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AN INDEPENDENT SYSTEM OF UNITS IN CERTAIN ALGEBRAIC NUMBER FIELDS

CLAUDE LEVESQUE

0. Introduction. For $K_n = \mathbf{Q}(\omega)$ a real algebraic number field of degree n over $\mathbf{Q}$ such that

$$\omega^n = M_n = \sum_{i=0}^{n} \left[\binom{n-1-i}{i-1} + \binom{n-i}{i} \right] D^{n-2i} d^i$$

with $D \in \mathbf{N}$, $d \in \mathbf{Z}$, $d | D^2$, and $D^2 + 4d > 0$, we proved in [5] (by using the approach of Halter-Koch and Stender [6]) that if

$$\epsilon_{nk} = 1 - \frac{M_k}{(-d)^k} \omega^k + \frac{\omega^2 k}{(-d)^k}$$

with

$$M_k = \sum_{i=0}^{k} \left[\binom{k-1-i}{i-1} + \binom{k-i}{i} \right] D^{k-2i} d^i,$$

then

$$S_0 = \{\epsilon_{nk} | k \in \mathbf{N}, k|n, k \neq n\}$$

is an independent system of units of K_n.

Noting that

$$\omega^n = M_n = \alpha^n + \beta^n,$$

where

$$\alpha = \frac{1}{2}(D + \sqrt{D^2 + 4d}), \beta = \frac{1}{2}(D - \sqrt{D^2 + 4d}),$$

and that in the quadratic extension $L_{2n} = K_n(\sqrt{D^2 + 4d})$ of K_n, we have the factorization

$$\epsilon_{nk} = e_{nk} u_{nk}$$

with

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$$e_{nk} = \frac{\omega^k - \alpha^k}{\beta^k} \quad \text{and} \quad u_{nk} = \frac{\omega^k - \beta^k}{\alpha^k},$$

we also proved in [5] that

$$S_1 = \left\{ e_{nk}, u_{nk}, \frac{\beta}{\alpha} \,\middle|\, k \in \mathbf{N}, k|n, k \neq n \right\}$$

is an independent system of units of

$$L_{2n} = \mathbf{Q}(\sqrt{D^2 + 4d}, \omega);$$

the proof was by induction on the number of prime divisors of n including multiplicity and rested on the fact that S_0 is independent.

The field L_{2n} may be viewed as the field

$$L_{2n} = \mathbf{Q}(\sqrt{D^2 + 4d}, \sqrt[n]{\alpha^n + \beta^n}).$$

A natural problem is then to exhibit an independent system of units in the field

$$F_{2n} = \mathbf{Q}(\sqrt{D^2 + 4d}, \sqrt[n]{\alpha^n - \beta^n}).$$

A solution to this problem is given by the following result.

MAIN THEOREM. *Let* $F_{2n} = \mathbf{Q}(\theta)$ *be a real algebraic number field of degree* $2n$ *over* $\mathbf{Q}$ *such that*

$$\theta = \sqrt[2n]{M_{2n} - 2(-d)^n} = \sqrt[2n]{(\alpha^n - \beta^n)^2} > 1,$$

where

$$M_{2n} = \alpha^{2n} + \beta^{2n} = \sum_{i=0}^{2n} \left[\binom{2n - 1 - i}{i - 1} + \binom{2n - i}{i} \right] D^{2n-2i} d^i$$

with $D \in \mathbf{N}$, $d \in \mathbf{Z}$, $d|D^2$, $D^2 + 4d > 0$ *and where*

$$\alpha = \frac{1}{2}(D + \sqrt{D^2 + 4d}) \quad \beta = \frac{1}{2}(D - \sqrt{D^2 + 4d}).$$

For any positive divisor t *of* n *and for any positive divisor* k *of* n *with* n/k *odd, define* ξ_{nt}, ψ_{nk} *and* η *by*

$$\xi_{nt} = \frac{\theta^t - \alpha^t}{\beta^t}, \psi_{nk} = \frac{\theta^k + \beta^k}{\alpha^k} \quad \text{and} \quad \eta = \frac{\beta}{\alpha}.$$

Then

$$S = \left\{ \xi_{nt}, \psi_{nk}, \eta \,\middle|\, t, k \in \mathbf{N}, t|n, k|n, t \neq n, k \neq n, 2 \nmid \frac{n}{k} \right\}$$

is an independent system of units of F_{2n}.

Examples. For $1 \leqq n \leqq 6$, here are the values of θ^{2n} and S:

$$n = 1 : D^2 + 4d,\ S = \{\eta\};$$

$$n = 2 : D^4 + 4D^2d,\ S = \left\{\frac{\theta - \alpha}{\beta}, \eta\right\};$$

$$n = 3 : D^6 + 6D^4d + 9D^2d^2 + 4d^3,$$

$$S = \left\{\frac{\theta - \alpha}{\beta}, \frac{\theta + \beta}{\alpha}, \eta\right\};$$

$$n = 4 : D^8 + 8D^6d + 20D^4d^2 + 16D^2d^3,$$

$$S = \left\{\frac{\theta - \alpha}{\beta}, \frac{\theta^2 - \alpha^2}{\beta^2}, \eta\right\};$$

$$n = 5 : D^{10} + 10D^8d + 35D^6d^2 + 50D^4d^3 + 25D^2d^4 + 4d^5,$$

$$S = \left\{\frac{\theta - \alpha}{\beta}, \frac{\theta + \beta}{\alpha}, \eta\right\};$$

$$n = 6 : D^{12} + 12D^{10}d + 54D^8d^2 + 112D^6d^3 + 105D^4d^4 + 36D^2d^5,$$

$$S = \left\{\frac{\theta - \alpha}{\beta}, \frac{\theta^2 - \alpha^2}{\beta^2}, \frac{\theta^3 - \alpha^3}{\beta^3}, \frac{\theta^2 + \beta^2}{\alpha^2}, \eta\right\}.$$

Let us recall that if K is an algebraic extension of degree $m = r + 2s$ over the rationals $\mathbf{Q}$ with r real (resp. $2s$ complex) embeddings in the field $\mathbf{C}$ of complex numbers, then by Dirichlet's theorem, the *unit group* $\mathcal{U}_K$ of K is a direct product of cyclic groups,

$$\mathcal{U}_K = W_K \times C_1 \times \ldots \times C_{r+s-1},$$

where W_K is the finite group of roots of unity in K and where the C_i's are copies of $\mathbf{Z}$. A *fundamental system of units* of K is a set of $r + s - 1$ generators of the C_i's. Finally, a finite set $S = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_t\}$ of t units of K is said to be an *independent system of units* if

$$\prod_{i=1}^{t} \epsilon_i^{a_i} = 1 \quad (\text{with } a_i \in \mathbf{Z}) \text{ implies } a_i = 0 \text{ for all } i.$$

After a section of preliminaries, we obtain in Section 2 the units ξ_{nt}, ψ_{nk} and calculate relative norms of these units.

Then supposing n odd, we obtain in Section 3 the independence of

$$\{\xi_{nk}\psi_{nk} | k \in \mathbf{N}, k|n, k \neq n\},$$

a fact which will prove useful, as can be seen just before formula (4.2).

Sections 4 and 5 are devoted to the proof of the main theorem for n odd and for n even respectively. The proof is by induction on the number of prime divisors of n including multiplicity: we assume that $n > 1$ is not a prime number and that the result holds true for all divisors m of n with $m \neq n$. Starting with a linear relation of the form

$$\left(\prod_{n \neq t|n} \xi_{nt}^{\nu(t)}\right)\left(\prod_{\substack{n \neq k|n \\ n/k \text{ odd}}} \psi_{nk}^{\lambda(k)}\right)\eta^a = 1,$$

and taking the relative norm $N_{F_{2n}/F_{2m}}$ with respect to some subfields, we obtain linear relations among certain units for which we are able to apply the induction hypothesis or some previous results.

In short, we have to prove the independence of a system of units for F_{2n}. We consider in Section 4 the case where n is odd, so the induction hypothesis can be used for the subfields F_{2m} with $m|n$ because m is still odd. In Section 5, we suppose n even, so when we come across the subfields F_{2m} with $m|n$, either we apply the induction hypothesis when m is still even, or we apply the results of Section 4 when m is odd.

Some parts of the paper may be skipped on a first reading: for instance, the technical lemmas 3.2, 4.1 and 5.2.

1. Preliminaries. Let us recall the definitions and properties of certain recursive sequences of second order defined in [**5**]. For $m, n \in \mathbf{Z}$, $0! = 1$ and

$$\binom{n}{m} = \begin{cases} \dfrac{n!}{(n-m)!m!} \text{ if } n \geqq m \geqq 0, \\ 1 \text{ if } n = -1 = m, \\ 0 \text{ otherwise.} \end{cases}$$

Definition 1.1. For $n \geqq 0$,

$$M_n = M_n(D, d) = \sum_{i=0}^{n} \left[\binom{n-1-i}{i-1} + \binom{n-i}{i}\right] D^{n-2i} d^i.$$

Definition 1.2. For $n \geqq -1$,

$$G_{n+1} = G_{n+1}(D, d) = \sum_{i=0}^{n} \binom{n-i}{i} D^{n-2i} d^i.$$

PROPOSITION 1.3. *For all* $r, s \in \mathbf{N}$,

(i) $M_{sr}(D, d) = M_r(M_s, -(-d)^s)$,

(ii) $G_{sr}(D, d) = G_s(D, d)G_r(M_s, -(-d)^s)$.

PROPOSITION 1.4. *If* $\alpha = \frac{1}{2}(D+\sqrt{D^2 + 4d})$, *and* $\beta = \frac{1}{2}(D-\sqrt{D^2 + 4d})$ (*with* $\alpha + \beta = D$, $\alpha - \beta = \sqrt{D^2 + 4d} \neq 0$ *and* $\alpha\beta = -d$), *then for any* $k \in \mathbf{N}$,

(i) $M_k = \alpha^k + \beta^k$,

(ii) $G_k = \dfrac{\alpha^k - \beta^k}{\alpha - \beta} = \dfrac{\alpha^k - \beta^k}{\sqrt{D^2 + 4d}}$,

(iii) $M_k^2 = (D^2 + 4d)G_k^2 + 4(-d)^k$,

(iv) $\alpha^k = \frac{1}{2}(M_k + G_k\sqrt{D^2 + 4d})$,

and

(v) $\beta^k = \frac{1}{2}(M_k - G_k\sqrt{D^2 + 4d})$.

Both of these propositions were proven in **[5]** and we will use them without explicitly referring to them.

Throughout this paper, μ stands for the Möbius function defined for a positive integer n by the rules

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } p^2 | n \text{ for some prime } p, \\ (-1)^r & \text{if } n \text{ is square-free with } r \text{ prime factors.} \end{cases}$$

2. Some units in F_{2n}. Let $F_{2n} = \mathbf{Q}(\theta)$ be a real algebraic number field of degree $2n$ over $\mathbf{Q}$ such that

$$\theta^{2n} = M_{2n}(D, d) - 2(-d)^n > 1$$

with $D \in \mathbf{N}$, $d \in \mathbf{Z}$, $d|D^2$, and $D^2 + 4d > 0$. Then

$$\begin{aligned} (2.1) \qquad \theta^{2n} &= M_{2n} - 2(-d)^n = M_n^2 - 4(-d)^n \\ &= (D^2 + 4d)G_n^2 = (\alpha^n - \beta^n)^2, \end{aligned}$$

where

$$\alpha = \frac{1}{2}(D + \sqrt{D^2 + 4d}) = \frac{1}{2}\left(D + \frac{\theta^n}{G_n}\right)$$

and

$$\beta = \frac{1}{2}(D + \sqrt{D^2 + 4d}) = \frac{1}{2}\left(D - \frac{\theta^n}{G_n}\right).$$

Assume that $n = mt$, and consider the subfields $F_{2m} = \mathbf{Q}(\theta^t)$ and $F_m = \mathbf{Q}(\theta^{2t})$ where

$$(2.2) \quad (\theta^t)^{2m} = M_{2m}(M_t, -(-d)^t) - 2(-d)^{tm}$$
$$= ((\alpha^t)^m - (\beta^t)^m)^2,$$

so that $F_{2m} = \mathbf{Q}(\theta^t)$ is a field of degree $2m$ over $\mathbf{Q}$.

$$\begin{tikzcd}
F_{2n} = \mathbf{Q}(\theta) \arrow[d, dash, "t"'] \arrow[dr, dash, "2"] & \\
F_{2m} = \mathbf{Q}(\theta^t) \arrow[d, dash, "m"'] \arrow[dr, dash, "2"] & F_n = \mathbf{Q}(\theta^2) \arrow[d, dash, "t"] \\
F_2 = \mathbf{Q}(\theta^n) = \mathbf{Q}(\sqrt{D^2 + 4d}) \arrow[dr, dash, "2"'] & F_m = \mathbf{Q}(\theta^{2t}) \arrow[d, dash, "m"] \\
& \mathbf{Q}
\end{tikzcd}$$

Figure 1

PROPOSITION 2.1. *For any positive divisor t of n with $n = mt$, the algebraic numbers*

$$\eta = \frac{\beta}{\alpha},$$

$$\xi_{nt} = \frac{\theta^t - \alpha^t}{\beta^t},$$

and

$$\psi_{nt} = \frac{\theta^t + \beta^t}{\alpha^t} \quad \textit{if } m \textit{ is odd},$$

are units in $F_{2m} \subseteq F_{2n}$.

Proof. (i) We saw in [**5**] that η is a unit of

$$\mathbf{Q}(\sqrt{D^2 + 4d}) = F_2.$$

(ii) Proceeding as in page 134 of [**5**] and noting that

$$(\theta^t)^m = \alpha^{mt} - \beta^{mt},$$

we see that ξ_{nt} is a root of the polynomial

$$\sum_{i=0}^{m-1} \binom{m}{i} (\eta^{-1})^{ti} z^{m-i} + 1$$

with coefficients in the ring of algebraic integers of $F_2 = \mathbf{Q}(\sqrt{D^2 + 4d})$.

(iii) Similarly, when $m = n/t$ is odd, ψ_{nt} is a root of the polynomial

$$\sum_{i=0}^{m-1} \binom{m}{i}(-\eta)^{ti} z^{m-i} - 1.$$

PROPOSITION 2.2. *Let m, k be positive divisors of n with*

$$n = mt, \ r = (k, t), \ h = \frac{k}{r}, \text{ and } l = \frac{t}{r}.$$

(i) *Then*

$$N_{F_{2n}/F_{2m}}(\xi_{nk}) = (-1)^{t+r}\xi_{n,kl}^r = (-1)^{t+r}\xi_{mh}^r.$$

(ii) *If n/k is odd, then*

$$N_{F_{2n}/F_{2m}}(\psi_{nk}) = \psi_{n,kl}^r = \psi_{mh}^r.$$

(iii) *If m is odd, then*

$$N_{F_{2n}/F_m}(\xi_{nk}) = (-1)^r\xi_{n,kl}^r\psi_{n,kl}^r = (-1)^r\xi_{mh}^r\psi_{mh}^r.$$

(iv) *If m and n/k are odd, then*

$$N_{F_{2n}/F_m}(\psi_{nk}) = (-1)^r\xi_{n,kl}^r\psi_{n,kl}^r = (-1)^r\xi_{mh}^r\psi_{mh}^r.$$

Proof. (i) Let ζ be a primitive t-th root of unity. Then ζ^k is a primitive l-th root of unity and

$$\begin{aligned}
N_{F_{2n}/F_{2m}}(\xi_{nk}) &= N_{F_{2n}/F_{2m}}\left(\frac{\theta^k - \left(\frac{D}{2} + \frac{\theta^n}{2G_n}\right)^k}{\left(\frac{D}{2} - \frac{\theta^n}{2G_n}\right)^k}\right) \\
&= \prod_{i=0}^{t-1}\left(\frac{\zeta^{ki}\theta^k - \left(\frac{D}{2} + \frac{\zeta^{ni}\theta^n}{2G_n}\right)^k}{\left(\frac{D}{2} - \frac{\zeta^{ni}\theta^n}{2G_n}\right)^k}\right) \\
&= \prod_{i=0}^{l-1}\left(\frac{(\zeta^k)^i\theta^k - \alpha^k}{\beta^k}\right)^r = (-1)^{lr}\left(\frac{\alpha^{kl} - \theta^{kl}}{\beta^{kl}}\right)^r \\
&= (-1)^{t+r}\,\xi_{n,kl}^r = (-1)^{t+r}\xi_{mh}^r.
\end{aligned}$$

(ii) Let n/k be odd and let ζ be a primitive t-th root of unity. Then l is odd and

$$N_{F_{2n}/F_{2m}}(\psi_{nk}) = \sum_{i=0}^{l-1} \left(\frac{(\zeta^k)^i\theta^k + \beta^k}{\alpha^k}\right)^r$$
$$= \left(\frac{\theta^{kl} + \beta^{kl}}{\alpha^{kl}}\right)^r = \psi^r_{n,kl} = \psi^r_{mh}.$$

(iii) Whenever m is odd, we have h odd and

$$N_{F_{2n}/F_m}(\xi_{nk}) = N_{F_{2m}/F_m}(N_{F_{2n}/F_{2m}}(\xi_{nk}))$$
$$= N_{F_{2m}/F_m}\left((-1)^{t+r}\left(\frac{(\theta^t)^h - (\alpha^t)^h}{(\beta^t)^h}\right)^r \right)$$
$$= \left(\frac{(\theta^t)^h - \left(\dfrac{M_t}{2} + \dfrac{G_t(\theta^t)^m}{2G_n}\right)^h}{\left(\dfrac{M_t}{2} - \dfrac{G_t(\theta^t)^m}{2G_n}\right)^h}\right)^r$$
$$\times \left(\frac{(-\theta^t)^h - \left(\dfrac{M_t}{2} + \dfrac{G_t(-\theta^t)^m}{2G_n}\right)^h}{\left(\dfrac{M_t}{2} - \dfrac{G_t(-\theta^t)^m}{2G_n}\right)^h}\right)^r$$
$$= \left(\frac{\theta^{ht} - \alpha^{ht}}{\beta^{ht}}\right)^r \cdot \left(\frac{-\theta^{ht} - \beta^{ht}}{\alpha^{ht}}\right)^r$$
$$= (-1)^r\xi^r_{n,kl}\psi^r_{n,kl}.$$

We used here the facts that

$$\alpha^t = \frac{1}{2}(M_t + G_t\sqrt{D^2 + 4d}),$$
$$\beta^t = \frac{1}{2}(M_t - G_t\sqrt{D^2 + 4d}),$$
$$\sqrt{D^2 + 4d} = \theta^n/G_n,$$

and $ht = kl$.

(iv) Similarly, if m and n/k are odd, we have that

$$N_{F_{2n}/F_m}(\psi_{nk}) = N_{F_{2m}/F_m}\left(\frac{\theta^{ht} + \beta^{ht}}{\alpha^{ht}}\right)^r$$
$$= \left(\frac{\theta^{ht} + \beta^{ht}}{\alpha^{ht}}\right)^r\left(\frac{-\theta^{ht} + \alpha^{ht}}{\beta^{ht}}\right)^r$$
$$= (-1)^r\psi_{n,ht}\xi_{n,ht}.$$

Note. In Proposition 3.1.3 of [**5**], $(-1)^{t(k+1)}$ should be replaced by $(-1)^{t+r}$. This does not change the proof, because in Lemma 3.3.1 of [**5**], we consider absolute values.

Whenever n is odd, we have from Proposition 2.1 that for any divisor k of n,

$$(2.3)\qquad \eta_{nk} = \xi_{nk}\psi_{nk} = \frac{\theta^k - \alpha^k}{\beta^k} \cdot \frac{\theta^k + \beta^k}{\alpha^k}$$

is a unit of $\mathbf{Q}(\theta^k) \subseteq F_{2n}$. More precisely,

$$\eta_{nk} = -1 + \frac{G_k}{d^k}\sqrt{D^2 + 4d}\,\theta^k - \frac{\theta^{2k}}{d^k} = -1 + \frac{G_k\theta^{n+k}}{d^k G_n} - \frac{\theta^{2k}}{d^k}$$

is a unit of $\mathbf{Q}(\theta^{2k})$, because

$$\theta^{n+k} = \theta^{k(n/k+1)}.$$

We shall now evaluate the norm $N_{\mathbf{Q}(\theta^2)/\mathbf{Q}(\theta^{2t})}$ of these units η_{nk} for $n = mt$.

PROPOSITION 2.3. *Let n be odd and let m, k be positive divisors of n with*

$$n = mt,\ r = (k, t),\ h = \frac{k}{r},\ \textit{and}\ l = \frac{t}{r}.$$

Then

$$N_{F_n/F_m}(\eta_{nk}) = \eta^r_{n,kl} = \eta^r_{mh}.$$

Proof. Let ζ be a primitive t-th root of unity. Then

$$N_{F_n/F_m}(\eta_{nk}) = \prod_{i=0}^{t-1}\left(\frac{-d^k + \frac{G_k}{G_n}(\zeta^i\theta^2)^{(n+k)/2} - (\zeta^i\theta^2)^k}{d^k}\right).$$

Now, since n is odd, ζ^2 is also a primitive t-th root of unity.

Moreover ζ^k is a primitive l-th root of unity. Therefore

$$\begin{aligned}
N_{F_n/F_m}(\eta_{nk}) &= \prod_{i=0}^{t-1}\left(\frac{-d^k + \frac{G_k}{G_n}\zeta^{(n+k)i}\theta^{n+k} - \zeta^{2ki}\theta^{2k}}{d^k}\right)\\
&= \prod_{i=0}^{t-1}\left(\frac{-d^k + G_k\sqrt{D^2+4d}\,\zeta^{ki}\theta^k - \zeta^{2ki}\theta^{2k}}{d^k}\right)\\
&= \prod_{i=0}^{t-1}\left(\frac{\zeta^{ki}\theta^k - \alpha^k}{\beta^k}\right)\left(\frac{\zeta^{ki}\theta^k + \beta^k}{\alpha^k}\right)\\
&= \xi^r_{n,kl}\psi^r_{n,kl} = \eta^r_{n,kl} = \eta^r_{mh}.
\end{aligned}$$

3. A secondary result. In this section, we assume that n is always odd. For any positive divisor t of n, we defined η_{nt} by

$$\eta_{nt} = \xi_{nt}\psi_{nt} = \frac{\theta^t - \alpha^t}{\beta^t} \cdot \frac{\theta^t + \beta^t}{\alpha^t}. \tag{3.1}$$

We plan to show that $\{\eta_{nt} | t \in \mathbf{N}, t|n, t \neq n\}$ is an independent system of units of $\mathbf{Q}(\theta^2) \subseteq F_{2n}$. This result will prove useful in the proof of the main theorem.

We need two lemmas which we shall prove under the hypotheses of Theorem 3.3.

LEMMA 3.1. *Let n be odd. Then*
(i) $\alpha > \theta$ *if and only if* $d < 0$;
(ii) $\alpha > 2|\beta|$;
(iii) $|\beta| < \theta$.

Proof. (i) $\theta^n = \alpha^n - \beta^n < \alpha^n$ if and only if $\beta > 0$ if and only if $d < 0$.
(ii) See part (iii) of Lemma 2.3.1 in [5].
(iii) For $d < 0$, we have $\theta^n = \alpha^n - \beta^n > \beta^n$ from part (ii).
If $d > 0$, $\theta > -\beta$, because $\theta^n = \alpha^n - \beta^n > -\beta^n$.

LEMMA 3.2. *Let $v = p_1 \dots p_s$, where $p_1, \dots, p_s$ are the distinct prime factors of n. Then*

$$\prod_{\substack{t|v \\ \mu(t)=-1}} |\eta_{nt}|^{v/t} > \prod_{\substack{t|v \\ \mu(t)=1}} |\eta_{nt}|^{v/t}.$$

Proof. We shall consider two cases separately.

Case A: $d > 0$. The previous lemma yields

$$\theta > \alpha > -2\beta > -\beta > 0; \tag{3.2}$$

hence

$$|\eta_{n,t}| = \frac{(\theta^t - \alpha^t)(\theta^t + \beta^t)}{\alpha^t|\beta^t|} = \left(\left(\frac{\theta}{\alpha}\right)^t - 1\right)\left(\left(\frac{\theta}{-\beta}\right)^t - 1\right).$$

Let

$$1 + \delta_1 = \Delta_1 = \frac{\theta}{\alpha} \text{ and } 1 + \delta_2 = \Delta_2 = \frac{\theta}{-\beta}. \tag{3.3}$$

Since

$$1 < (1 + \delta_1)^n = \Delta_1^n = 1 + \left(\frac{-\beta}{\alpha}\right)^n < 1 + \left(\frac{1}{2}\right)^n$$

by Lemma 3.1, we conclude that $0 < \delta_1 < 1$. In addition, $\theta > \alpha > -2\beta$ implies $\theta/(-\beta) > 2$, i.e.,

$$\delta_2 = \Delta_2 - 1 = \frac{\theta}{-\beta} - 1 > 1.$$

Proceeding as in page 129 of [**5**], we will have the conclusion if we can prove

$$(3.4) \qquad \prod_{\substack{t \mid v \\ \mu(t) = -1}} (t\delta_1\delta_2^t)^{v/t} > \prod_{\substack{t \mid v \\ \mu(t) = 1}} (t\delta_1\Delta_1^t\Delta_2^t)^{v/t}.$$

(i) Suppose $s \geqq 2$. As in page 130 of [**5**], we see that it suffices to prove

$$(3.5) \qquad \Delta_1 = 1 + \delta_1 < 1 + \frac{2}{(\Delta_1\Delta_2)^n}.$$

Since $-\beta < \dfrac{1}{2}\alpha$ and $n > 1$, we have

$$(\alpha^n - \beta^n)^2 = \left(1 + \left(\frac{-\beta}{\alpha}\right)^n\right)^2 \alpha^{2n} < \left(1 + \frac{1}{2^n}\right)^2 \alpha^{2n} < 2\alpha^{2n},$$

from which we conclude

$$\frac{1}{\alpha^n} < \frac{2\alpha^n}{(\alpha^n - \beta^n)^2}.$$

This last inequality leads to

$$\Delta_1^n = 1 + \frac{(-\beta)^n}{\alpha^n} < 1 + \frac{2\alpha^n(-\beta)^n}{(\alpha^n - \beta^n)^2} < \left(1 + \frac{2}{(\Delta_1\Delta_2)^n}\right)^n,$$

from which we can obtain inequality (3.5).

(ii) Suppose $s = 1$, i.e., $n = p^l$ for p a prime. Then the inequality which corresponds to (3.4) and which we want to prove is

$$p\delta_1\delta_2^p > (\delta_1\Delta_1\Delta_2)^p, \text{ i.e., } p\delta_2^p > \delta_1^{p-1}\Delta_1^p\Delta_2^p.$$

Since $p\delta_2^p > 2$ and

$$\delta_1(\Delta_1\Delta_2)^n \geqq \delta_1^{p-1}(\Delta_1\Delta_2)^p,$$

it suffices to prove

$$\delta_1 < \frac{2}{(\Delta_1\Delta_2)^n},$$

which follows from inequality (3.5).

Case B: $d < 0$. From the preceeding lemma, we have

$$\alpha > \theta > \beta > 0 \text{ and } \alpha > 2\beta, \tag{3.6}$$

hence

$$|\eta_{nt}| = \frac{(\alpha^t - \theta^t)(\theta^t + \beta^t)}{\alpha^t \beta^t} = \frac{\left(\frac{\alpha^t}{\theta^t} - 1\right)\left(1 + \frac{\beta^t}{\theta^t}\right)\theta^{2t}}{(-d)^t}.$$

Let

$$1 + \delta_1 = \Delta_1 = \frac{\alpha}{\theta} \text{ and } \Delta_2 = \frac{\beta}{\theta}. \tag{3.7}$$

Here $0 < \Delta_2 < 1$ and $\alpha > 2\beta$ implies

$$\alpha^n < 2\alpha^n - 2\beta^n,$$

i.e.,

$$\Delta_1^n = \frac{\alpha^n}{\alpha^n - \beta^n} < 2,$$

i.e.,

$$\delta_1 < 1.$$

Proceeding as in page 131 of **[5]**, we need only to prove

$$\prod_{\substack{t|v \\ \mu(t)=-1}} (t\delta_1)^{v/t} > \prod_{\substack{t|v \\ \mu(t)=1}} (t\delta_1\Delta_1^t 2^t)^{v/t}. \tag{3.8}$$

(i) Suppose $s \geqq 2$. Then it suffices to prove

$$\frac{\alpha^n}{\alpha^n - \beta^n} < 1 + \frac{2n(\alpha^n - \beta^n)}{2^n\alpha^n}, \tag{3.9}$$

i.e.,

$$2n\alpha^{2n} > (2^n + 4n)(-d)^n - 2n\beta^{2n}.$$

We will show

$$2n\alpha^{2n} > (2^n + 4n)(-d)^n.$$

Here $d < 0$ and $D^2 + 4d > 0$ imply $D > 2\sqrt{-d}$, and $d| (D^2 + 4d)$ implies $D^2 + 4d > -d$. Hence

$$\alpha = \frac{1}{2}(D + \sqrt{D^2 + 4d}) > \frac{1}{2}(2\sqrt{-d} + \sqrt{-d}) = \frac{3}{2}\sqrt{-d},$$

from which we conclude that

$$\alpha^2 > -\frac{9}{4}d > -2d,$$

and that

$$2n\alpha^{2n} > 2n(-2d)^n = 2^{n+1}n(-d)^n > (2^n + 4n)(-d)^n.$$

(ii) Suppose $s = 1$, i.e., $n = p^l$ for p a prime. Then the inequality which corresponds to (3.8) and which we want to prove is

$$p\delta_1 > (2\delta_1\Delta_1)^p, \text{ i.e., } p > \delta_1^{p-1}(2\Delta_1)^p.$$

Since $p \geqq 2$ and $\delta_1(2\Delta_1)^n \geqq \delta_1^{p-1}(2\Delta_1)^p$, it is sufficient to prove

$$\delta_1 < \frac{2}{(2\Delta_1)^n}, \text{ i.e., } \Delta_1^n = (1 + \delta_1)^n < \left(1 + \frac{2}{(2\Delta_1)^n}\right)^n,$$

which follows from inequality (3.9).

Lemma 3.2 will be used in the last part of the proof of the following result.

THEOREM 3.3. *Let $F_{2n} = \mathbf{Q}(\theta)$ be a real algebraic number field of degree $2n$ over $\mathbf{Q}$ with $n > 1$ odd such that*

$$\theta = \sqrt[2n]{M_{2n} - 2(-d)^n} > 1,$$

where

$$M_{2n} = \sum_{i=0}^{2n} \left[\binom{2n-1-i}{i-1} + \binom{2n-i}{i}\right] D^{2n-2i}d^i,$$

with $D \in \mathbf{N}$, $d \in \mathbf{Z}$, $d|D^2$, and $D^2 + 4d > 0$. For any positive divisor t of n, define η_{nt} by

$$\eta_{nt} = -1 + \frac{G_t}{d^t}\sqrt{D^2 + 4d}\,\theta^t - \frac{\theta^{2t}}{d^t}$$

where

$$G_t = \sum_{i=0}^{t-1} \binom{t-1-i}{i} D^{t-1-2i}d^i.$$

Then

$$S_0 = \{\eta_{nt} | t \in \mathbf{N},\ t|n,\ t \neq n\}$$

is an independent system of units of $F_n = \mathbf{Q}(\theta^2) \subseteq F_{2n}$.

Proof. Since we have seen that S_0 is a set of units of F_n, so it remains to show the independence; this will be achieved by induction on the number of prime divisors of n including multiplicity. We shall use the factorisation of η_{nt} in (3.1) and proceed as in [**5**].

If n is a prime number, the proof is immediate. We now assume that n is not a prime number and that the theorem holds for all divisors m of n such that $m \neq n$. Suppose that there exists a linear relation

$$(3.10)\quad \prod_{n \neq t | n} \eta_{nt}^{\nu(t)} = 1 \quad \text{with } \nu(t) \in \mathbf{Z}.$$

By applying the norm N_{F_n/F_m} with respect to the field $F_m = \mathbf{Q}(\theta^{2p})$ where $n = mp$ for p a prime divisor of n, we obtain from Proposition 2.3

$$N_{F_n/F_m}(\eta_{nt}) = \begin{cases} \eta_{m,t/p}^{p} & \text{if } p | t, \\ \eta_{mt} & \text{if } p \nmid t. \end{cases}$$

Proceeding as in page 127 of [**5**], we conclude that either S_0 is independent or (3.10) reduces to

$$(3.11)\quad \Bigg(\prod_{\substack{n \neq t | \nu \\ \mu(t) = 1}} \eta_{nt}^{1/t} \Bigg) \Bigg(\prod_{\substack{n \neq t | \nu \\ \mu(t) = -1}} \eta_{nt}^{-1/t} \Bigg) = \pm 1.$$

In (3.11), we can omit the condition $t \neq n$ since it is superfluous whenever $n \neq \nu$, and it adds the factor $\eta_{nn} = -1$ for $n = \nu$. So (3.11) implies

$$(3.12)\quad \prod_{\substack{t | \nu \\ \mu(t) = 1}} \eta_{nt}^{\nu/t} = \pm \prod_{\substack{t | \nu \\ \mu(t) = -1}} \eta_{nt}^{\nu/t},$$

which contradicts Lemma 3.2.

4. Proof of the main theorem for n odd. In this section, n is always odd. Under the hypotheses of the main theorem, we shall first prove a lemma which will be used in the last part of the proof of the fact that

$$S = \{\xi_{nk}, \psi_{nk}, \eta \,|\, k \in \mathbf{N}, k | n, k \neq n\}$$

is an independent system of units of F_{2n}.

LEMMA 4.1. *Let $\nu = p_1 \ldots p_s$, where $p_1, \ldots, p_s$ are the distinct prime factors of n. Then*

$$\Bigg(\prod_{\substack{t | \nu \\ \mu(t) = -1}} |\xi_{nt}|^{\nu/t} \Bigg) \Bigg(\prod_{\substack{t | \nu \\ \mu(t) = 1}} |\psi_{nt}|^{\nu/t} \Bigg) > \Bigg(\prod_{\substack{t | \nu \\ \mu(t) = 1}} |\xi_{nt}|^{\nu/t} \Bigg) \Bigg(\prod_{\substack{t | \nu \\ \mu(t) = -1}} |\psi_{nt}|^{\nu/t} \Bigg).$$

Proof. We shall use the same notation as that used in the proof of Lemma 3.2.

Case A. $d > 0$. We must prove

$$\Big(\prod_{\substack{t|v\\ \mu(t)=-1}} (\Delta_1^t - 1)^{v/t}\Big)\Big(\prod_{\substack{t|v\\ \mu(t)=1}} (\Delta_2^t - 1)^{v/t}\Big)$$
$$> \Big(\prod_{\substack{t|v\\ \mu(t)=1}} (\Delta_1^t - 1)^{v/t}\Big)\Big(\prod_{\substack{t|v\\ \mu(t)=-1}} (\Delta_2^t - 1)^{v/t}\Big)$$

and we see that this follows directly from inequality (3.4).

Case B. $d < 0$. We need to prove

$$\Big(\prod_{\substack{t|v\\ \mu(t)=-1}} (\Delta_1^t - 1)^{v/t}\Big)\Big(\prod_{\substack{t|v\\ \mu(t)=1}} (\Delta_2^t + 1)^{v/t}\Big)$$
$$> \Big(\prod_{\substack{t|v\\ \mu(t)=1}} (\Delta_1^t - 1)^{v/t}\Big)\Big(\prod_{\substack{t|v\\ \mu(t)=-1}} (\Delta_2^t + 1)^{v/t}\Big),$$

which follows from (3.8).

Let us show that S is independent when n is odd. If $n = 1$, then $S = \{\eta\}$ is independent. Assume that $n \neq 1$ and that there exists a linear relation

$$(4.1) \quad \Big(\prod_{n\neq k|n} \xi_{nk}^{\nu(k)}\Big)\Big(\prod_{n\neq k|n} \psi_{nk}^{\lambda(k)}\Big)\eta^a = 1$$

with $\nu(k)$, $\lambda(k)$ and $a \in \mathbf{Z}$. Using Proposition 2.2, we apply the norm N_{F_{2n}/F_2} to obtain $\eta^{na} = \pm 1$, i.e., $a = 0$. Applying now the norm N_{F_{2n}/F_n} and using Theorem 3.3, we find

$$\nu(k) + \lambda(k) = 0,$$

whereupon relation (4.1) becomes

$$(4.2) \quad \Big(\prod_{n\neq k|n} \xi_{nk}^{\nu(k)}\Big)\Big(\prod_{n\neq k|n} \psi_{nk}^{-\nu(k)}\Big) = 1.$$

The proof is again by induction on the number of prime divisors of n including multiplicity. If n is a prime number p, then (4.2) becomes

$$\left(\frac{\xi_{p1}}{\psi_{p1}}\right)^{\nu(1)} = 1,$$

i.e.,

$$\xi_{p1} = \pm\psi_{p1},$$

i.e.,

$$(\alpha \mp \beta)\theta = \alpha^2 \pm \beta^2,$$

i.e.,

$$\theta^{p+1}/G_p = D^2 + 2d \quad \text{or} \quad \theta = \theta^p/G_p;$$

since this is impossible, S is independent whenever $n = p$.

Applying the norm $N_{F_{2n}/F_{2m}}$, where $n = mp$ for p a prime divisor of n and proceeding as in [**5**], we have that either S is independent or that (4.2) implies

$$\Big(\prod_{\substack{k|\nu \\ \mu(k)=-1}} \xi_{nk}^{\nu/k}\Big)\Big(\prod_{\substack{k|\nu \\ \mu(k)=1}} \psi_{nk}^{\nu/k}\Big) = \pm\Big(\prod_{\substack{k|\nu \\ \mu(k)=1}} \xi_{nk}^{\nu/k}\Big)\Big(\prod_{\substack{k|\nu \\ \mu(k)=-1}} \psi_{nk}^{\nu/k}\Big),$$

which contradicts Lemma 4.1.

5. Proof of the main theorem for n even.

Throughout this section, n is always even. In order to show that

$$S = \left\{\xi_{nt}, \psi_{nk}, \eta \,\middle|\, t, k \in \mathbf{N}, t|n, k|n, t \neq n, k \neq n, 2 \nmid \frac{n}{k}\right\}$$

is an independent system of units of F_{2n}, we shall need two lemmas which we shall prove under the hypotheses of the main theorem.

LEMMA 5.1. *Let n be even. Then*
(i) $\theta < \alpha$;
(ii) $\alpha > 2\,|\beta|$;
(iii) $|\beta| < \theta$.

Proof. This is similar to that of Lemma 3.1.

LEMMA 5.2. *Let $\nu = p_1 \dots p_s$, where $p_1, \dots, p_s$ are the distinct prime factors of n. Then*

$$\prod_{\substack{t|\nu \\ \mu(t)=-1}} |\xi_{nt}|^{\nu/t} > \prod_{\substack{t|\nu \\ \mu(t)=1}} |\xi_{nt}|^{\nu/t}.$$

Proof. Lemma 5.1 yields

(5.1) $\quad \alpha > \theta > |\beta| \quad \text{and} \quad \alpha > 2|\beta| > 0;$

hence

$$|\xi_{nt}| = \frac{\theta^t}{|\beta|^t}\left(\frac{\alpha^t}{\theta^t} - 1\right).$$

Let

(5.2) $\quad 1 + \delta = \Delta = \dfrac{\alpha}{\theta}.$

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Here

$$\Delta^n = -\frac{\alpha^n}{\alpha^n - \beta^n} < 2,$$

because $2\beta^n < \alpha^n$. Hence $0 < \delta < 1$. As in [**4**], we have the inequalities

$$t\delta < \Delta^t - 1 < t\delta\Delta^t. \tag{5.3}$$

Proceeding as in [**4**], we see that it suffices to prove

$$\prod_{\substack{t|v \\ \mu(t)=-1}} (t\delta)^{v/t} > \prod_{\substack{t|v \\ \mu(t)=1}} (t\delta\Delta^t)^{v/t}. \tag{5.4}$$

Since $\Delta^v \leqq \Delta^n < 2$, we need only to prove

$$\prod_{\substack{t|v \\ \mu(t)=-1}} (t\delta)^{v/t} > 2^{2^s-1}\Big(\prod_{\substack{t|v \\ \mu(t)=-1}} (t\delta)^{v/t}\Big), \tag{5.5}$$

which is inequality (15) of [**4**].

Let us prove the independence of S when n is even. Suppose there exists a linear relation

$$\Big(\prod_{n\neq t|n} \xi_{nt}^{\nu(t)}\Big)\Big(\prod_{\substack{n\neq k|n \\ n/k\ \mathrm{odd}}} \psi_{nk}^{\lambda(k)}\Big)\eta^a = 1 \tag{5.6}$$

with $\nu(t)$, $\lambda(k)$, and $a \in \mathbf{Z}$. When we apply the norm N_{F_{2n}/F_2}, we obtain $\eta^{na} = \pm 1$, i.e., $a = 0$, whereupon relation (5.6) becomes

$$\Big(\prod_{n\neq t|n} \xi_{nt}^{\nu(t)}\Big)\Big(\prod_{\substack{n\neq k|n \\ n/k\ \mathrm{odd}}} \psi_{nk}^{\lambda(k)}\Big) = 1. \tag{5.7}$$

Suppose that n is a prime number. Since n is even, $n = 2$. So $S = \{\xi_{n1}, \eta\}$. Now $a = 0$ implies $\nu(1) = 0$, since

$$\xi_{n1} = \frac{\theta - \alpha}{\beta} \neq \pm 1,$$

i.e.,

$$\theta \neq \alpha \pm \beta,$$

i.e.,

$$\theta \neq D \quad \text{and} \quad \theta \neq \theta^2/G_2.$$

The proof is once more by induction on the number of prime divisors of n including multiplicity. Assume that n is not a prime number and that the theorem holds for all divisors m of n with $m \neq n$. Therefore, whenever

$n = mt$, with $m \neq n$,

$$S = \left\{ \xi_{mt}, \psi_{mk}, \eta \,\middle|\, t, k \in \mathbf{N},\ t|m,\ k|m,\ t \neq m,\ k \neq m,\ 2 \nmid \frac{m}{k} \right\}$$

is an independent system of units of $F_{2m} = \mathbf{Q}(\theta^t)$; in fact, if m is even, this is the induction hypothesis and if m is odd, this is a result we proved in Section 4.

If we start with the linear relation (5.6), we saw that it reduces to relation (5.7).

We now apply the norm $N_{F_{2n}/F_{2m}}$ with respect to the field $F_{2m} = \mathbf{Q}(\theta^p)$ where $n = pm$ for p a prime divisor of n. From Proposition 2.2, we have

$$N_{F_{2n}/F_{2m}}(\xi_{nt}) = \begin{cases} \xi^p_{m,t/p} & \text{if } p|t, \\ (-1)^{p+1}\xi_{mt} & \text{if } p \nmid t; \end{cases}$$

when n/k is odd (so k is even) and when $p|k$, we have

$$N_{F_{2n}/F_{2m}}(\psi_{nk}) = \psi^p_{m,k/p}.$$

Let us first take $p = 2$. We then obtain

$$\begin{aligned} 1 &= \Bigg(\prod_{\substack{n \neq t|n \\ 2\nmid t}} N_{F_{2n}/F_{2m}}(\xi_{nt}^{\nu(t)}) \Bigg) \Bigg(\prod_{\substack{n \neq t|n \\ 2|t}} N_{F_{2n}/F_{2m}}(\xi_{nt}^{\nu(t)}) \Bigg) \\ &\quad \times \Bigg(\prod_{\substack{n \neq k|n \\ n/k \text{ odd}}} N_{F_{2n}/F_{2m}}(\psi_{nk}^{\lambda(k)}) \Bigg) \\ &= \pm \Bigg(\prod_{\substack{n \neq t|n \\ 2\nmid t}} \xi_{mt}^{\nu(t)} \Bigg) \Bigg(\prod_{\substack{n \neq t|n \\ 2|t \\ 4\nmid t}} \xi_{m,t/2}^{2\nu(t)} \Bigg) \Bigg(\prod_{\substack{n \neq t|n \\ 4|t}} \xi_{m,t/2}^{2\nu(t)} \Bigg) \Bigg(\prod_{\substack{n \neq k|n \\ n/k \text{ odd}}} \psi_{m,k/2}^{2\lambda(k)} \Bigg) \\ &= \pm \Bigg(\prod_{\substack{m \neq t|m \\ 2\nmid t}} \xi_{mt}^{\nu(t)+2\nu(2t)} \Bigg) \Bigg(\prod_{\substack{m \neq t|m \\ 2|t}} \xi_{mt}^{2\nu(2t)} \Bigg) \Bigg(\prod_{\substack{m \neq k|m \\ m/k \text{ odd}}} \psi_{mk}^{2\lambda(2k)} \Bigg). \end{aligned}$$

(As usual, an empty product is 1.) Whatever the parity of m, we conclude that $\lambda(2k) = 0$ for any $k \neq n/2$ with $k|(n/2)$ and $n/2k$ odd, i.e.,

(5.8) $\quad \lambda(k) = 0$ for any $k \neq n$ with $k|n$ and n/k odd.

It seems amazing that we eliminated all the integers $\lambda(k)$. This is best explained by the fact that if $n = 2m$ and if

$$A = \left\{ k \in \mathbf{N} \,\middle|\, k|n,\ k \neq n,\ 2 \nmid \frac{n}{k} \right\},$$

$$B = \left\{ k \in \mathbf{N} \,\middle|\, k|m,\ k \neq m,\ 2 \nmid \frac{m}{k} \right\},$$

then A and B have the same cardinality.

Next, we apply the norm $N_{F_{2n}/F_{2m}}$ with respect to the field $F_{2m} = \mathbf{Q}(\theta^p)$ where $n = mp$ for p a prime divisor of n. Thus, we obtain

$$1 = \prod_{n \neq t | n} N_{F_{2n}/F_{2m}}(\xi_{nt}^{\nu(t)})$$

$$= \pm \Big(\prod_{\substack{m \neq t | m \\ p \nmid t}} \xi_{mt}^{\nu(t)+p\nu(tp)} \Big) \Big(\prod_{\substack{m \neq t | m \\ p | t}} \xi_{mt}^{p\nu(tp)} \Big),$$

whereupon

(5.9) $\quad \nu(t) = 0$ for any $t \neq n$ with $p^2 | t | n$,

(5.10) $\quad \nu(t) = -p\nu(tp)$ for any $t \neq \dfrac{n}{p}$ with $p \nmid t | \dfrac{n}{p}$.

From (5.9) and (5.10), we conclude that by applying this procedure to all primes $p | n$, we obtain $\nu(t) = 0$ for all proper divisors t of n containing a non-linear prime factor.

For all divisors t of n $(t \neq n)$ that are products of r distinct prime factors, we obtain from (5.10) that

(5.11) $\quad \nu(1) = (-1)^r t\nu(t)$.

If $\nu(1) = 0$, then $\nu(t) = 0$ and the theorem is proven. Let $\nu(1) \neq 0$. Then (5.11) implies

$$\Big(\prod_{\substack{n \neq t | \nu \\ \mu(t) = 1}} \xi_{nt}^{\nu/t} \Big) \Big(\prod_{\substack{n \neq t | \nu \\ \mu(t) = -1}} \xi_{nt}^{-\nu/t} \Big) = \pm 1,$$

which contradicts Lemma 5.2.

6. Conclusion. It is obvious that instead of η in the independent system S of units of F_{2n}, we can take η_0, the fundamental unit of $\mathbf{Q}(\sqrt{D^2 + 4d})$.

For $n = 1, 2, 3$, S is a maximal independent system of units of $F_{2n} = \mathbf{Q}(\theta)$, in the sense that the cardinality of S is equal to the rank of $\mathcal{U}_k / W_k$. For $n = 1$, the fundamental unit of $\mathbf{Q}(\theta)$ is known from a result of Degert [7].

In a forthcoming paper, we will show that for $n = 2, 3$, S can be taken, under certain hypotheses, as a fundamental system of units of F_{2n}.

For $n = 2$, it will suffice to show, according to the method of Ljunggren described in [7], that ξ_{21} is the smallest unit (>1) of F_4 such that

$$N_{F_4/F_2}(\xi_{21}) = 1,$$

and that $\xi_{21}\eta_0$ is not a square of F_4.

In the case where $n = 3$, we will use Stender's method: starting with η_0 (>1) the fundamental unit of F_2, we will show that $\eta_3 = \xi_{31}\psi_{31}$ is the fundamental unit of F_3; then we will prove that $E_0 = -\xi_{31}^{-1}\psi_{31}$ is the smallest unit >1 of F_6 such that

$$N_{F_6/F_3}(E_0) = \pm 1, N_{F_6/F_2} = \pm 1;$$

once it is established that neither $E_0\eta_2$ nor $E_0\eta_2^2$ is a cube of F_6, the conclusion will be drawn from the fact that the group generated by $\{\xi_{31}, \psi_{31}, \eta_0\}$ is equal to the group generated by $\{E_0, \eta_3, \eta_0\}$.

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Université, Laval,
Québec, Québec