism and the *local Schottky problem* is the problem of describing its image. We say that the *local Torelli theorem* holds at a point x of $\mathcal{M}$ if the derivative per_* of per is injective at x and that the *generic local Torelli theorem* holds if it holds at every generic point of $\mathcal{M}$. We also say that the *generic Torelli*

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theorem holds if *per* has degree 1 onto its image. The *Schottky problem* asks for a description of the image of the period map.

As explained on p. 228 of [Gri84], if the generic local Torelli theorem holds and if it can be proved that a variety X can be recovered from knowledge of the period point $per(X)$ and the subspace $per_*(T_{\mathcal{M}}(X))$ of the tangent space $T_{\mathcal{P}}(X)$, then the generic Torelli theorem also holds.

In this paper we consider the problem for elliptic surfaces $f : X \to C$ with no multiple fibres (such surfaces we shall call *simple*) and show that the situation is closely parallel to that for curves, as follows.

Suppose that the geometric genus of X is h and its irregularity q, that $b_1(X)$ is even (so that, by a result of Miyaoka [Miy74], X is Kähler), that $10h > 12(q-1)$ and that $h \geq q+3$. Assume also that X is *general*, in a sense to be made precise later. Then we prove the following results, the first two of which are well-known tautologies.

For $a \in C$ we let E_a denote the fibre $f^{-1}(a)$ and $\omega_a^\vee$ the line in $H^{2,0}(X)^\vee$ whose kernel is $H^0(X, \Omega_X^2(-E_a))$.

Assume that $10h > 12(q-1)$.

(1) There is a classifying morphism $\phi = \phi_f : C \to \overline{\mathcal{M}}_1$, where $\overline{\mathcal{M}}_1$ is the stack of stable curves of genus 1. Set $Z = \mathrm{Ram}_\phi \subset C$, the ramification t divisor. (Up to noise which is removed by the language of stacks, this is the locus where the derivative of the j-invariant vanishes.)
(2) If also X is algebraic, then the tangent space at the point X to the stack of algebraic elliptic surfaces is naturally isomorphic to an invertible sheaf on Z.
(3) Every choice of a point a in Z and of a local coordinate on C at a defines a 1-parameter variation of X. This is based on the construction of the version of *Schiffer variations* that is described on p. 443 of [Gar49].
(4) The derivative of the period map of this variation, which is a linear map $\overline{\nabla}_a : H^{2,0}(X) \to H^{1,1}_{\mathrm{prim}}(X)$, is of rank 1.
(5) There is a meromorphic 2-form $\eta_a \in H^0(X, \Omega_X^2(2E_a))$ of the second kind (that is, the residue of η_a along E_a vanishes) such that $\overline{\nabla}_a = \omega_a^\vee \otimes [\eta_a]$.
(6) Assume also that $h \geq q+3$. Then the canonical model of X is a copy of C embedded as a curve of degree $h+q-1$ in a projective space $\mathbb{P}(H^{2,0}(X)^\vee) = \mathbb{P}^{h-1}$, and the set Z can be recovered, as a finite point set in $\mathbb{P}^{h-1}$, from the finite subset $\{\omega_a^\vee \otimes [\eta_a]\}_{a \in Z}$ of $\mathbb{P}((H^{2,0}(X)^\vee \otimes H^{1,1}_{\mathrm{prim}}(X))^\vee)$. Indeed, we exploit this set of N points in projective space as an analogue of the theta divisor on the Jacobian of a curve.
(7) Assume that $h \geq q+3$ and that $4h > 5(q-1)$. Then the curve C can be recovered from Z in $\mathbb{P}^{h-1}$ via quadratic interpolation.
(8) Given C and Z, we then prove a generic Torelli theorem for Jacobian elliptic surfaces.

Remark. It is clear that some of these constructions can be still be made when the phrase 'elliptic curve' is replaced by 'Calabi–Yau variety whose compactified moduli stack is a smooth 1-dimensional Deligne–Mumford stack whose first Chern class is positive'.

An essential difference between the case of curves and that of elliptic surfaces, however, is that for curves these variations arise for any point a on C while for surfaces they only arise for points of the ramification divisor Z. Indeed, for other points x of C there is no meromorphic 2-form of the second kind with double poles along E_x. (I am grateful to Richard Thomas for explaining this to me.)

We now give some more details.

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DEFINITION 1.1. An elliptic surface is *Jacobian* if it has a specified section.

Jacobian implies simple, but not conversely.

In this paper the things of primary concern are the stacks $\mathcal{SE}$ and $\widetilde{\mathcal{JE}}$ of simple and Jacobian elliptic surfaces $f : X \to C$ that are smooth and relatively minimal. We also consider the stack $\mathcal{JE}$ whose objects are the relative canonical models of surfaces in $\widetilde{\mathcal{JE}}$; given $f : X \to C$ in $\widetilde{\mathcal{JE}}$ the relative canonical model is obtained by contracting all vertical (-2)-curves in $\mathbb{X}$ that are disjoint from the given section. There is an obvious morphism $\widetilde{\mathcal{JE}} \to \mathcal{JE}$ that is a bijection on geometric points. At the level of miniversal deformation spaces, this morphism can be described by taking the geometric quotient by an action of the relevant Weyl group, as was shown by Artin and Brieskorn.

We shall say that a surface X in $\mathcal{SE}$ is *general* if its j-invariant $C \to \mathbb{P}^1_j$ is non-constant and its singular fibres are all of type I_1. We let $\mathcal{SE}^{\mathrm{gen}}$ and $\mathcal{JE}^{\mathrm{gen}}$ denote the stacks of general simple surfaces and general Jacobian surfaces; these are open substacks of $\mathcal{SE}$ and $\widetilde{\mathcal{JE}}$. Note that $\mathcal{JE}^{\mathrm{gen}}$ maps isomorphically to its image in $\mathcal{JE}$ so is also naturally an open substack of $\mathcal{JE}$. Then to give a point in $\mathcal{SE}^{\mathrm{gen}}$ is equivalent to giving a classifying morphism $F : C \to \overline{\mathcal{M}}_1$ that is non-constant and unramified over $j = \infty$. The stack $\overline{\mathcal{M}}_1$ is not the same as the stack $\overline{\mathcal{E}\ell\ell}$ of stable generalized elliptic curves; these stacks will be discussed in more detail in § 2. Giving a point in $\mathcal{JE}^{\mathrm{gen}}$ is equivalent to giving a morphism $\phi : C \to \overline{\mathcal{E}\ell\ell}$ that is non-constant over the j-line and unramified over $j = \infty$.

Assume that $f : X \to C$ is general in $\mathcal{JE}$. Let Z denote the ramification divisor in C of ϕ. Say $h = p_g(X)$ and $q = h^1(\mathcal{O}_X)$, so that q is also the genus of C. We shall assume throughout this paper that

$$10h > 12(q-1) \quad \text{and} \quad h > 0. \tag{1.2}$$

These assumptions ensure that $\deg \phi^* T_{\overline{\mathcal{E}\ell\ell}} > 2q - 2$, which in turn ensures the vanishing of certain obstruction spaces. From § 6 onwards we shall make the stronger assumptions that

$$8h > 10(q-1) \quad \text{and} \quad h > q + 2. \tag{1.3}$$

These assumptions make it possible to apply theorems of Mumford and Saint-Donat about the defining equations of linearly normal projective curves.

Write $N = 10h + 8(1-q)$. Then, as is well known, $\mathcal{SE}$ is smooth at the point corresponding to X and

$$\deg Z = N, \quad \dim \mathcal{SE} = N + h, \quad h^{1,1}(X) = N + 2.$$

Then we shall prove effective forms of both a generic local Torelli theorem and a generic Torelli theorem for the weight 2 Hodge structure on X, in the following sense.

If X is a surface in $\mathcal{JE}$ with specified section σ and fibre ξ, then $H^2_{\mathrm{prim}}(X)$ and $H^{1,1}_{\mathrm{prim}}(X)$ will denote the orthogonal complement $\langle \sigma, \xi \rangle^{\perp}$. If X is in $\mathcal{SE}$ but is not necessarily algebraic, then $H^2_{\mathrm{prim}}(X)$ and $H^{1,1}_{\mathrm{prim}}(X)$ will denote $\xi^{\perp}/\mathbb{Z}\xi$; these two definitions are equivalent for $X \in \mathcal{JE}$. Observe that $h^{1,1}_{\mathrm{prim}}(X) = N$. In fact $\dim \mathcal{JE} = N$ also, so that $\dim \mathcal{JE} = h^{1,1}_{\mathrm{prim}}(X)$. We shall use this coincidence in § 5 to enhance the local structure of the derivative of the period map.

From the description of $\mathcal{JE}^{\mathrm{gen}}$ as the stack that parametrizes those non-constant morphisms from curves to $\overline{\mathcal{E}\ell\ell}$ that are unramified over $j = \infty$ we shall prove the following theorem, which appears as Theorem 4.10. It is the main result of the paper; everything else follows from it.

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Theorem 1.4. *Fix a surface $f : X \to C$ that is a point X of $\mathcal{JE}^{\mathrm{gen}}$ and corresponds to $\phi : C \to \overline{\mathcal{E}\ell\ell}$. For $P \in C$ put $E_P = f^{-1}(P)$ and let Z denote the ramification divisor Ram_ϕ.*

(1) *Given a point a of Z there is a tangent line $\mathbb{C}v_a$ to $\mathcal{JE}$ at the point X.*
(2) *There is a meromorphic 2-form $\eta_a \in H^0(X, \Omega^2_X(2E_a))$ of the second kind (that is, its residue $\mathrm{Res}_{E_a} \eta_a$ vanishes).*
(3) *The corresponding map*

$$\overline{\nabla}_{v_a} : H^{2.0}(X) \to H^{1,1}_{\mathrm{prim}}(X)$$

is of rank 1. Its kernel is the space $H^0(X, \Omega^2_X(-E_a))$ of 2-forms that vanish along E_a and its image is the line generated by the class $[\eta_a]$ of η_a, modulo ξ.

In order to prove this theorem we shall use Schiffer variations to construct, for each point $a \in \mathrm{Ram}_\phi$ where the ramification index of $\phi : C \to \overline{\mathcal{E}\ell\ell}$ is m (so that a is of multiplicity $m-1$ in Z) an $(m-1)$-parameter deformation $\mathcal{C} \to \Delta^{m-1}$ of C whose derivative can be calculated. Thus, when X is a point of $\mathcal{JE}^{\mathrm{gen}}$, we have a detailed description of an N-dimensional subspace of the tangent space to moduli inside the tangent space to the period domain as the subspace spanned by certain explicit tensors of rank 1. (Masa-Hiko Saito [Sai83] has proved the local Torelli theorem for simple elliptic surfaces with non-constant j-invariant and for many surfaces with constant j-invariant. We shall extend his result slightly; see Theorem 2.10.)

It is a matter of linear algebra to recover Z as a subset of the projective space $\mathbb{P}^{h-1}$ in which C is embedded as the canonical model of X, under the assumption that Z is reduced. We then use a theorem of Mumford [Mum70] and Saint-Donat [Sai72], to the effect that linearly normal curves of genus q and degree at least $2q+2$ are intersections of quadrics, to show that C is determined by quadratic interpolation through Z. We go on to prove that from the pair (C, Z) we can recover the classifying morphism $\phi : C \to \overline{\mathcal{E}\ell\ell}$, modulo the automorphism group $\mathbb{G}_m$ of $\overline{\mathcal{E}\ell\ell}$ provided that ϕ is generic. This recovery of C and ϕ from the period data we regard as an effective theorem. It shows that any failure of generic Torelli for Jacobian surfaces can be detected in a pencil that is the closure of the $\mathbb{G}_m$-orbit thorough a generic point of $\mathcal{JE}^{\mathrm{gen}}$. (The fact that the automorphisms of $\overline{\mathcal{E}\ell\ell}$ obstruct a direct deduction of generic Torelli from knowledge of C and Z was observed by Cox and Donagi [CD86].)

Once we know that the base curve C is determined by Hodge-theoretical data of weight 2 we go on to prove the generic Torelli theorem for Jacobian elliptic surfaces via ideas similar to those used by Chakiris [Cha82, Cha84] to prove generic Torelli when $C = \mathbb{P}^1$, but reinforced by the minimal model program.

There is also some further structure on the period map for $\mathcal{JE}^{\mathrm{gen}}$: the relevant vector bundles and homomorphisms between them can be described in terms of line bundles on the universal ramification divisor $\mathcal{Z}^{\mathrm{gen}}$ over $\mathcal{JE}^{\mathrm{gen}}$ of the universal classifying morphism to $\overline{\mathcal{E}\ell\ell}$. This can be seen as a local solution to the Schottky problem. The details are stated in Theorem 4.11.

We also give a variational form of a partial solution to the global Schottky problem.

If $\mathcal{X}$ is a Deligne–Mumford stack, then $[\mathcal{X}]$ will denote its geometric quotient.

2. Preliminaries on stacks and tangent spaces

Everything in the next two sections is well known; if it is not due to either Kas [Kas66] or Kodaira [Kod63], then it is folklore.

The stack $\overline{\mathcal{E}\ell\ell}$ is the Deligne–Mumford stack over $\mathbb{C}$ of *stable generalized elliptic curves*; that is, an S-point of $\overline{\mathcal{E}\ell\ell}$ is a flat projective morphism $Y \to S$ with a section S_0 contained in the relatively smooth locus of $Y \to S$ and whose geometric fibres are reduced and irreducible nodal

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curves of arithmetic genus 1. Such a curve is then, locally on S, a plane cubic with affine equation

$$y^2 = 4x^3 - g_4 x - g_6,$$

where g_4 and g_6 are not both zero, so that $\overline{\mathcal{E}\ell\ell}$ is the quotient stack $\mathbb{P}(4,6) = (\mathbb{A}^2 - \{0\})/\mathbb{G}_m$, where $\mathbb{G}_m$ acts on $\mathbb{A}^2$ with weights $4, 6$. Note that $\mathbb{G}_m$ acts on $\mathbb{A}^2$ via a homomorphism $\mathbb{G}_m \to \mathbb{G}_m^2$ (whose kernel is μ_2) and the standard action of $\mathbb{G}_m^2$ on $\mathbb{A}^2$, so that there is a residual action of $\mathbb{G}_m$ on $\overline{\mathcal{E}\ell\ell}$. This exhibits $\mathbb{G}_m$ as the full automorphism group of $\overline{\mathcal{E}\ell\ell}$; cf. Theorem 8.1 of [BN06] and the calculation there on p. 139.

The geometric quotient $[\overline{\mathcal{E}\ell\ell}]$ of $\overline{\mathcal{E}\ell\ell}$ is the compactified j-line $\mathbb{P}^1_j$; if $\rho : \overline{\mathcal{E}\ell\ell} \to \mathbb{P}^1_j$ is the quotient morphism, then the automorphism group of each fibre of ρ is $\mathbb{Z}/2$, except over $j = j_6 = 0$, where it is $\mathbb{Z}/6$, and over $j = j_4 = 1728$, where it is $\mathbb{Z}/4$. So $\deg \rho = 1/2$. As is well known, it is possible to write down a generalized elliptic curve over the open locus U of $\mathbb{P}^1_j$ defined by $j \neq 0, 1728$, so that there is a section of ρ over U. Moreover, $\rho^{-1}(U)$ is isomorphic to $U \times B(\mathbb{Z}/2)$, but there is no global section of ρ.

There are two obvious line bundles on $\overline{\mathcal{E}\ell\ell}$: the bundle M of modular forms of weight 1, which is identified with the conormal bundle of the zero-section of the universal stable generalized elliptic curve, and the tangent bundle $T_{\overline{\mathcal{E}\ell\ell}}$.

Lemma 2.1. *We have* $T_{\overline{\mathcal{E}\ell\ell}} \cong M^{\otimes 10}$, $\deg M = 1/24$ *and* $\deg T_{\overline{\mathcal{E}\ell\ell}} = 5/12$.

Proof. Quite generally the Picard group of $\mathbb{P}(a,b)$ is generated by $\mathcal{O}(1)$, which has degree $1/ab$, and $T_{\mathbb{P}(a,b)}$ is isomorphic to $\mathcal{O}(a+b)$, which then has degree $(a+b)/ab$. □

The objects of the stack $\overline{\mathcal{M}}_1$ are *stable curves of genus* 1; the geometric fibres are isomorphic to stable generalized elliptic curves, but no section is given. This is an Artin stack, but not Deligne–Mumford. Indeed, the word 'stable' in this context is an abuse of language, but I am optimistic that it will cause no confusion.

Let $\mathcal{C} \to \overline{\mathcal{M}}_1$ and $\mathcal{E} \to \overline{\mathcal{E}\ell\ell}$ denote the universal objects and let $\mathcal{G} \to \overline{\mathcal{E}\ell\ell}$ denote the Néron model of $\mathcal{E} \to \overline{\mathcal{E}\ell\ell}$, so that $\mathcal{G}$ is the open substack of $\mathcal{E}$ obtained by deleting the singular point of the fibre over $j = \infty$.

The next result is due to Altman and Kleiman [AK80], although reformulated here in the language of stacks. We have chosen to include a slightly different proof that emphasizes automorphism groups rather than Picard varieties so that the relevant classifying stacks enter more easily.

Theorem 2.2. *There is a morphism* $\pi : \overline{\mathcal{M}}_1 \to \overline{\mathcal{E}\ell\ell}$ *via which* $\overline{\mathcal{M}}_1$ *is isomorphic to the classifying stack* $B\mathcal{G}$ *over* $\overline{\mathcal{E}\ell\ell}$.

Proof. Let $G \to \overline{\mathcal{M}}_1$ denote the connected component of the relative automorphism group scheme of $\mathcal{C} \to \overline{\mathcal{M}}_1$. Thus, $G \to \overline{\mathcal{M}}_1$ is elliptic over the open substack M_1 of $\overline{\mathcal{M}}_1$ defined by $j \neq \infty$ and over $j = \infty$ the fibre of G is the multiplicative group $\mathbb{G}_m$. □

Lemma 2.3. *There is a unique open embedding* $G \hookrightarrow \widetilde{G}$, *over* $\overline{\mathcal{M}}_1$, *where* $\widetilde{G} \to \overline{\mathcal{M}}_1$ *is a stable generalized elliptic curve and* G *is its relative smooth locus.*

Proof. To construct $\widetilde{G}$ we need to patch the puncture of $G \to \overline{\mathcal{M}}_1$ that lies over $j = \infty$. In a suitable neighbourhood S of the locus $j = \infty$ in $\overline{\mathcal{M}}_1$ the process of patching the puncture is a matter of 'reversing the process of deleting a closed point from a normal 2-dimensional analytic space (or scheme)', so the patch is unique if it exists. Therefore, it is enough to exhibit the patch locally on $\overline{\mathcal{M}}_1$, in the neighbourhood S.

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Put $\mathcal{C}_S = \mathcal{C} \times_{\overline{\mathcal{M}}_1} S$. Since S is local and $\mathcal{C}_S \to S$ is generically smooth, there is a section over S of $\mathcal{C}_S \to S$ that is contained in the relative smooth locus. Use this section to put the structure of a stable generalized elliptic curve on $\mathcal{C}_S \to S$. This structure on $\mathcal{C}_S \to S$ provides the patch for $G_S = G \times_{\overline{\mathcal{M}}_1} S \to S$ and yields $\widetilde{G} \to S$; the lemma is proved. □

Sending $\mathcal{C} \to \overline{\mathcal{M}}_1$ to $\widetilde{G} \to \overline{\mathcal{M}}_1$ defines a morphism $\pi : \overline{\mathcal{M}}_1 \to \overline{\mathcal{E}\ell\ell}$.

A *gerbe* is a morphism of stacks that is locally surjective on both objects and morphisms. A *neutral gerbe* is a gerbe with a section.

Lemma 2.4. *The morphism π is a neutral gerbe.*

Proof. It is enough to show that:

(1) π has a section (then it is certainly locally surjective on objects); and
(2) π is locally surjective on morphisms.

For part (1), use the forgetful morphism $\overline{\mathcal{E}\ell\ell} \to \overline{\mathcal{M}}_1$ to get a section of π.

Part (2) is equally clear: a morphism of stable generalized elliptic curves is, in particular, a morphism of the underlying stable curves.

Finally, suppose that $\mathcal{X} \to \mathcal{Y}$ is a neutral gerbe with section $s : \mathcal{Y} \to \mathcal{X}$. The stabilizer group scheme is a group scheme $\mathcal{H}' \to \mathcal{X}$; define $\mathcal{H} \to \mathcal{Y}$ to be the pull back of $\mathcal{H}'$ via s. If $\mathcal{H} \to \mathcal{Y}$ is flat, then [LM00] there is an isomorphism $\mathcal{X} \to B\mathcal{H}$. Since the Néron model $\mathcal{G} \to \overline{\mathcal{E}\ell\ell}$ is flat, the theorem is proved. □

Note that $G \to \overline{\mathcal{M}}_1$ is isomorphic to the pull back under π of the Néron model $\mathcal{G} \to \overline{\mathcal{E}\ell\ell}$.

As already remarked, a general simple surface $f : X \to C$ determines, and is determined by, a morphism $F = F_f : C \to \overline{\mathcal{M}}_1$ that is unramified over $j = \infty$. Let $\phi = \phi_f = \pi \circ F : C \to \overline{\mathcal{E}\ell\ell}$ denote the composite, so that the induced Jacobian elliptic surface is the compactified relative automorphism group scheme of $f : X \to C$.

For example, if $f : X \to \mathbb{P}^1$ is a primary Hopf surface, then ϕ_f is constant: the relative automorphism group scheme is a constant relative group scheme $E \times \mathbb{P}^1 \to \mathbb{P}^1$.

Lemma 2.5. *If $X \to C$ has non-constant j-invariant, then the irregularity q of X equals the geometric genus of C.*

Proof. This is well known, and easy. □

Lemma 2.6. *Suppose that $f : X \to C$ is a general simple surface. Then*

(1) $\deg \phi = 2c_2(X) = 24\chi(X, \mathcal{O}_X)$;
(2) $\deg \phi^* M = \chi(X, \mathcal{O}_X)$;
(3) $\deg \phi^* T_{\overline{\mathcal{E}\ell\ell}} = 10\chi(X, \mathcal{O}_X)$.

Proof. Since $c_2(X)$ equals the total number of nodes in the singular fibres of $X \to C$, part (1) follows from consideration of the inverse image of the locus $j = \infty$, the fact that $\deg \rho = 1/2$ and Noether's formula. The rest follows immediately. □

We next consider various tangent spaces.

Regard $B\mathcal{G}$ as the quotient of $\overline{\mathcal{E}\ell\ell}$ by $\mathcal{G}$. Since $\overline{\mathcal{M}}_1$ is isomorphic to $B\mathcal{G}$, locally on $\overline{\mathcal{E}\ell\ell}$, the tangent complex $T^\bullet_{\overline{\mathcal{M}}_1}$ is a 2-term complex, which is obtained by descending a 2-term complex on $\overline{\mathcal{E}\ell\ell}$. In degree 0, this complex is $T_{\overline{\mathcal{E}\ell\ell}}$, in degree -1 it is the adjoint bundle $\operatorname{Ad}\mathcal{G}$ and the differential is the derivative of the action of $\mathcal{G}$ on $\overline{\mathcal{E}\ell\ell}$.

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Since this action is trivial, the differential in $T^{\bullet}_{\overline{\mathcal{M}}_1}$ is zero. Moreover, since $\mathcal{G}$ has no characters (it is generically an elliptic curve) it follows that $T^{\bullet}_{\overline{\mathcal{M}}_1}$ is quasi-isomorphic to the pull back of the complex $\operatorname{Ad}\mathcal{G}[1] \oplus T_{\overline{\mathcal{E}\ell\ell}}[0]$ on $\overline{\mathcal{E}\ell\ell}$.

Note that $(\operatorname{Ad}\mathcal{G})^{\vee}$ is exactly the line bundle M of modular forms of weight 1.

Now fix a point X of $\mathcal{SE}$. That is, we fix a general simple elliptic surface $f : X \to C$. This equals the datum of a morphism $F : C \to \overline{\mathcal{M}}_1$. Set $\phi = \pi \circ F : C \to \overline{\mathcal{E}\ell\ell}$. Then $F^*T^{\bullet}_{\overline{\mathcal{M}}_1}$ is quasi-isomorphic to $\phi^* \operatorname{Ad}\mathcal{G}[1] \oplus \phi^* T_{\overline{\mathcal{E}\ell\ell}}$. There is a distinguished triangle

$$T_C \to F^*T^{\bullet}_{\overline{\mathcal{M}}_1} \to K^{\bullet},$$

where $K^{\bullet}$ is a 2-term complex of coherent sheaves on C, $K^{-1} = \phi^* \operatorname{Ad}\mathcal{G}$, K^0 is the skyscraper sheaf $\operatorname{coker}(T_C \to \phi^* T_{\overline{\mathcal{E}\ell\ell}})$ and the differential in $K^{\bullet}$ is zero.

PROPOSITION 2.7. *The tangent space $T_{\mathcal{SE}}(X)$ is naturally isomorphic to the hypercohomology group $\mathbb{H}^0(C, K^{\bullet})$ and the obstructions to the smoothness of $\mathcal{SE}$ at X lie in $\mathbb{H}^1(C, K^{\bullet})$.*

Proof. This follows from the identification of the points of $\mathcal{SE}$ with morphisms from curves to $\overline{\mathcal{M}}_1$. In this latter context the result is well known. □

Let Z denote the ramification divisor $Z = \operatorname{Ram}_{\phi} = \operatorname{Ram}_F$ on C. Then K^0 is an invertible sheaf on Z, so that there is a non-canonical isomorphism $K^0 \xrightarrow{\cong} \mathcal{O}_Z$.

The distinguished triangle just mentioned gives an exact sequence

$$\begin{aligned} 0 \to H^0(C, T_C) \to \mathbb{H}^0(C, F^*T^{\bullet}_{\overline{\mathcal{M}}_1}) \to \mathbb{H}^0(C, K^{\bullet}) \\ \to H^1(C, T_C) \to \mathbb{H}^1(C, F^*T^{\bullet}_{bsm_1}) \to \mathbb{H}^1(C, K^{\bullet}) \to 0. \end{aligned}$$

Then $H^2(C, \phi^* \operatorname{Ad} G) = 0$ and $H^1(C, \phi^* T_{\overline{\mathcal{E}\ell\ell}}) = 0$, since

$$\deg \phi^* T_{\overline{\mathcal{E}\ell\ell}} = 10\chi(X, \mathcal{O}_X) > 2q - 2,$$

from the assumption (1.2), so that $\mathbb{H}^1(C, K^{\bullet}) = \mathbb{H}^1(C, F^*T^{\bullet}_{\overline{\mathcal{M}}_1}) = 0$ and there is an exact commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & H^0(C, T_C) & \xrightarrow{=} & H^0(C, T_C) & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & H^0(C, \phi^* T_{\overline{\mathcal{E}\ell\ell}}) & \to & \mathbb{H}^0(C, F^*T^{\bullet}_{\overline{\mathcal{M}}_1}) & \to & H^1(C, \phi^* \operatorname{Ad}\mathcal{G}) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 0 & \to & H^0(C, K^0) & \to & \mathbb{H}^0(C, K^{\bullet}) & \to & H^1(C, \phi^* \operatorname{Ad}\mathcal{G}) & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & H^1(C, T_C) & \xrightarrow{=} & H^1(C, T_C) & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

in which the two middle rows are canonically split.

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PROPOSITION 2.8.

(1) *The stack $\mathcal{SE}$ is smooth at the point X.*
(2) *Its dimension there is $11h + 8(1-q)$.*
(3) *The degree of the ramification divisor $\mathrm{Ram}_\phi = \mathrm{Ram}_F$ is N.*
(4) *We have $\phi^*M \cong f_*\omega_{X/C} \cong (R^1 f_*\mathcal{O}_X)^\vee$ and $\deg \phi^*M = \chi(X, \mathcal{O}_X) = h + 1 - q$.*

Proof. Except for part (4), which is well known, this follows from the preceding discussion. □

Suppose that $f : X \to C$ is a point of $\mathcal{JE}^{\mathrm{gen}}$ and defines $\phi : C \to \overline{\mathcal{E}\ell\ell}$. Set $Z = \mathrm{Ram}_\phi$.

PROPOSITION 2.9.

(1) *There is a short exact sequence*

$$0 \to H^0(C, \phi^* T_{\overline{\mathcal{E}\ell\ell}}) \to T_{\mathcal{JE}}(X) \to H^1(C, T_C) \to 0.$$

(2) *The tangent space $T_{\mathcal{JE}}(X)$ is naturally isomorphic to $H^0(Z, K^0)$.*
(3) *The moduli stack $\mathcal{JE}$ is smooth at the point X and its dimension there is N.*

Proof. This is proved in the same way as Propositions 2.7 and 2.8. □

Since Z is 0-dimensional and K^0 is an invertible sheaf on Z, $H^0(Z, K^0)$ is non-canonically isomorphic to $H^0(Z, \mathcal{O}_Z)$.

We shall usually write $\phi^*M = L = f_*\omega_{X/C}$. Then $\phi^* T_{\overline{\mathcal{E}\ell\ell}} \cong L^{\otimes 10}$, so that

$$\mathcal{O}_C(\mathrm{Ram}_\phi) \cong \phi^* T_{\overline{\mathcal{E}\ell\ell}} \otimes T_C^\vee \cong L^{\otimes 10} \otimes \mathcal{O}_C(K_C).$$

At this point we give a slight refinement of Saito's local Torelli theorem. The argument is essentially his.

THEOREM 2.10. *Suppose that $f : X \to C$ is a simple elliptic surface with r singular fibres. Put $L = f_*\omega_{X/C}$. Assume that $r \geq \deg L + 3$ and that $\deg L \geq 2$. Then the local Torelli theorem holds for X.*

Proof. From the main result of [Sai83] it is enough to consider the situation where the j-invariant is constant. Following [Sai83] it is enough to show that the natural homomorphisms

$$\mu_1 : H^0(C, f_*\Omega_X^2) \otimes H^1(C, f_*\Omega_X^1) \to H^1(C, f_*(\Omega_X^1 \otimes \Omega_X^2))$$

and

$$\mu_2 : H^0(C, f_*\Omega_X^2) \otimes H^0(C, R^1 f_*\Omega_X^1) \to H^0(C, R^1 f_*(\Omega_X^1 \otimes \Omega_X^2))$$

are surjective. Recall that $K_X \sim f^*(K_C + L)$, so that $f_*(\Omega_X^1 \otimes \Omega_X^2) \cong \mathcal{O}_C(K_C + L) \otimes f_*\Omega_X^1$, and that $|K_C + L|$ has no base points. □

LEMMA 2.11. *If $\mathcal{F}, \mathcal{G}$ are coherent sheaves on a 1-dimensional projective scheme C over a field k and if $\mathcal{F}$ is generated by $H^0(C, \mathcal{F})$, then the natural multiplication*

$$H^0(C, \mathcal{F}) \otimes_k H^1(C, \mathcal{G}) \to H^1(C, \mathcal{F} \otimes_{\mathcal{O}_C} \mathcal{G})$$

is surjective.

Proof. There is an exact sequence

$$H^0(C, \mathcal{F}) \otimes_k \mathcal{O}_C \to \mathcal{F} \to 0;$$

tensoring this with $\mathcal{G}$ gives an exact sequence

$$H^0(C, \mathcal{F}) \otimes_k \mathcal{G} \to \mathcal{F} \otimes \mathcal{G} \to 0.$$

Taking H^1 of this sequence gives the result.

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In particular, taking $\mathcal{F} = \Omega^1_C \otimes L = f_*\Omega^2_X$ and $\mathcal{G} = f_*\Omega^1_X$ shows that μ_1 is surjective.

Now consider μ_2. Set $A = \sum_1^r a_i$, the critical subset of C. The exact sequences (4.17) and (4.18) of [Sai83] are

$$0 \to \Omega^1_C \to f_*\Omega^1_X \to \mathcal{O}_C(L - A) \to 0$$

and

$$0 \to \mathcal{O}_C(K_C + A - L) \to R^1 f_*\Omega^1_X \to \mathcal{O}_C \oplus \mathcal{T}^1 \to 0,$$

where $\mathcal{T}^1$ is a skyscraper sheaf. The second sequence then gives

$$0 \to \mathcal{O}_C(K_C + A - L + \delta) \to R^1 f_*\Omega^1_X \to \mathcal{O}_C \to 0,$$

where $\delta \geq 0$, via the process of saturating a subsheaf. Since $\deg A > \deg L$, by assumption, this last sequence splits and

$$R^1 f_*\Omega^1_X = \mathcal{O}_C \oplus \mathcal{O}_C(K_C + A - L + \delta).$$

From its definition, μ_2 is then the direct sum

$$\mu_2 = \mu_2' \oplus \mu_2''$$

of multiplication maps

$$\mu_2' : H^0(C, \mathcal{O}_C(K_C + L)) \otimes \mathbb{C} \to H^0(C, \mathcal{O}_C(K_C + L))$$

and

$$\mu_2'' : H^0(C, \mathcal{O}_C(K_C + L)) \otimes H^0(C, \mathcal{O}_C(K_C - L + A + \delta)) \to H^0(C, \mathcal{O}_C(2K_C + A + \delta)).$$

The first of these is obviously surjective, while the surjectivity of the second follows from [Mum70, Theorem 6, p. 52] and the facts that $\deg(K_C + L) \geq 2q$ and $\deg(K_C - L + A + \delta) \geq 2q + 1$. □

3. The comparison between $\mathcal{SE}^{\mathrm{gen}}$ and $\mathcal{JE}^{\mathrm{gen}}$

The morphism $\pi : \overline{\mathcal{M}}_1 \to \overline{\mathcal{E}\ell\ell}$ defines a morphism $\Pi : \mathcal{SE}^{\mathrm{gen}} \to \mathcal{JE}^{\mathrm{gen}}$. If we fix a point $f : X \to C$ of $\mathcal{JE}^{\mathrm{gen}}$, then $\Pi^{-1}(X)$ is identified with $H^1(C, \mathcal{H})$, where $\mathcal{H} \to C$ is the Néron model of $X \to C$.

The sheaves $L^\vee$ and $\mathcal{H}$ are group schemes over C. Define $\mathcal{F} = R^1 f_*\mathbb{Z}$; this is a constructible sheaf of $\mathbb{Z}$-modules on C that is of generic rank 2.

The following results are due, in essence, to Kodaira. In particular, Proposition 3.2 is a variant of [Kod63, Theorem 11.7, p. 1341]; there he proves only that $H^2(C, \mathcal{F})$ is finite, but he does not assume that $f : X \to C$ is general.

LEMMA 3.1. *There is a short exact sequence*

$$0 \to \mathcal{F} \to L^\vee \to \mathcal{H} \to 0$$

of sheaves of commutative groups on C.

Proof. This follows from the exponential exact sequence on X and the identification $\mathcal{H} = \ker(R^1 f_*\mathcal{O}^*_X \to R^2 f_*\mathbb{Z})$. □

PROPOSITION 3.2. *We have $H^2(C, \mathcal{F}) = 0$ and the homomorphism $H^1(C, L^\vee) \to H^1(C, \mathcal{H})$ is surjective.*

Proof. It is enough to show that $H^2(C, \mathcal{F}) = 0$.

For any ring A, there is a Leray spectral sequence

$$E^{pq}_{2,A} = H^p(C, R^q f_* A) \Rightarrow H^{p+q}(X, A).$$

Since the fibres of f are irreducible curves and $X \to C$ has a section, the hypotheses of Théorème 1.1 of [DK, XVIII] are satisfied, so that this degenerates at E_2. Take $A = \mathbb{Z}$; then

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there is a short exact sequence

$$0 \to H^2(C, \mathcal{F}) \to H^3(X, \mathbb{Z}) \to H^1(C, \mathbb{Z}) \to 0.$$

Thus, $H^2(C, \mathcal{F})$ is identified with the torsion subgroup $\operatorname{Tors} H^3(X, \mathbb{Z})$ of $H^3(X, \mathbb{Z})$.

Suppose that ℓ is a prime dividing the order of $\operatorname{Tors} H^3(X, \mathbb{Z})$. Since $H^4(X, A)$ is isomorphic to A, it follows from taking cohomology of the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{.\ell} \mathbb{Z} \to \mathbb{Z}/\ell \to 0$$

that $H^3(X, \mathbb{Z}/\ell) \cong H^3(X, \mathbb{Z}) \otimes \mathbb{Z}/\ell$. In addition, Poincaré duality gives an isomorphism $H^3(X, \mathbb{Z}/\ell) \to H^1(X, \mathbb{Z}/\ell)^\vee$, where $^\vee$ denotes the dual $\mathbb{Z}/\ell$-vector space.

The spectral sequence $E^{pq}_{r, \mathbb{Z}/\ell}$ shows that $\beta : H^1(C, \mathbb{Z}/\ell) \to H^1(X, \mathbb{Z}/\ell)$ is injective, so if it is not surjective, then the map $i^* : H^1(X, \mathbb{Z}/\ell) \to H^1(\sigma, \mathbb{Z}/\ell)$ induced by the inclusion $i : \sigma \hookrightarrow X$ of the zero section is not injective. Then there is an étale $\mathbb{Z}/\ell$-cover $\alpha : \widetilde{X} \to X$ that is split over σ. Thus, $\widetilde{X} \to C$ is elliptic and has a section $\widetilde{\sigma}$ such that $N_{\widetilde{\sigma}, \widetilde{X}} \cong N_{\sigma/X}$. However,

$$\deg N_{\widetilde{\sigma}, \widetilde{X}} = -\chi(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = -\ell\chi(X, \mathcal{O}_X) = \ell \deg N_{\sigma/X}.$$

Thus, β is an isomorphism, so that $H^3(X, \mathbb{Z}/\ell) \cong (\mathbb{Z}/\ell)^{2q}$ and, therefore, $H^3(X, \mathbb{Z})$ is torsion-free. □

Let $\mathcal{SE}_{h,q}$ denote the substack of $\mathcal{SE}$ that consists of surfaces whose geometric genus is h and whose irregularity is q. This is a union of connected components of $\mathcal{SE}$.

The next result is an immediate corollary of Proposition 3.2.

Corollary 3.3.

(1) *The closed substack* $\Pi^{-1}(X)$ *is irreducible.*
(2) *The substack* $\mathcal{SE}_{h,q}$ *is irreducible.*

Now suppose that $Y \in \Pi^{-1}(X)$. According to [Kod63, Theorem 11.5, p. 1338], Y is algebraic if and only if it defines a torsion element of $H^1(C, \mathcal{H})$.

Proposition 3.4. *The algebraic surfaces are dense in* $\Pi^{-1}(X)$.

Proof. We must show that the image of $H^1(C, R^1 f_* \mathbb{Z}) \otimes \mathbb{Q}$ in $H^1(C, R^1 f_* \mathcal{O}_X)$ is dense.

Let $\xi^\perp$ denote the orthogonal complement of ξ in $H^2(X, \mathbb{Z})$. Then, via the Leray spectral sequence, there is a commutative square

$$\begin{array}{ccc} H^1(C, R^1 f_* \mathbb{Z}) \otimes \mathbb{Q} & \xrightarrow{\alpha} & H^1(C, R^1 f_* \mathcal{O}_X) \\ \cong \downarrow & & \downarrow \cong \\ (\xi^\perp / \mathbb{Z}\xi) \otimes \mathbb{Q} & \xrightarrow{\beta} & H^2(X, \mathcal{O}_X) \end{array}$$

where the vertical arrows are isomorphisms. Then β has dense image, from the Kähler property of X, and the proposition is proved. □

4. Schiffer variations for elliptic surfaces and the derivative of the period map

Fix a curve C, a point $a \in C$, a local coordinate z on C at a, an integer $e \geq 2$ and a real number δ such that $0 < \delta \ll 1$ and the region $|z| < \delta$ is an open disc $D = D_z$ in C. We begin by recalling, from [Gar49, p. 443], the construction of certain variations that we shall call *Schiffer variations*:

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these are families $\pi : \mathcal{C} \to \Delta^{e-1} = \Delta_{\underline{t}}^{e-1}$ over $(e-1)$-dimensional polydiscs with coordinates $\underline{t} = (t_2, \dots, t_e)$ whose closed fibre $\pi^{-1}(\underline{0})$ is C. They are constructed as follows.

In $\mathbb{C}_{\underline{t}}^{e-1}$ take the polydisc $\Delta_{\underline{t}}^{e-1}$ defined by $|t_j| < \delta^{4e}$ for all j. In $C \times \Delta_{\underline{t}}^{e-1}$ take the complement U of the closed subset $\bigcup_j (|z|^{4e} \le |t_j|)$. Thus, U is a thickening of the punctured curve $C - \{a\}$. Then put $G = U \cap (|z| < \delta)$.

In $\mathbb{C}_{z,w}^2 \times \Delta_{\underline{t}}^{e-1}$ take the subset F defined by

$$|t_j| < |z|^{4e} \ \forall j, \ |z| < \delta, \ |w - z| < |z|^{2e} \text{ and } w^e + e\sum_0^{e-2} t_{e-j} w^j = z^e.$$

Then there are projections $p : F \to U$ and $q : F \to \mathbb{C}_w \times \Delta_{\underline{t}}^{e-1}$.

LEMMA 4.1. *The projections p and q are open embeddings.*

Proof. It is enough to show that they are unramified and separate points.

The ramification locus of p is defined by

$$w^{e-1} + \sum j t_{e-j} w^{j-1} = 0.$$

Since $|t_{e-j}| < |z|^{4e}$ this gives $|w| < |z|^4$. Then $|z| - |z|^{2e} < |z|^4$, so that $z = 0$ and p is unramified.

To check the separation of points, suppose that

$$w_\alpha^e + e\sum t_{e-j} w_\alpha^j - z^e = 0$$

for $\alpha = 1, 2$ and $w_1 \ne w_2$. Then

$$\prod_{r=1}^{e-1} (w_1 - \zeta_e^r w_2) = e\sum_{j=1}^{e-2} t_{e-j} \prod_{s=1}^{j-1} (w_1 - \zeta_j^s w_2),$$

where, for any integer n, ζ_n is a primitive nth root of unity. Since $|w_1 - z| < |z|^{2e}$ and $|\zeta_j^s w_2 - \zeta_j^s z| < |z|^{2e}$ it follows that

$$|w_1 - \zeta_j^s w_2| < 2|z|^{2e} + |z - \zeta_j^s z| = 2|z|^{2e} + |1 - \zeta_j^s||z| < 4|z|.$$

Thus,

$$\left| e\sum_{j=1}^{e-2} t_{e-j} \prod_{s=1}^{j-1} (w_1 - \zeta_j^s w_2) \right| \le 4e(e-2)|z|^{4e+1}.$$

On the other hand, $|w_1 - z| < |z|^{2e}$ and $|\zeta_e^r - \zeta_e^r z| < |z|^{2e}$, so that

$$|w_1 - \zeta_e^r w_2| \ge |z - \zeta_e^r z| - 2|z|^{2e} \ge \lambda |z|,$$

where $\lambda = |\sin(2\pi/e)|/2$. Thus,

$$\left| \prod_{r=1}^{e-1} (w_1 - \zeta_e^r w_2) \right| \ge \lambda^{e-1} |z|^{e-1}$$

and, therefore,

$$\lambda^{e-1} \le 4e(e-2)|z|^{3e+2}.$$

This contradiction proves the result for p.

The argument for q is similar but easier, so we omit it. □

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LEMMA 4.2. *The image of p is G.*

Proof. This is a matter of showing that, given $(z, \underline{t}) \in G$, we can solve the equation $w^e + e\sum t_{e-j}w^j - z^e = 0$ with $|w - z| < |z|^{2e}$. Without the inequality there are e solutions; if $|w - z| \geq |z|^{2e}$ for all of them, then we get a contradiction to $z^e - w^e = e\sum t_{e-j}w^j$ and the inequalities satisfied by the t_{e-j}. □

For each $\underline{t} \in \Delta_{\underline{t}}^{e-1}$ the intersection $q(F) \cap (\mathbb{C}_w \times \{\underline{t}\})$ is an annulus $A_{\underline{t}}$ in $\mathbb{C}_w \times \{\underline{t}\}$ that surrounds zero. Let $R_{\underline{t}}$ denote the open simply connected region in $\mathbb{C}_w \times \{\underline{t}\}$ that contains 0 and has the same outer boundary as $A_{\underline{t}}$, and put $H = \bigsqcup_{\underline{t}} R_{\underline{t}}$. Thus, H is a tubular neighbourhood of $\{0\} \times \Delta_{\underline{t}}^{e-1}$ in $\mathbb{C}_w \times \Delta_{\underline{t}}^{e-1}$ and $q(F) \subseteq H$.

Define $\mathcal{C}$ to be the result of glueing U and H together along F via the maps p and q. This is Hausdorff and, after shrinking $\Delta_{\underline{t}}^{e-1}$ if necessary, the morphism $\pi : \mathcal{C} \to \Delta_{\underline{t}}^{e-1}$ is proper and is the morphism that we sought. This is sometimes expressed by saying that $\mathcal{C}_{\underline{t}}$ is constructed from C by deleting a small z-disc around a and glueing in a w-disc, where w is defined implicitly by $w^e + e\sum_0^{e-2} t_{e-j}w^j = z^e$.

Until after Theorem 4.11 we fix a point $f : X \to C$ of $\mathcal{JE}^{\text{gen}}$ and a point a in the ramification divisor $Z = \text{Ram}_\phi$ of the classifying morphism $\phi : C \to \overline{\mathcal{E}\ell\ell}$. In particular, ϕ is unramified over $j = \infty$. Put $E_a = f^{-1}(a)$ and denote by $e = e(a)$ the ramification index at a of ϕ, so that a has multiplicity $e - 1$ in Z. Assume that the disc D is sufficiently small, so that it contains no other points of Z. We use the Schiffer variations of C that we have just described to construct variations of X.

PROPOSITION 4.3. *For some choice of local coordinate z on C at a the morphism $\phi : C \to \overline{\mathcal{E}\ell\ell}$ lifts to a morphism $\Phi : \mathcal{C} \to \overline{\mathcal{E}\ell\ell}$ in such a way that the restriction of Φ to U factors through the projection $U \to C - \{a\}$.*

Proof. Given a local coordinate s on $\overline{\mathcal{E}\ell\ell}$ we have $\phi^* s = z^e$ for some local coordinate z on C. Then we define Φ on H by

$$\Phi^* s = w^e + e\sum_{j=0}^{e-2} t_{e-j}w^j$$

and on U we define Φ by composing ϕ with the projection $U \to C - \{a\}$. □

Any Jacobian deformation of the surface X determines, for each point $a \in Z$, a deformation of the finite scheme $V_{e(a)} = \mathbf{Spec}\mathbb{C}[t]/(t^{e(a)})$, so that there is a morphism of local analytic deformation spaces $\Psi : \text{Def}_X \to \prod_{a \in Z} \text{Def}_{V_{e(a)}}$. Recall that $\text{Def}_{V_{e(a)}}$ is smooth of dimension $e(a) - 1$.

PROPOSITION 4.4.

(1) *The morphism Ψ is a local analytic isomorphism.*
(2) *The universal ramification divisor $\mathcal{Z}$ is smooth over $\mathbb{C}$.*

Proof. Part (1) is a an immediate consequence of the formula used to define $\Phi^* s$, and part (2) follows at once. □

Thus, we have morphisms $\mathcal{X} \xrightarrow{F} \mathcal{C} \to \Delta_{\underline{t}}^{e-1}$ and $F : \mathcal{X} \to \mathcal{C}$ is a family, parametrized by $\Delta_{\underline{t}}^{e-1}$, of Jacobian elliptic surface whose fibre over $0 \in \Delta_{\underline{t}}^{e-1}$ is $f : X \to C$.

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Fix a suitable basis of $H_2(X,\mathbb{Z})$, which we identify with $H_2(\mathcal{X}_{\underline{t}},\mathbb{Z})$. Then there are holomorphic $(e+1)$-forms $\Omega^{(1)},\dots,\Omega^{(h)}$ on $\mathcal{X}$ such that the residues

$$\omega^{(j)}(\underline{t}) = \mathrm{Res}_{\mathcal{X}_{\underline{t}}}\begin{pmatrix}\Omega^{(j)}\\ \underline{t}\end{pmatrix}$$

form a normalized basis of $H^0(\mathcal{X}_{\underline{t}},\Omega^2_{\mathcal{X}_{\underline{t}}})$ for all $\underline{t}$. In particular, there are 2-cycles $A_1,\dots,A_h$ on $\mathcal{X}_{\underline{t}}$ such that $\int_{A_i}\omega^{(j)}(\underline{t}) = \delta_i^j$, the Kronecker delta.

Since the line bundle $\Omega^{e+1}_{\mathcal{X}}$ pulls back from a line bundle on $\mathcal{C}$, we can expand $\Omega^{(j)}$ as

$$\Omega^{(j)} = (-1)^{e-1}\sum_{p\ge 0,\underline{q}\ge\underline{0}} b^{(j)}_{p,\underline{q}} w^p \underline{t}^{\underline{q}}\, dw\wedge d\underline{t}\wedge dv,$$

where v is a fibre coordinate.

LEMMA 4.5.

(1) *We have* $w = z(1-\sum_{i=0}^{e-2} t_{e-i}z^{i-e}) = z(1-\sum_{i=2}^{e} t_i z^{-i})$ *modulo* $(\underline{t})^2$.
(2) *We have* $dw\wedge d\underline{t}\wedge dv = (1+\sum_{i=2}^{e}(i-1)t_i z^{-i})\,dz\wedge d\underline{t}\wedge dv$ *modulo* $(\underline{t})^2$.

Proof. Immediate from $w^e + e\sum t_{e-i}w^i = z^e$. □

Remark. Lemma 4.5(1) shows that, in terms of $H^1(C,T_C)$, the first-order deformation obtained from $\mathcal{C}\to\Delta_{\underline{t}}^{e-1}$ is the one that arises from integrating the space of Čech 1-cocycles with values in T_C, with respect to the cover $\{D,\ C-\{a\}\}$ of C, that is generated by the vector fields $z^{-1}\,d/dz,\dots,z^{-(e-1)}\,d/dz$ on $D-\{a\}$. However, there are different kinds of Schiffer variation that lead to the same space of cocycles; this feature is part of their strength.

We substitute this into the expansion of $\Omega^{(j)}$. Since

$$\Big(1-\sum t_i z^{-i}\Big)^p\Big(1+\sum(i-1)t_i z^{-i}\Big) = 1+\sum(i-p-1)t_i z^{-i}$$

modulo $(\underline{t})^2$, we get

$$\omega^{(j)}(\underline{t}) = \sum b^{(j)}_{p,\underline{q}} z^p\Big(1+\sum(i-p-1)t_i z^{-i}\Big)\underline{t}^{\underline{q}}\,dz\wedge dv. \tag{4.6}$$

modulo $(\underline{t})^2$.

Next, write $\omega^{(j)}(t) = \omega^{(j)} + \sum_{i=0}^{e-2} t_{e-i}\eta^{(j)}_{e-i}$ modulo $(\underline{t})^2$, where

$$\omega^{(j)} = \omega^{(j)}|_{\underline{t}=\underline{0}} \quad\text{and}\quad \eta_i^{(j)} = \frac{\partial}{\partial t_i}\omega^{(j)}|_{\underline{t}=\underline{0}}.$$

Moreover, every 2-cycle γ on X that is disjoint from E is identified, via a C^∞ collapsing map, with a 2-cycle on $\mathcal{X}_{\underline{t}}$ and

$$\int_\gamma \omega^{(j)}(t) = \int_\gamma \omega^{(j)} + \sum_i t_i\int_\gamma \eta_i^{(j)} \quad\text{for all such } \gamma. \tag{4.7}$$

Consider the class $[\omega^{(j)}(t)]\in H^2(\mathcal{X}_{\underline{t}},\mathbb{C})$. Let $\mathcal{E}\subset\mathcal{X}_{\underline{t}}$ be a fibre. Then $[\omega^{(j)}(t)]\in\mathcal{E}^\perp$. Moreover, from the exact sequence

$$\mathbb{Z}\xi = \mathbb{Z}[E_a]\to H^2(X,\mathbb{Z})\to H^2(X-E_a,\mathbb{Z})$$

and the formula (4.7) it follows that $\eta_i^{(j)}$ defines a class

$$[\eta_i^{(j)}]\in\xi^\perp/\mathbb{Z}\xi = H^2_{\mathrm{prim}}(X)$$

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such that

$$\frac{\partial}{\partial t_i}([\omega^{(j)}(t)])|_{t=0} = [\eta_i^{(j)}] \pmod{\mathbb{Z}\xi}.$$

Griffiths transversality shows that, in fact, $[\eta_i^{(j)}]$ lies in $Fil^1 = Fil^1 H^2_{\text{prim}}(X)$. (We let Fil^i refer to the i'th piece of the Hodge filtration of $H^2_{\text{prim}}(X)$.)

We have constructed, for each $a \in Z$ of ramification index $e(a) = e$, an $(e-1)$-parameter variation $\mathcal{X} \to \Delta^{e-1}$ of X such that the tangent space $T_{\Delta^{e-1}}(\underline{0})$ is identified with the cyclic skyscraper sheaf L_a of length $e-1$ on C that is supported at a and determined by Z. For any $v \in L_a$ consider the derivative

$$\nabla_v : H^{2,0}(X) = Fil^2 \to Fil^1.$$

Let

$$\overline{\nabla}_v : H^{2,0} \to H^{1,1}_{\text{prim}}(X)$$

denote the composite of ∇_v with the projection $Fil^1 \to H^{1,1}_{\text{prim}}(X)$. If L_a is of length 1, then we write ∇_a rather than ∇_v.

Assume that the geometric genus h of X is not zero. In addition, given $\omega \in H^0(X, \Omega^2_X)$ and $P \in C$, put $\omega(P) = (\omega/dz \wedge dv)(Q)$ for an arbitrary point $Q \in E_P$ and identify ω with the pullback of a tensor on C. Denote by $\sigma \subset X$ the given zero-section of $f : X \to C$.

Proposition 4.8. *Suppose that $0 \neq \omega \in H^0(X, \Omega^2_X)$ and that $(\omega)_0$ is disjoint from Z. Then the cup product $\omega : H^1(X, \Theta_X(-\log \sigma)) \to H^{1,1}_{\text{prim}}(X)$ is injective.*

Proof. Up to now we have identified the tangent space $T_{\mathcal{JE}}(X)$ with a line bundle on Z; we can also identify it with $H^1(X, \Theta_X(-\log \sigma))$. Since $\Theta_X(-\log \sigma) \otimes \Omega^2_X$ is isomorphic to $\Omega^1_X(\log \sigma)(-\sigma)$ we get a short exact sequence

$$0 \to \Theta_X(-\log \sigma) \to \Omega^1_X(\log \sigma)(-\sigma) \to \bigoplus_{P \in (\omega)_0} \mathcal{F}_P \to 0,$$

where $\mathcal{F}_P$ is a rank 2 vector bundle on E_P that fits into a short exact sequence

$$0 \to \mathcal{O}_{E_P}(-\sigma_P) \to \mathcal{F}_P \to \mathcal{O}_{E_P} \to 0.$$

The coboundary map $H^0(E_P, \mathcal{O}_{E_P}) \to H^1(E_P, \mathcal{O}_{E_P}(-\sigma_P))$ is identified with the Kodaira–Spencer map, so is an isomorphism from our assumption about $(\omega)_0$. Thus, $H^0(E_P, \mathcal{F}_P) = 0$ and then the homomorphism

$$H^1(X, \Theta_X(-\log \sigma)) \to H^1(X, \Omega^1_X(\log \sigma)(-\sigma))$$

is injective. This homomorphism factors through the cup product with which we are concerned, and the proposition is proved. □

Write $b^{(j)}_{p,\underline{0}} = b_p$. Then $\omega^{(j)} = \sum b^{(j)}_{p \geq 0} z^p dz \wedge dv$ and $b^{(j)}_0 = \omega^{(j)}(a)$.

Theorem 4.9.

(1) *We have $\eta_i^{(j)} \in H^0(X, \Omega^2_X(iE_a))_{\text{2nd kind}}$.*
(2) *We have $\eta_i^{(j)} = \sum_{p \geq 0}(i - p - 1) b_p^{(j)} z^{p-i}\, dz \wedge dv$ modulo $H^0(X, \Omega^2_X)$.*
(3) *The classes $[\eta_2^{(1)}], \ldots, [\eta_2^{(h)}]$ span a line $\mathbb{C}[\eta_{a,2}]$ in Fil^1/Fil^2 where $\eta_{a,2} \in H^0(X, \Omega^2_X(2E_a))_{\text{2nd kind}}$.*

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(4) *More generally, for each* ℓ *with* $2 \le \ell \le e(a)$, *there exists a form* $\eta_{a,\ell} \in H^0(X, \Omega^2_X(\ell E_a))_{\text{2nd kind}}$ *such that the classes* $[\eta_k^{(j)}]_{j\in[1,h],k\in[2,\ell]}$ *span an* $(\ell-1)$*-dimensional subspace* $\mathbb{C}.\{[\eta_{a,2},\dots,[\eta_{a,\ell}]\}$ *of* $H^{1,1}_{\text{prim}}(X)$.

Proof. This follows from further consideration of the expansion (4.6). To begin, write $b^{(j)}_{p,\underline{0}} = b^{(j)}_0$. Then we get

$$\omega^{(j)}(\underline{t}) = \sum_{p,\underline{q}} b^{(j)}_{p,\underline{q}} z^p \left(1 + \sum_i (i-p-1) t_i z^{-i}\right) dz \wedge dv,$$

modulo $(\underline{t}).H^0(X, \Omega^2_X)$, so that

$$\omega^{(j)} = \sum_p b^{(j)}_p z^p \, dz \wedge dv \quad \text{and} \quad \eta^{(j)}_i = \sum_p (i-p-1) b^{(j)}_p z^{p-i} \, dz \wedge dv,$$

where the second equality holds modulo $H^0(X, \Omega^2_X)$. Observe that

$$\dim H^0(X, \Omega^2_X(iE_a)) \le h + i \quad \text{and} \quad H^0(X, \Omega^2_X(E_a))_{\text{2nd kind}} = H^0(X, \Omega^2_X),$$

so that $\dim H^0(X, \Omega^2_X(iE_a))_{\text{2nd kind}} \le h + i - 1$. Then inspection of these coefficients and the linear independence provided by Proposition 4.8 are enough to prove the theorem. □

We can restate all this in more intrinsic terms, as follows. Let $\underline{\omega}$ denote the vector $[\omega^{(1)}, \dots, \omega^{(h)}]$ and $\underline{\omega}_{(i)}$ its ith derivative with respect to z.

THEOREM 4.10. *For each* $a \in Z$ *and each* $k = 2, \dots, e(a)$ *there is an explicit meromorphic* 2*-form*

$$\eta_{a,k} \in H^0(X, \Omega^2_X(kE_a))_{\text{2nd kind}}$$

such that $[\eta_{a,k}]$ *lies in* Fil^1 *and the image of* $\overline{\nabla}_{\partial/\partial t_k} : Fil^2 \to Fil^1$ *is spanned by the classes* $[\eta_{a,2}], \dots, [\eta_{a,k}]$. *The kernel of* $\overline{\nabla}_{\partial/\partial t_k}$ *contains* $H^0(X, \Omega^2_X(-(k-1)E))$.

Moreover, given the identification $H^0(X, \Omega^2_X) \cong H^0(X, \Omega^2_X)^\vee$ *provided by the basis* $\underline{\omega}$, $\overline{\nabla}_{\partial/\partial t_k}$ *is, as an element of* $H^0(X, \Omega^2_X) \otimes Fil^1$, *a linear combination of the tensors* $\underline{\omega}(a) \otimes [\eta_{a,k}], \dots, \underline{\omega}_{(k-2)}(a) \otimes [\eta_{a,2}]$.

Finally, if $h + q - 1 \ge e(a)$, *then* $\overline{\nabla}_{\partial/\partial t_k}$ *is of rank* $k-1$.

Proof. As remarked, this is, except for the final statement, nothing more than a restatement in intrinsic terms of what we have just proved. For the final part we need to know that the vectors $\underline{\omega}(a), \dots, \underline{\omega}_{(e(a)-2)}$ are linearly independent. This follows from the cohomology of the exact sequence

$$0 \to \mathcal{O}_C(K_C + L - (e(a)-1)a) \to \mathcal{O}_C(K_C + L) \to \mathcal{O}_{(e(a)-1)a}(K_C + L) \to 0$$

and the assumption that $h + q - 1 \ge e(a)$. □

THEOREM 4.11.

(1) *If* $a, b \in Z$ *are distinct, then, for all* $v \in L_a$ *and for all* $\eta_{b,k}$ *with* $k \in [2, e(b)]$, $\overline{\nabla}_v \eta_{b,k}$ *is a linear combination of the classes* $\{[\eta_{a,i}]\}_{i\in[2,e(a)]}$ *and* $\{[\eta_{b,j}]\}_{j\in[2,e(b)]}$. *In particular,* $\overline{\nabla}_v \eta_{b,k}$ *lies in* Fil^1.
(2) *If* $a, b \in Z$ *are distinct, then the classes* $[\eta_{a,i}]$ *and* $[\eta_{b,k}]$ *are orthogonal.*
(3) *The classes* $[\eta_{a,i}]$ *form a basis of* $H^{1,1}_{\text{prim}}(X)$ *as* a *runs over the points of* Z *and* i *runs from* 2 *to* $e(a)$.
(4) *If* Z *is reduced, then the classes* $[\eta_{a,2}]$ *form an orthogonal basis of* $H^{1,1}_{\text{prim}}(X)$ *as* a *runs over the points of* Z.

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Proof. (1) We use the same family $\mathcal{X} \to \mathcal{C} \to \Delta^{e(a)-1}$ corresponding to the point a as before. This variation is constant outside a small neighbourhood of a and the morphism to $\overline{\mathcal{E}\ell\ell}$ is also constant outside this neighbourhood. Thus, the ramification locus Z is also constant there.

The point b moves in a family of points $b(\underline{t}) \in \mathcal{C}_{\underline{t}}$, each $b(\underline{t})$ being of constant multiplicity $e(b)-1$ in the ramification divisor $Z_{\underline{t}} \subset \mathcal{C}_{\underline{t}}$. The fibre E_b in X moves in a family $\mathcal{E}_{b(\underline{t})}$. Then, for each $k \in [2, e(b)]$,

$$\eta_{b(\underline{t}),k} = \eta_{b,k} + \sum t_i \nabla_{\partial/\partial t_i} \eta_{b(\underline{t}),k}$$

modulo t^2. Similarly to what we did before, we write

$$\eta_{b(\underline{t}),k} = \mathrm{Res}_{\mathcal{X}_{\underline{t}}} \begin{pmatrix} H_{b(\underline{t}),k} \\ \underline{t} \end{pmatrix}$$

for some meromorphic $(e(a)+1)$-form

$$H_{b(\underline{t}),k} \in H^0(\mathcal{X}, \Omega^{e(a)+1}_{\mathcal{X}}(k\mathcal{E}_{b(t)})).$$

Then the same kind of calculation in terms of a power series expansion as before shows that $\nabla_{\partial/\partial t_i}\eta_{b(\underline{t}),k}$ is an element of $H^0(X, \Omega^2_X(iE_a + kE_b))$ whose residues along both E_a and E_b are zero. Therefore, $\nabla_a(\eta_{b,k})$ is, modulo $H^{2,0}(X)$, a linear combination as described.

(2) Choose an element ω of $H^{2,0}(X)$ that does not vanish along E_a. Thus, $\nabla_{\partial/\partial t_i}\omega$ is a non-zero multiple of $\eta_{a,i}$. We can assume that $\nabla_{\partial/\partial t_i}\omega = \eta_{a,i}$. Now $\langle \omega, \eta_{b,k}\rangle = 0$, since $H^{2,0}(X)$ is orthogonal to $H^{1,1}(X)$, so that

$$0 = \langle \nabla_{\partial/\partial t_i}\omega, \eta_{b,k}\rangle + \langle \omega, \nabla_{\partial/\partial t_i}\eta_{b,k}\rangle = \langle \eta_{a,i}, \eta_{b,k}\rangle,$$

since, as we have just proved, $\nabla_{\partial/\partial t_i}\eta_{b,k} \in Fil^1$.

Parts (3) and (4) follow from the linear independence of the $[\eta_{a,i}]$ and the fact that there are N of them, where $N = \dim H^{1,1}_{\mathrm{prim}}(X)$. □

Remark. The fact that the $\eta_{a,i}$ form a basis of $H^{1,1}_{\mathrm{prim}}(X)$ follows from Theorem 3.24 and Remark 3.29 of [CZ79], or from Proposition 4.8. However, the orthogonality seems to be new.

Until now $f : X \to C$ has been a point in $\mathcal{JE}^{\mathrm{gen}}$. Now, however, allow X to have A_1-singularities, so that $f : X \to C$ is defined by a classifying morphism $\phi : C \to \overline{\mathcal{E}\ell\ell}$ that is simply ramified over $j = \infty$. Let $\widetilde{X} \to X$ be the minimal model, so that $\widetilde{X}$ has singular fibres of types I_1 and I_2 and $X \to C$ is the relative canonical model of $\widetilde{X} \to C$. Suppose that $D_1, \dots, D_r$ are the exceptional (-2)-curves on $\widetilde{X}$. Suppose that $a_1, \dots, a_r$ are the points in Z lying over $j = \infty$ and that $a_{r+1}, \dots, a_N$ are the other points of Z. For $a = a_i$ with $i \le r$ define $[\eta_a] = [D_i]$. The surface $\widetilde{X}$ is a point in the stack $\widetilde{\mathcal{JE}}$ whose points are minimal models of points of $\mathcal{JE}$. We can extend Theorem 4.11 to this context, as follows. For simplicity we state it with the assumption that Z is reduced.

Theorem 4.12. *Suppose that Z is reduced. Put $\eta_{a,2} = \eta_a$ for each $a \in Z$.*

(1) *Each point a in Z defines a line v_a in the tangent space $H^1(\widetilde{X}, T_{\widetilde{X}}) = T_{\mathcal{JE}}\widetilde{X}$ such that the covariant derivative*

$$\overline{\nabla}_a : H^2(\widetilde{X}, \Omega^2_{\widetilde{X}}) \to H^{1,1}_{\mathrm{prim}}(\widetilde{X})$$

is proportional to the rank one tensor $\omega_a^\vee \otimes \eta_a$.

(2) *The classes $[\eta_a]_{a\in Z}$ form an orthogonal basis of $H^{1,1}_{\mathrm{prim}}(\widetilde{X})$.*

Proof. We only need to prove part (1) when $a = a_i$ for $i \le r$. Regard the surface $\widetilde{X}$ as the specialization of a surface in $\mathcal{JE}^{\mathrm{gen}}$. It follows by continuity that $\overline{\nabla}_a = \omega_a^\vee \otimes \theta_a$ for some class θ_a. To see that θ_a is proportional to $[D_a]$ we argue as follows.

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Suppose that $\Gamma = \sum \Gamma_j$ is a configuration of (-2)-curves on a smooth surface $\widetilde{X}$ that contracts to a single du Val singularity $P \in X$. Then we identify $H^2(\widetilde{X}, \Omega^2_{\widetilde{X}}) = H^0(X, \omega_X)$ and put

$$V(P) = H^0(X, \omega_X)/\mathfrak{m}_P H^0(X, \omega_X).$$

There is a short exact sequence

$$0 \to H^1(\widetilde{X}, T_{\widetilde{X}}(-\log \Gamma)) \to H^1(X, T_X) \to \oplus H^1(\Gamma_j, \mathcal{N}_{\Gamma_j/\widetilde{X}}),$$

which gives the following commutative diagram.

$$\begin{array}{ccc} H^1(\widetilde{X}, T_{\widetilde{X}}) \otimes H^0(X, \omega_X) & \longrightarrow & H^1(\widetilde{X}, \Omega^1_{\widetilde{X}}) \\ \downarrow & & \downarrow \\ H^1(\widetilde{X}, T_{\widetilde{X}})/H^1(\widetilde{X}, T_{\widetilde{X}}(-\log \Gamma)) \otimes V(P) & \longrightarrow & \oplus \mathbb{C}[\Gamma_j]^\vee \end{array}$$

Taking $\Gamma = D_a$ leads to the fact that θ_a is proportional to $[D_a]$.

Part (2) is proved as in Theorem 4.11. □

5. A local Schottky theorem

In this section we use the coincidence that $\dim H^{1,1}_{\text{prim}} = \dim \mathcal{JE}$ to put further structure on the derivative of the period map.

The vector spaces $H^{1,1}_{\text{prim}}(X)$ fit together into a vector bundle $\mathcal{H} = \mathcal{H}^{1,1}_{\text{prim}}$ on $\mathcal{JE}^{\text{gen}}$. We shall restrict attention to the open substack $\mathcal{JE}^{ss}$ of $\mathcal{JE}^{\text{gen}}$ over which the universal ramification divisor $\mathcal{Z}$ is étale, so that, under the projection $\rho : \mathcal{Z}^{ss} \to \mathcal{JE}^{ss}$, the sheaf $\rho_* \mathcal{O}_{\mathcal{Z}^{ss}}$ is a sheaf of semi-simple rings on $\mathcal{JE}^{ss}$.

PROPOSITION 5.1. *On $\mathcal{JE}^{ss}$ the vector bundle $\mathcal{H}$ is naturally a line bundle $\mathcal{B}$ on $\mathcal{Z}^{ss}$.*

Proof. Essentially, we do this componentwise, using the orthogonal basis of $H^{1,1}_{\text{prim}}(X)$ that is provided by the classes $([\eta_a])_{a \in Z}$ described in the previous section.

Let $x \in \mathcal{JE}^{ss}$, and choose an analytic neighbourhood U of x such that $\rho^{-1}(U)$ is a disjoint union

$$\rho^{-1}(U) = \bigsqcup_{a \in Z} U_a$$

of copies of U.

Suppose $h \in \rho_* \mathcal{O}_{\mathcal{Z}}$. So $h|_{U_a}$ is identified with a holomorphic function h_a defined on U. If $s \in \Gamma(U, \mathcal{H})$, then we can write $s = \sum f_a[\eta_a]$ where f_a is a function on U. Now define hs by the formula

$$hs = \sum h_a f_a [\eta_a].$$ □

PROPOSITION 5.2. *There is a perfect $\mathcal{O}_{\mathcal{Z}^{ss}}$-bilinear pairing $\beta : \mathcal{B} \times \mathcal{B} \to \mathcal{O}_{\mathcal{Z}^{ss}}$ such that $Tr(\rho_* \beta)$ is the intersection product on $\mathcal{H}$.*

Proof. We define the pairing β in terms of the notation used in the proof of Proposition 5.1, as follows:

$$\beta(s,t) = \beta\left(\sum_a f_a[\eta_a], \sum_a g_a[\eta_a]\right) = \sum f_a g_a([\eta_a] \cup [\eta_a]).$$

It is clear that β is perfect. □

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Recall that, tautologically, the tangent bundle $T_{\mathcal{JE}^{\rm gen}}$ is naturally a line bundle $\mathcal{S}$ on $\mathcal{Z}$. That is, $T_{\mathcal{JE}^{\rm gen}} = \rho_*\mathcal{S}$. The derivative per_* of the period map is an $\mathcal{O}_{\mathcal{JE}^{\rm gen}}$-linear map

$$per_* : \rho_*\mathcal{S} \to \mathcal{H}om_{\mathcal{JE}^{\rm gen}}(\mathcal{H}^{2,0}, \rho_*\mathcal{B}).$$

Define

$$\mathcal{P} = \Omega^1_{\mathcal{C}/\mathcal{JE}^{\rm gen}} \otimes \Phi^* M,$$

where $\mathcal{C}$ is the pull back of the universal curve over $\mathcal{M}_q$ to $\mathcal{JE}^{\rm gen}$ and $\Phi : \mathcal{C} \to \overline{\mathcal{E}\ell\ell}$ is the classifying morphism. Then

$$\Omega^2_{\mathcal{X}/\mathcal{JE}^{\rm gen}} \cong F^*\mathcal{P},$$

where $F : \mathcal{X} \to \mathcal{C}$ is the universal Jacobian surface. There is an evaluation map

$$eval : \rho^*\mathcal{H}^{2,0} \to \mathcal{P}|_{\mathcal{Z}}$$

and this map is surjective.

PROPOSITION 5.3. *On the locus $\mathcal{JE}^{ss}$ there is a factorization of per_* given by a commutative diagram*

$$\begin{array}{ccc} \rho_*\mathcal{S} & \xrightarrow{\widetilde{per}_*} & \rho_*\mathcal{H}om_{\mathcal{Z}^{ss}}(\mathcal{P}|_{\mathcal{Z}^{ss}}, \mathcal{B}) \\ & \searrow^{per_*} & \downarrow{\scriptstyle eval^\vee} \\ & & \rho_*\mathcal{H}om_{\mathcal{Z}^{ss}}(\rho^*\mathcal{H}^{2,0}, \mathcal{B}) \end{array}$$

where $\widetilde{per}_*$ is $\rho_*\mathcal{O}_{\mathcal{Z}^{ss}}$-linear.

Proof. This is a consequence of part (3) of Theorem 4.10. □

We have proved almost all of the next result. It is a local Schottky theorem in that it gives a precise description of the image of the tangent bundle to moduli under the period map.

THEOREM 5.4 (Local Schottky). *There is an isomorphism*

$$\widetilde{\widetilde{per}}_* : \mathcal{S} \to \mathcal{H}om_{\mathcal{Z}^{ss}}(\mathcal{P}|_{\mathcal{Z}^{ss}}, \mathcal{B})$$

of line bundles on $\mathcal{Z}^{ss}$ such that

$$\widetilde{per}_* = \rho_*\widetilde{\widetilde{per}}_*.$$

Proof. It is clear from the preceding discussion that there is a homomorphism $\widetilde{\widetilde{per}}_*$ of line bundles on $\mathcal{Z}^{ss}$ with the desired properties. Since per_* is injective on fibres, so is $\widetilde{\widetilde{per}}_*$, and we are done. □

6. Recovering Ram_ϕ and C from the infinitesimal period data

For the rest of this paper we make the assumptions (1.3).

Suppose that $f_i : X_i \to C_i$ are two points of $\mathcal{JE}^{ss}$ that have equal geometric genus h and irregularity q, and that each classifying morphism $\phi_i : C_i \to \overline{\mathcal{E}\ell\ell}$ is simply ramified. Say $\mathrm{Ram}_{\phi_i} = Z_i$ and $\deg Z_i = N$, so that $N = 10h + 8(1-q)$. Recall that $L_= \phi_i^* M$. Our assumptions ensure that Z_i is reduced and the linear system $|K_{X_i}| = f_i^*|K_{C_i} + L_i|$ has no base points and embeds C_i in $\mathbb{P}^{h-1}$ as the canonical model of X_i. Note that $\deg C_i = h + q - 1$.

THEOREM 6.1. *Assume that both surfaces have the same infinitesimal variation of Hodge structure (IVHS). Then $C_1 = C_2$ and $Z_1 = Z_2$.*

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Proof. Our assumptions mean that the two surfaces give the same point and the same tangent space under the period map after each cohomology group $H^2(X_i, \mathbb{Z})$ has been appropriately normalized. In particular, such a normalization gives a normalized basis $\underline{\omega}_i = [\omega_i^{(1)}, \dots, \omega_i^{(h)}]$ of $H^0(X_i, \Omega^2_{X_i})$ for each i. We can therefore regard the curves C_i as embedded in the same projective space $\mathbb{P}^{h-1}$. That is, any point P of C_i is identified with the point $\underline{\omega}_i(\widetilde{P})$ in $\mathbb{P}^{h-1}$, where $\widetilde{P} \in X$ is any point of X that lies over P. The basis $\underline{\omega}_i$ also gives an identification of $H^0(X_i, \Omega^2_{X_i})$ with its dual.

We show first that $Z_1 = Z_2$. We then deduce, by quadratic interpolation, that $C_1 = C_2$.

By the results of § 4, especially Theorem 4.10, the tangent space $T_{\mathcal{JE}} X_1$ is of dimension N and its image in $H^0(X_1, \Omega^2_{X_1})^\vee \otimes H^{1,1}_{\text{prim}}(X_1)$ is a direct sum $\bigoplus_{a \in Z_1} L_a$ where L_a is a line described by Theorem 4.10. We make the identifications

$$H^0(X_1, \Omega^2_{X_1}) = H^0(X_2, \Omega^2_{X_2}) = U \quad \text{and} \quad H^{1,1}_{\text{prim}}(X_1) = H^{1,1}_{\text{prim}}(X_2) = V$$

and assume that the tangent spaces $T_{\mathcal{JE}} X_1$ and $T_{\mathcal{JE}} X_2 = \bigoplus_{b \in Z_2} M_b$ are equal as subspaces of $U \otimes V$. We know that L_a is spanned by the rank 1 tensor $u_0^{(i)} \otimes v_0^{(i)}$ (the image of $\overline{\nabla}_{\partial/\partial t_2}$). Similarly, M_b is spanned by a rank 1 tensor $p_0^{(j)} \otimes q_0^{(j)}$. □

LEMMA 6.2. *Suppose that:*

(1) *U and V are vector spaces such that $\dim U = h$ and $\dim V = N$;*
(2) *$u_1, \dots, u_N \in U$;*
(3) *no two of the u_i are linearly dependent;*
(4) *$v_1, \dots, v_N \in V$ and form a basis of V;*
(5) *$\xi \in U \otimes V$ and there is a linear relation*

$$\xi = \sum_1^N \lambda_k u_k \otimes v_k$$

and, finally;
(6) *$\xi = x \otimes y$, a tensor of rank 1.*

Then there is a unique index i such that ξ is proportional to $u_i \otimes v_i$.

Proof. Let $(v_i^\vee)$ be the dual basis of $V^\vee$. There is an index j such that $\langle y, v_j^\vee \rangle \neq 0$. Then

$$\langle y, v_j^\vee \rangle x = \sum \lambda_k \delta_{jk} u_k = \lambda_j u_j,$$

so x is a multiple of u_j. Since no two of the u_i are linearly dependent, this index j is unique, so $y = \mu_j v_j$ and then $x \otimes v_j = \sum \lambda'_k u_k \otimes v_k$.

Say $Z_1 = \{a_1, \dots, a_N\}$ and $Z_2 = \{a'_1, \dots, a'_N\}$. Take $U = H^{2,0}(X_1)$ and $V = H^{1,1}_{\text{prim}}(X_1)$ and write

$$x_k = \underline{\omega}_1(a_k),\ x'_k = \underline{\omega}_2(a'_k), \quad y_k = [\eta_{1,a_k}] \quad \text{and} \quad y'_k = [\eta_{2,a'_k}]. \tag{6.3}$$

From Theorem 4.10 and the assumption that the IVHS of the two surfaces X_1 and X_2 are isomorphic, the tensors $x'_l \otimes y'_l$ span the same N-dimensional subspace of $U \otimes V$ as do the tensors $x_k \otimes y_k$. In particular, each $x'_l \otimes y'_l$ is a linear combination of the tensors $x_k \otimes y_k$. Then, by Lemma 6.2, for each index l there is a unique index m such that x_l is proportional to x'_m. That is, $a_l = a'_m$, and therefore $Z_1 = Z_2 = Z$, say.

By the assumptions (1.3) each C_i is non-degenerately embedded in $\mathbb{P}^{h-1}$ by a complete linear system, and $\deg C_i \geq 2q + 2$, so that, by the results of [Mum70] and [Sai72], C_i is an intersection

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of quadrics. Since

$$\deg Z > 2 \deg C_i,$$

again by (1.3), each C_i equals the intersection of the quadrics through Z, so that $C_1 = C_2$. □

We can reformulate this as follows.

Theorem 6.4. *If the ramification divisor* Ram_ϕ *of the classifying morphism* $\phi : C \to \overline{\mathcal{E}\ell\ell}$ *is reduced, then the IVHS of the surface* X *determines the base curve* C*, the divisor* Ram_ϕ *and the line bundle* $L = \phi^* M$.

Proof. What remains to be done is to show that we can recover the bundle L. For this, observe that our argument has shown that the IVHS of X determines the embedding $i : C \hookrightarrow \mathbb{P}^{h-1}$ as the canonical model of X and that $i^*\mathcal{O}(1) = \mathcal{O}_C(K_C + L)$. □

Remark. (1) Theorems 6.1 and 6.4 lead to the problem of trying to recover a morphism $\phi : C \to \overline{\mathcal{E}\ell\ell}$ from knowledge of C, the divisor Ram_ϕ and the line bundle L. However, it is impossible to do more than recover ϕ modulo the action of the automorphism group $\mathbb{G}_m$ of $\overline{\mathcal{E}\ell\ell}$. In this direction, we shall prove Proposition 8.7.

(2) If instead of Ram_ϕ being reduced we assume only that $h + q - 1 \geq e(a) + e(b) - 1$ for all $a, b \in \mathrm{Ram}_\phi$, then a refinement of the argument given here shows that Theorem 6.4 still holds.

7. The structure of the tangent bundle to $\mathcal{SE}^{\mathrm{gen}}$

Recall from § 3 that we have a morphism $\pi : \overline{\mathcal{M}}_1 \to \overline{\mathcal{E}\ell\ell}$ that gives, locally on $\overline{\mathcal{E}\ell\ell}$, an isomorphism $\overline{\mathcal{M}}_1 \to B\mathcal{G}$, so that the tangent complex $T^\bullet_{\overline{\mathcal{M}}_1}$ is locally isomorphic to the 2-term complex

$$0 \to \pi^* M^\vee[1] \to \pi^* T_{\overline{\mathcal{E}\ell\ell}}[0] \to 0$$

whose differential is zero. The morphism π determines a morphism

$$\Pi : \mathcal{SE}^{\mathrm{gen}} \to \mathcal{JE}^{\mathrm{gen}}.$$

Suppose that $f : X \to C$ corresponds to $\psi : C \to \overline{\mathcal{M}}_1$ and maps under Π to $Y \to C$. Say $\phi = \pi \circ \psi$. Observe that $\mathrm{Ram}_\phi = \mathrm{Ram}_\psi = Z$, say. The description just given of $T^\bullet_{\overline{\mathcal{M}}_1}$ leads to a short exact sequence

$$0 \to H^1(C, -L) \to T_{\mathcal{SE}} X \to T_{\mathcal{JE}} Y \to 0,$$

where $L = \phi^* M$. The subobject $H^1(C, -L)$ in this sequence is identified with the tangent space $T_{\Pi^{-1}(Y)} X$ to the fibre of Π though X. Thus,

$$T_{\Pi^{-1}(Y)}(X) \cong H^1(C, -L) \cong H^0(C, K_C + L)^\vee,$$

which is, in turn, naturally isomorphic to both $H^{2,0}(X)^\vee$ and to $H^{2,0}(Y)^\vee$. Let ξ denote the class of a fibre of $X \to C$; then the period map gives the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_{\Pi^{-1}(Y)}(X) & \longrightarrow & T_{\mathcal{SE}}(X) & \longrightarrow & T_{\mathcal{JE}}(Y) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle per_{X,*}} & & \downarrow{\scriptstyle per_{Y,*}} & & \\
0 & \longrightarrow & H^{2,0}(X)^\vee \otimes \xi & \longrightarrow & H^{2,0}(X)^\vee \otimes \xi^\perp & \longrightarrow & H^{2,0}(Y)^\vee \otimes H^{1,1}_{\mathrm{prim}}(Y) & \longrightarrow & 0
\end{array}$$

Theorem 7.1. *If* Z *is reduced and disjoint from the locus* $j = \infty$*, then the IVHS of the surface* X *determines the ramification locus* Z*, the base curve* C *and the line bundle* $\phi^* M$.

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Proof. Put $H^{2,0}(X)^\vee = H^{2,0}(Y)^\vee = U$ and $\xi^\perp = V$, so that $\dim U = h$ and $\dim V = N+1$. The image of $T_{\mathcal{SE}}(X)$ under $per_{X,*}$ is an $(N+h)$-dimensional subspace W of $U \otimes V$ such that W contains a subspace $U \otimes \xi$ and $W/U \otimes \xi$, which is the image of $T_{\mathcal{JE}}(Y)$ under $per_{Y,*}$, is spanned by rank 1 tensors

$$x_1 \otimes y_1, \dots, x_N \otimes y_N$$

where the vectors $y_1, \dots, y_N$ form a basis of $V/\xi = H^{1,1}_{\text{prim}}(Y)$. Let $\pi : V \to V/\xi$ denote the projection.

Suppose that the same IVHS arises also from another surface X'. Then there is a vector $\xi' \in V$ that arises from X such that $(1_U \otimes \pi)(U \otimes \xi')$ is a subspace of $U \otimes V/\xi$ and lies in the subspace of $U \otimes V/\xi$ that is spanned by the $x_i \otimes y_i$. However, by Lemma 6.2, the $x_i \otimes y_i$ are the only rank 1 tensors in $U \otimes V/\xi$, so that $(1_U \otimes \pi)(U \otimes \xi') = 0$ and, therefore, ξ' is proportional to ξ. Consider the vectors x_i', y_i' that arise from X'; then the tensors $x_i \otimes y_i$ and $x_i' \otimes y_i'$ lie in the same vector space $U \otimes V/\xi$, and then we can use Lemma 6.2 again to conclude the proof. □

Compare the case of $\mathcal{M}_q$: over the non-hyperelliptic locus Schiffer variations give a cone structure in the tangent bundle, where at each point C of $\mathcal{M}_q$ the corresponding cone is the cone over the bicanonical model of C, the generators of the cone map, under the period map, to tensors (quadratic forms) of rank 1 and, again, these account for all the rank 1 tensors in the image.

8. Recovering information from C and Z

We shall show (Proposition 8.7) that $\phi_1 : C \to \overline{\mathcal{E}\ell\ell}$ is generic and if $\phi_2 : C \to \overline{\mathcal{E}\ell\ell}$ is another morphism such that $\text{Ram}_{\phi_1} = \text{Ram}_{\phi_2}$ and $\phi_1^* M \cong \phi_2^* M$, then ϕ_1 and ϕ_2 are equivalent modulo the action of $\text{Aut}_{\overline{\mathcal{E}\ell\ell}} = \mathbb{G}_m$ provided that also there are sufficiently many points $a_i \in Z$ such that $\phi_1(a_i)$ is isomorphic to $\phi_2(a_i)$. Therefore an effective form of generic Torelli holds for Jacobian elliptic surfaces modulo this action of $\mathbb{G}_m$.

To begin, we rewrite some results of Tannenbaum [Tan84] in the context of Deligne–Mumford stacks. Assume that $\mathcal{S}$ is a smooth 2-dimensional Deligne–Mumford stack (the relevant example will be $\mathcal{S} = \overline{\mathcal{E}\ell\ell} \times \overline{\mathcal{E}\ell\ell}$), that C is a smooth projective curve and that $\pi : C \to \mathcal{S}$ is a morphism that factors as

$$\begin{array}{ccc} C & & \\ {\scriptstyle \pi'}\downarrow & \searrow^{\pi} & \\ D & \xrightarrow{i} & \mathcal{S} \end{array}$$

where D is a projective curve with only cusps, π' is birational and i induces surjections of henselian local rings at all points. (We shall say that 'π is birational onto its image'.) Then there is a conormal sheaf $\mathcal{N}_D^\vee$, a line bundle on D, which is generated by the pull back under i of the kernel $\mathcal{I}_D$ of $\mathcal{O}_\mathcal{S} \to i_*\mathcal{O}_D$. It fits into a short exact sequence

$$0 \to \mathcal{N}_D^\vee \to i^*\Omega^1_\mathcal{S} \to \Omega^1_D \to 0.$$

Thus, the adjunction formula (or duality for the morphism $i : D \to \mathcal{S}$) gives an isomorphism

$$\mathcal{N}_D \xrightarrow{\cong} \omega_D \otimes i^*\omega_\mathcal{S}^\vee.$$

There is also a homomorphism $\mathcal{N}_D \to T^1(D, \mathcal{O}_D)$; let $\mathcal{N}_D'$ denote its kernel. The space $H^0(D, \mathcal{N}_D')$ is the tangent space to the functor that classifies deformations of the morphism $i : D \to \mathcal{S}$ that are locally trivial in the étale (or analytic) topology.

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These conditions imply that there is a reduced effective divisor R on C such that $T_C(R) = T_C \otimes \mathcal{O}_C(R)$ is the saturation of T_C in $\pi^* T_{\mathcal{S}}$. Define $\mathcal{N}'_\pi$ by

$$\mathcal{N}'_\pi = \operatorname{coker}(T_C(R) \to \pi^* T_{\mathcal{S}});$$

this is a line bundle on C.

Define $\mathcal{J} \subset \mathcal{O}_{\mathcal{S}}$ to be the Jacobian ideal of the ideal $\mathcal{I}_D$. The next result is a slight variant of Lemma 1.5 of [Tan84].

LEMMA 8.1. *We have* $\pi'_* \mathcal{N}'_\pi = \mathcal{N}'_D$.

Proof. We have $\Omega^1_D = \mathcal{J}.\omega_D$, by direct calculation, and, from the definition of $\mathcal{N}'_D$, we have $\mathcal{N}'_D = \mathcal{J}.\mathcal{N}_D$.

From the definition, $\omega_C^\vee(R) \otimes \mathcal{N}'_\pi \cong \pi^* \omega_{\mathcal{S}}^\vee$. Let $\mathcal{C}$ denote the conductor ideal. Then $\omega_C \cong \mathcal{C} \otimes \pi'^* \omega_D$, so that

$$\mathcal{N}'_\pi \cong \pi'^* \omega_D \otimes \mathcal{C} \otimes \mathcal{O}_C(-R - K_{\mathcal{S}}).$$

Now $\mathcal{C} = \mathcal{O}_C(-R)$, by the nature of a cusp, and $\mathcal{J}.\mathcal{O}_C = \mathcal{O}_C(-2R)$ for the same reason. Thus,

$$\mathcal{N}'_\pi \cong \pi'^* (\omega_D \otimes i^* \omega_S^\vee) \otimes \mathcal{O}_C(-2R)$$

and, therefore,

$$\pi'_* \mathcal{N}'_\pi \cong \mathcal{J}.(\omega_D \otimes \omega_{\mathcal{S}}^\vee) \cong \mathcal{J}.\mathcal{N}_D = \mathcal{N}'_D. \qquad \square$$

COROLLARY 8.2. *The tangent space* $H^0(C, \mathcal{N}'_\pi)$ *is isomorphic to the tangent space of the deformation functor that classifies those deformations of the morphism* $\pi : C \to \mathcal{S}$ *where the length of the* $\mathcal{O}_C$*-module* $\operatorname{coker}(\pi^* \Omega^1_{\mathcal{S}} \to \Omega^1_C)$ *is preserved.*

Proof. By the lemma, $H^0(C, \mathcal{N}'_\pi)$ is the tangent space to locally trivial deformations of the morphism $i : D \to \mathcal{S}$. Since D has only cusps, a deformation of i is locally trivial if and only if it preserves the length of $\operatorname{coker}(\pi^* \Omega^1_{\mathcal{S}} \to \Omega^1_C)$. $\square$

PROPOSITION 8.3. *Fix a generic morphism* $\phi_1 : C \to \overline{\mathcal{E}\ell\ell}$ *and consider the morphisms* $\phi_2 : C \to \overline{\mathcal{E}\ell\ell}$ *such that* $(\phi_1, \phi_2) : C \to \overline{\mathcal{E}\ell\ell} \times \overline{\mathcal{E}\ell\ell}$ *is birational onto its image.*

(1) *If* $q \geq 2$, *there are no such morphisms* ϕ_2.
(2) *If* $q = 1$, *there are only finitely many such* ϕ_2.
(3) *If* $q = 0$, *then these morphisms* ϕ_2 *form a family of dimension at most* 3.

Proof. Assume that ϕ_2 exists. Take $\overline{\mathcal{E}\ell\ell} \times \overline{\mathcal{E}\ell\ell} = \mathcal{S}$ and $(\phi_1, \phi_2) = \pi$. Then there is a factorization

$$\begin{array}{ccc} C & & \\ {\scriptstyle \pi'}\downarrow & \searrow^{\pi} & \\ D & \xrightarrow{i} & \mathcal{S} \end{array}$$

as before, and the divisor R is $R = \operatorname{Ram}_{\phi_1} = \operatorname{Ram}_{\phi_2}$. We get

$$\begin{aligned} \deg \mathcal{N}'_\pi &= 2 \deg \phi_1^* T_{\overline{\mathcal{E}\ell\ell}} + 2q - 2 - N \\ &= 2.24(h + 1 - q).\frac{5}{12} + 2q - 2 - N \\ &= 10h + 10(1 - q), \end{aligned}$$

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so that

$$h^0(C, \mathcal{N}'_\pi) = 10h + 11(1-q) = N - (3q-3).$$

However, $\dim \mathcal{JE} = N$, and the proposition is proved. □

COROLLARY 8.4. *Fix $r \geq 0$ and suppose that $\phi_1 : C \to \overline{\mathcal{E}\ell\ell}$ is generic. Consider the morphisms $\phi_2 : C \to \overline{\mathcal{E}\ell\ell}$ such that $\mathrm{Ram}_{\phi_1} = \mathrm{Ram}_{\phi_2} = Z$, say, and there exist distinct points $a_1, \ldots, a_r \in Z$ such that $\phi_1(a_i)$ is isomorphic to $\phi_2(a_i)$ for all $i = 1, \ldots, r$ and, moreover, $(\phi_1, \phi_2) : C \to \overline{\mathcal{E}\ell\ell} \times \overline{\mathcal{E}\ell\ell}$ is birational onto its image.*

Then no such ϕ_2 exist provided that one of the following is true: $q \geq 2$; $q = 1$ and $r \geq 1$; or $q = 0$ and $r \geq 4$.

Proof. The further constraints on ϕ_2 imply that, in the notation of the proof of Proposition 8.3, the curve D has r cusps that lie on the diagonal of $\mathcal{S}$. Therefore, there are r further constraints on ϕ_2 and the corollary follows. □

LEMMA 8.5. *If $\phi : C \to \overline{\mathcal{E}\ell\ell}$ is generic, then there is no non-trivial factorization through a curve of the composite morphism $\gamma : C \to \mathbb{P}^1_j$.*

Proof. Suppose that

$$C \xrightarrow{\beta} \Gamma \xrightarrow{\alpha} \mathbb{P}^1_j$$

is a non-trivial factorization of $\gamma = \pi \circ \phi$. Say $\deg \alpha = a$ and $\deg \beta = b$. By the assumption of genericity, the divisor $Z = \mathrm{Ram}_\phi$ is reduced and its image $B = \gamma_* Z$ in $\mathbb{P}^1_j$ consists of distinct points, none of which equals j_4 or j_6.

Suppose α is branched over $y \in \mathbb{P}^1_j$ and $y \neq j_4, j_6$. Then $\gamma^{-1}(y) \subset Z$, so that $Z \to B$ is not one-to-one. Thus, α is branched only over j_4, j_6, and so is the cyclic cover of $\mathbb{P}^1_j$ that is branched at these two points and is of degree a. Since ϕ is étale over j_4, and over j_6 it follows that $a|2$ and $a|3$, a contradiction. □

Suppose that Δ is any irreducible curve of bidegree $(1,1)$ in $\mathbb{P}^1_j \times \mathbb{P}^1_j$, let $\widetilde{\Delta}'$ denote the fibre product

$$\widetilde{\Delta}' = \Delta \times_{\mathbb{P}^1_j \times \mathbb{P}^1_j} \overline{\mathcal{E}\ell\ell} \times \overline{\mathcal{E}\ell\ell}$$

and let $\widetilde{\Delta}$ be the normalization of $\widetilde{\Delta}'$. Note that, generically, $\widetilde{\Delta}$ is isomorphic to $\Delta \times B(\mathbb{Z}/2 \times \mathbb{Z}/2)$, so that $\deg(\widetilde{\Delta} \to \Delta) = \frac{1}{4}$.

LEMMA 8.6. *We have $\deg T_{\widetilde{\Delta}} = 5/24$ if Δ contains the points (j_4, j_4) and (j_6, j_6) and $\deg T_{\widetilde{\Delta}} \leq 1/12$ otherwise.*

Proof. We consider separately the cases according to which of the special points (j_4, j_4), (j_4, j_6), (j_6, j_4) and (j_6, j_6) lie on Δ and summarize the results in Table 1. Each case depends on local calculation. For example, suppose that (j_4, j_4) and $(j_6, j_6) \in \Delta$. Then $\widetilde{\Delta}'$ has a double point with an action of $(\mathbb{Z}/4)^2$ lying over (j_4, j_4) and a triple point with an action of $(\mathbb{Z}/6)^2$ over (j_6, j_6). In each case the branches are permuted transitively by the group, so that $\widetilde{\Delta}$ has one point with an action of $\mathbb{Z}/4 \times \mathbb{Z}/2$ and another with an action of $\mathbb{Z}/6 \times \mathbb{Z}/2$. Thus,

$$\begin{aligned}\deg T_{\widetilde{\Delta}} = \deg T_\Delta . \deg B((\mathbb{Z}/2)^2) + (-\deg B((\mathbb{Z}/2)^2) + \deg B(\mathbb{Z}/4 \times \mathbb{Z}/2)) \\ + (-\deg B((\mathbb{Z}/2)^2) + \deg B(\mathbb{Z}/6 \times \mathbb{Z}/2)) = 1/8 + 1/12 = 5/24.\end{aligned}$$

□

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Table 1.

Special points on Δ	$\deg T_{\widetilde{\Delta}}$
None	$-1/12$
(j_4, j_4)	$1/24$
One of (j_4, j_6) and (j_6, j_4)	0
(j_6, j_6)	$1/24$
(j_4, j_6) and (j_6, j_4)	$1/12$
(j_4, j_4) and (j_6, j_6)	$5/24$

Proposition 8.7. *Suppose that $\phi_1 : C \to \overline{\mathcal{E}\ell\ell}$ is generic and that $\phi_2 : C \to \overline{\mathcal{E}\ell\ell}$ is another morphism such that $\mathrm{Ram}_{\phi_1} = \mathrm{Ram}_{\phi_2} = Z$ and there are r points $a_i \in Z$ such that $\phi_1(a_i)$ is isomorphic to $\phi_2(a_i)$ for all i. Assume also that one of the following is true: $q \geq 2$; $q = 1$ and $r \geq 1$; or $q = 0$ and $r \geq 4$.*

(1) *The image of the composite morphism $C \to \mathbb{P}^1_j \times \mathbb{P}^1_j$ is a curve Δ of bidegree $(1, 1)$.*
(2) *The curve Δ passes through the points (j_4, j_4) and (j_6, j_6).*
Suppose also that $\phi_1^* M \cong \phi_2^* M$.
(3) *The morphism $(\phi_1, \phi_2) : C \to \overline{\mathcal{E}\ell\ell} \times \overline{\mathcal{E}\ell\ell}$ factors through the graph of an automorphism of $\overline{\mathcal{E}\ell\ell}$.*
(4) *We have that ϕ_1 and ϕ_2 are equivalent under the action of $\mathrm{Aut}_{\overline{\mathcal{E}\ell\ell}}$.*

Proof. Part (1) follows from Proposition 8.3 and Lemma 8.5.

(2) Suppose this is false. Then, by Lemma 8.6,

$$\dim_{(\phi_1,\phi_2)} \mathrm{Mor}(C, \widetilde{\Delta}) \leq 2 \deg \phi_1 \times \tfrac{1}{12} + 1 - q.$$

However, $\dim_{(\phi_1,\phi_2)} \mathrm{Mor}(C, \widetilde{\Delta}) \geq \dim_{\phi_1} \mathrm{Mor}(C, \overline{\mathcal{E}\ell\ell})$, a contradiction.

For parts (3) and (4) we can now use the $\mathbb{G}_m$-action to ensure that Δ is the diagonal. Then the two surfaces $X_i \to C$ have the same j-invariant, so that one is a quadratic twist of the other via a quadratic cover $\widetilde{C} \to C$. Since both surfaces are semistable, this quadratic cover is étale. Suppose it corresponds to the 2-torsion class P on C. From the identification of $\phi_i^* M$ with the conormal bundle of the zero section, it follows that $\phi_2^* M = \phi_1^* M + P$, so that $P = 0$, and the result is proved. □

9. Generic Torelli for Jacobian surfaces

Define $r(q) = 2$ if $q \geq 1$ and $r(0) = 4$. Let $\widetilde{\mathcal{JE}}_{C,L}$ (or $\mathcal{JE}_{C,L}$) denote the closed substack of $\widetilde{\mathcal{JE}}$ (or $\mathcal{JE}$) defined by the properties that the base of the elliptic fibration is C and $\phi^* M$ is isomorphic to L. Let $\widetilde{\mathcal{JE}}_r$ denote the substack of $\widetilde{\mathcal{JE}}$ with r fibres of type I_2 and put $\widetilde{\mathcal{JE}}_{C,L,r} = \widetilde{\mathcal{JE}}_{C,L} \cap \mathcal{JE}_r$.

Let $\mathcal{Z}' \to \mathcal{JE}_r$ be the restriction of the universal ramification divisor $\rho : \mathcal{Z} \to \mathcal{JE}$. Then $\mathcal{Z}'$ has a closed substack $\mathcal{Z}_r$ obtained by discarding the part of $\mathcal{Z}$ that lies over $j = \infty$.

Lemma 9.1. *The substacks $\widetilde{\mathcal{JE}}_r$, $\mathcal{JE}_r$, $\widetilde{\mathcal{JE}}_{C,L,r}$, $\mathcal{JE}_{C,L,r}$ and $\mathcal{Z}_r$ are irreducible when $r = r(q)$.*

Proof. We show first that $\mathcal{JE}_{C,L,r}$ is irreducible. For this, refer to the Weierstrass equation

$$y^2 = 4x^3 - g_4 x - g_6$$

of a Jacobian surface X over C. Fix distinct points $P_1, \ldots, P_r \in C$. Then X has fibres of type I_2 (or worse) over each P_i if and only if

$$g_4^3 - 27 g_6^2 = 2 g_4 g_6' - 3 g_6 g_4' = 0,$$

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where the prime denotes the derivative with respect to a local coordinate z at P_i. (For this, we use the description of $\mathcal{Z}$ in terms of transvectants that is given in § 10.) Expand g_n in terms of z to get

$$g_4 = a_0 + a_1 z + \text{h.o.t.}, \quad g_6 = b_0 + b_1 z + \text{h.o.t.};$$

then the condition on the fibres of X is expressed by a pair of equations (one pair for each P_i)

$$a_0^3 - 27b_0^2 = 2a_0b_1 - 3a_1b_0 = 0.$$

Since $r \leq 4$ these equations define an irreducible subvariety V_r of $H^0(C, 4L) \times H^0(C, 6L)$ and the irreducibility of $\mathcal{JE}_{C,L,r}$ is proved. The irreducibility of $\mathcal{JE}_r$ follows at once, and then so too does the irreducibility of $\widetilde{\mathcal{JE}}_{C,L,r}$ and $\widetilde{\mathcal{JE}}_r$.

Since $\mathcal{Z}_r \cap \mathcal{JE}_{C,L,r}$ is dominated by V_r, the irreducibility of $\mathcal{Z}_r$ is also proved. □

THEOREM 9.2.

(1) *Suppose that $\phi_i : C \to \overline{\mathcal{E}\ell\ell}$ is a morphism corresponding to the Jacobian surface $f_i : X_i \to C$, for $i = 1, 2$. Assume that $\mathrm{Ram}_{\phi_1} = \mathrm{Ram}_{\phi_2} = Z$, say, that Z is reduced and that $\phi_1^* M = \phi_2^* M$. Suppose that there are $r = r(q)$ points $a_1, \ldots, a_r \in Z$ such that $\phi_i(a_k)$ lies over $j = \infty$ for $i = 1, 2$ and $k = 1, \ldots, r$. Then X_1 is isomorphic to X_2.*
(2) *A generic Jacobian elliptic surface with $r(q)$ singular fibres of type I_2 is determined by its IVHS.*
(3) *Generic Torelli holds for Jacobian surfaces with $r(q)$ singular fibres of type I_2.*

Proof. Part (1) is a consequence of Proposition 8.7. Part (2) is merely a restatement of part (1), and then part (3) follows in the usual way. □

Next, we use the minimal model program to prove a version of the good reduction result from [Cha82] and [Cha84].

THEOREM 9.3. *Suppose that $\mathcal{X} \to \Delta$ is a semistable 1-parameter degeneration of Jacobian elliptic surfaces over a fixed curve C of genus q and that there is no monodromy on the second Betti cohomology of $\mathcal{X}_{\bar{\eta}}$. Assume also that, under the period map, the image of $0 \in \Delta$ is the primitive weight 2 Hodge structure of a Jacobian elliptic surface $g : V \to C$ such that $p_g(V) = p_g(\mathcal{X}_{\bar{\eta}})$ and this Hodge structure is irreducible.*

Then $\mathcal{X} \to \Delta$ has good reduction.

Proof. We can follow the proof of Theorem 9.1 of [She21] as far as the end of Lemma 9.9 of [She21]. This leads to a model $\mathcal{X} \to \Delta$ such that:

(1) $\mathcal{X}$ has $\mathbb{Q}$-factorial canonical singularities and $\mathcal{X}_0$ has semi log canonical (slc) singularities;
(2) there is a surface S with a proper semi–stable morphism $g : S \to \Delta$ and a projective morphism $f : \mathcal{X} \to S$ with only 1-dimensional fibres such that $K_{\mathcal{X}/\Delta}$ is the pullback under f of a g-ample $\mathbb{Q}$-line bundle L on S.

Then S_0 is a tree, and $S_0 = \sum C_i$ where $C_0 \cong C$ and $C_j \cong \mathbb{P}^1$ for all $j > 0$. Since S has only singularities of type A, each curve C_i is a $\mathbb{Q}$-Cartier divisor on S. Let $f_i : X_i = f^{-1}(C_i) \to C_i$ be the morphism induced by f. Thus, X_i is $\mathbb{Q}$-Cartier on $\mathcal{X}$ and $K_{\mathcal{X}/\Delta}|_{X_i} \sim f_i^* L_i$ for some $\mathbb{Q}$-line bundle L_i on C_i of degree $\alpha_i > 0$, where $\sum \alpha_i = h + q - 1$.

Since the Hodge structure on $H^2_{\mathrm{prim}}(V)$ is irreducible and, from the absence of monodromy, $\sum p_g(X_i) \leq p_g(V)$, there is a unique index i_0 such that the Hodge structure on $H^2(X_{i_0}, \mathbb{Q})$ maps onto that on $H^2(V, \mathbb{Q})$. It follows that $i_0 = 0$.

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Suppose that X_0 meets $X_1, \dots, X_s$. Since X_0 is $\mathbb{Q}$-Gorenstein, and even Gorenstein in codimension one, the adjunction formula shows that there exist $\delta_1, \dots, \delta_s \in \mathbb{Q}_{>0}$ such that $K_{X_0} \sim_{\mathbb{Q}} f_0^*(\alpha_0 - \sum \delta_i)$.

Let $\gamma : \widetilde{X} \to X_0$ denote the minimal resolution. Then there is an effective $\mathbb{Q}$-divisor D on $\widetilde{X}$ such that

$$K_{\widetilde{X}} \sim_{\mathbb{Q}} \gamma^* K_{X_0} - D \sim_{\mathbb{Q}} (f_0 \circ \gamma)^* \left(\alpha_0 - \sum \delta_i \right) - D.$$

Thus,

$$p_g(\widetilde{X}) \leq \lfloor \alpha_0 - \sum \delta_i \rfloor + 1 - q \leq \sum \alpha_j + 1 - q.$$

On the other hand,

$$p_g(\widetilde{X}) = p_g(V) = h = \sum \alpha_j + 1 - q.$$

Therefore, $s = 0$, so that $\mathcal{X}_0 = X_0$, and $D = 0$. Therefore, X_0 has only du Val singularities and the result is reduced to the existence of a simultaneous resolution for such singularities. □

Remark. There are degenerating families of elliptic surfaces $\mathcal{X} \to \mathcal{C} \to \Delta$ with no monodromy on H^2 but where $\mathcal{C}$ has bad reduction. For example, take an elliptic K3 surface $X \to \mathbb{P}^1$ with two isomorphic fibres of type $\overline{D}_4$. Then this can be plumbed to itself to give such a family: the generic fibre has $p_g = q = 1$ and $\mathcal{C} \to \Delta$ is the Tate curve.

Corollary 9.4. *The period map* $[per] : [\mathcal{JE}] \to [D/\Gamma]$ *of geometric quotients is proper and one-to-one over the image of the generic point of the locus of surfaces with* $r(q)$ *fibres of type* I_2.

Lemma 9.5. *Suppose that* $V \hookrightarrow Y$ *is a closed embedding of normal analytic spaces. Suppose also that* H *is a finite group and* G *a subgroup of* H *such that* H *acts on* Y, *that* G *preserves* V *and that, moreover,* $G = \{g \in H : g(V) = V\}$.

Then the morphism $[V/G] \to [Y/H]$ *of geometric quotients is bimeromorphic onto its image.*

Proof. We can replace Y by the largest open subspace Y^0 of Y such that Y^0 is preserved by H and $Y^0 \to [Y^0/H]$ is étale. Then the result is clear. □

Lemma 9.6. *A generic pair* (C, Z) *has no automorphisms.*

Proof. We consider the various values of q separately:

(1) $q \geq 3$; a generic curve has no automorphisms;
(2) $q = 2$; then Z is invariant under the hyperelliptic involution of C; in this case the dimension of the set Σ of pairs (C, Z) is at most $\dim \mathcal{M}_2 + N/2$, which is less than $\dim \mathcal{JE}$;
(3) $q = 1$; either Z is invariant under an involution, and then we argue as in the case where $q = 2$, or Z is invariant under a translation; in this case Z is determined by any one of its points and $\dim \Sigma \leq 2$;
(4) $q = 0$; suppose $1 \neq g \in PGL_2(\mathbb{C})$; if g fixes one point, then $\dim \Sigma \leq 2$, and if g fixes 2 points, then $\dim \Sigma \leq 3$. □

We now take V to be a miniversal deformation space of a surface $\widetilde{X}$ with $r = r(q)$ fibres of type I_2 and Y a germ of the domain D at the period point of $\widetilde{X}$.

Suppose that $H \subset O(H^2_{\text{prim}}(\widetilde{X}, \mathbb{Z}))$ is the automorphism group of the polarized Hodge structure of the $\widetilde{X}$ and that $G \subset H$ is the subgroup consisting of those elements of H that preserve the image of V in Y. Since H is finite, we can identify the germs Y and V, which are smooth, with their tangent spaces. In turn, $V = H^{2,0}(\widetilde{X})^\vee \otimes H^{1,1}_{\text{prim}}(\widetilde{X})$ and H acts on both components

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of this tensor product. Let $W = W(A_1^r)$ denote the Weyl group generated by reflexions $\sigma_{[\eta_a]}$ in the classes $[\eta_a]$ where a lies over $j = \infty$; then $\sigma_{[\eta_a]}$ lies in H, so that $W \subset H$. Also (± 1) is a subgroup of H and acts trivially on D.

PROPOSITION 9.7. *We have $G = (\pm 1) \times W$.*

Proof. Let $g \in G$. We know that the subspace V of Y is based by the rank 1 tensors $\omega_a^\vee \otimes [\eta_a]$ and that these are the only rank 1 tensors in V. Therefore, g permutes the lines $\mathbb{C}\omega_a^\vee$ and so acts on the pair (C, Z), where C is embedded in $\mathbb{P}^{h-1}$ via $|K_C + L|$. This action is trivial, by the previous lemma, and so g acts as a scalar on $H^{2,0}(\widetilde{X})$. We can then take this scalar to be 1. Then g fixes each line $\mathbb{C}[\eta_a]$. The classes $[\eta_a]$ are orthogonal and can be normalized by imposing the condition that $[\eta_a]^2 = -2$ for all $a \in Z$. This normalization is unique up to a choice of sign for each a. Since $\mathcal{Z}_r$ is irreducible, one choice of sign, for a point in Z_r, determines all the other signs, except when a lies over $j = \infty$. This last ambiguity is exactly taken care of by the Weyl group and so $G \subset (\pm 1) \times W$.

It is clear that $(\pm 1) \times W \subset G$. □

THEOREM 9.8 (Generic Torelli). *The period map $[per] : [\mathcal{JE}] \to [D/\Gamma]$ is birational onto its image.*

Proof. The theorem follows from Corollary 9.4 and the fact that, at the level of miniversal deformation spaces, the morphism $\widetilde{\mathcal{JE}} \to \mathcal{JE}$ is a quotient map by the relevant Weyl group. We conclude via Proposition 9.7 and the fact that (± 1) acts trivially. □

10. The variational Schottky problem for Jacobian surfaces

The Schottky problem is that of determining the image of a moduli space under a period map. As explained by Donagi [Don84], there is a variational approach to this. For curves of genus q that are neither hyperelliptic, trigonal nor plane quintics, his approach leads (see p. 257 of [Don84]) to the statement that the image of the variational period map lies in the Grassmannian that parametrizes $3q-3$-dimensional quotient spaces W of $\mathrm{Sym}^2 V$, where V is a fixed q-dimensional vector space, and the kernel of $\mathrm{Sym}^2 V \to W$ defines a smooth linearly normal curve C of genus q in $\mathbb{P}(V)$. (It follows from this that the embedding $C \hookrightarrow \mathbb{P}(V)$ can be identified with the canonical embedding of C.)

For Jacobian elliptic surfaces of geometric genus h and irregularity q we get something equally concrete.

Let $\mathcal{Z} \to \mathcal{JE}^{\mathrm{gen}}$ be the universal ramification locus, of degree $10 + 8(1-q)$ over $\mathcal{JE}^{\mathrm{gen}}$. The image of $\mathcal{Z}$ lies in a tensor product $U \otimes V$ where $\dim U = h$ and $\dim V = 10h + 8(1-q)$. Projecting to $\mathbb{P}(U) = \mathbb{P}^{h-1}$ leads to the following variational partial solution to the Schottky problem. The solution is only partial because this projection factors through the quotient stack $\mathcal{JE}^{\mathrm{gen}}/\mathbb{G}_m$.

Recall that $\mathcal{JE}^{\mathrm{gen}}$ can be described as follows.

Suppose that $\mathcal{P} = \mathcal{P}^{h+1-q} \to \mathcal{M}_q$ is the universal Picard variety of degree $h+1-q$ line bundles L on a curve C of genus q. Then writing the equation of X in affine Weierstrass form, namely as

$$y^2 = 4x^3 - g_4 x - g_6,$$

shows that $g_n \in H^0(C, nL)$ and $\mathcal{JE}^{\mathrm{gen}}$ is birationally equivalent to a $B(\mathbb{Z}/2)$-gerbe over a bundle over $\mathcal{P}$ whose fibre is the quotient stack

$$(H^0(C, 4L) \oplus H^0(C, 6L) - \{(0,0)\})/\mathbb{G}_m.$$

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The action of $\mathbb{G}_m$ on $\overline{\mathcal{E}\ell\ell}$ leads to an action of $\mathbb{G}_m$ on $\mathcal{JE}^{\mathrm{gen}}$ and the quotient $\mathcal{JE}^{\mathrm{gen}}/\mathbb{G}_m$ is birationally equivalent to a $B(\mathbb{Z}/2)$-gerbe over the universal $|4L| \times |6L|$-bundle over $\mathcal{P}$.

Theorem 10.1. *The image of the variational period map for Jacobian elliptic surfaces lies in the locus $\mathcal{V} = \mathcal{V}_{h,q}$ of zero-cycles Z in $\mathbb{P}^{h-1}$ such that:*

(1) $\deg Z = 10h + 8(1-q)$;
(2) *the intersection of the quadrics through Z is a curve C of genus q and degree $h+q-1$;*
(3) *the divisor Z on C is linearly equivalent to $10L + K_C$ and the hyperplane class H on C is $H = L + K_C$; thus, $Z \sim 10H - 9K_C$.*

Proof. This follows at once from the results of the previous section. □

We can make this more precise. First, recall the idea of *transvectants*: if N is a line bundle on a variety V over a field k and $m, n \in \mathbb{Z}$, then there is a homomorphism

$$N^{\otimes m} \otimes_k N^{\otimes n} \to \Omega^1_V \otimes_{\mathcal{O}_V} N^{\otimes m+n}$$

of sheaves on V defined, in terms of a local generator s of N, by

$$fs^{\otimes m} \otimes_k gs^{\otimes n} \mapsto (mf\,dg - ng\,df) \otimes_{\mathcal{O}_V} s^{\otimes m+n}.$$

Suppose that V is projective; then at the level of global sections this defines a k-linear homomorphism

$$H^0(V, N^{\otimes m}) \otimes_k H^0(V, N^{\otimes n}) \to H^0(V, \Omega^1_V \otimes_{\mathcal{O}_V} N^{\otimes m+n})$$

of finite-dimensional vector spaces. This morphism has several names, depending on the context; for example, the first transvectant, the first Ueberschiebung, the Jacobian determinant, the Gauss map, the Wahl map and the first Rankin–Cohen bracket. Its relevance for us lies in the case where $V = C$, $N = L$, $m = 4$ and $n = 6$.

Lemma 10.2. *The ramification divisor Z, which is a point in $|10L + K_C|$, lies in the image of the projectivized first transvectant, which is a rational bilinear map*

$$V_L : |4L| \times |6L| - \ - \to |10L + K_C|.$$

Proof. Consider the elliptic surface given, in affine terms, by the equation $y^2 = 4x^3 - g_4x - g_6$, where g_n is a section of $L^{\otimes n}$. Then the j-invariant is a fractional linear function of the quantity $j_1 = g_4^3/g_6^2$, and the lemma follows from calculating the zero locus of dj_1/dz when z is a local coordinate on C. □

Corollary 10.3. *The universal ramification divisor $\mathcal{Z}$ is irreducible.*

Proof. The universal ramification divisor $\mathcal{Z}$ is dominated by $|6L| \times |4L|$. □

Let $V(L)$ denote the image of the projectivized first transvectant.

Theorem 10.4. *The image of the variational period map for Jacobian elliptic surfaces equals the locus $\mathcal{T}$ of triples (C, L, Z) where:*

(1) $C \in \mathcal{M}_q$;
(2) $L \in \mathrm{Pic}^{h+1-q}_C$;
(3) $Z \in V(L)$.

This locus is irreducible.

Proof. The only thing left to do is to observe that $\mathcal{T}$ is irreducible. However, $\mathcal{T}$ is dominated by $\mathcal{Z}$. □

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11. Multiple surfaces

An elliptic surface $f : X \to C$ is *multiple* if it is not simple; that is, if it has multiple fibres. As far as the period map is concerned there is not much to be said about these. Kodaira proved that, given $f : X \to C$, there is a simple elliptic surface $g : Y \to C$ such that X is obtained from Y by logarithmic transformations; in particular, there is a finite subset S of C such that, if $C_0 = C - S$, $X_0 = f^{-1}(C_0)$ and $Y_0 = g^{-1}(C_0)$, then X_0 and Y_0 are isomorphic relative to C_0, but are not usually bimeromorphic. In this situation the sheaves $f_*\omega_{X/C}$ and $g_*\omega_{Y/C}$ are isomorphic [Sch73, p. 234]. It follows that $f_*\Omega^2_X$ and $g_*\Omega^2_Y$ are isomorphic and then that the weight 2 Hodge structures on X and Y are isomorphic.

We deduce that the period map can detect neither the presence nor the location of multiple fibres on an elliptic surface.

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Conflicts of Interest

None.

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