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FROBENIUS-AFFINE STRUCTURES AND TANGO CURVES

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Abstract. In a previous paper, we discussed *Frobenius-projective structures* on projective smooth curves in positive characteristic and established a relationship between *pseudo-coordinates* and *Frobenius-indigenous structures* by means of Frobenius-projective structures. In the present paper, we discuss an "affine version" of this study of Frobenius-projective structures. More specifically, we discuss *Frobenius-affine structures* and establish a similar relationship between *Tango functions* and *Frobenius-affine-indigenous structures* by means of Frobenius-affine structures. Moreover, we also consider a relationship between these objects and *Tango curves*.

§1. Introduction

In the previous paper [7], we discussed *Frobenius-projective structures* on projective smooth curves in positive characteristic and established a relationship between certain rational functions (i.e., *pseudo-coordinates*) and certain $\mathbb{P}^1$-bundles equipped with sections (that may be regarded as an analogue, in positive characteristic, of indigenous bundles in the classical theory of Riemann surfaces; i.e., *Frobenius-indigenous structures*) by means of Frobenius-projective structures. In the present paper, we discuss an "*affine version*" of this study of Frobenius-projective structures. More specifically, we discuss *Frobenius-affine structures* and establish a similar relationship between *Tango functions* and *Frobenius-affine-indigenous structures*. Moreover, we also consider a relationship between these objects and *Tango curves* (cf., e.g., [8], [9]).

Let p be a prime number, let k be an algebraically closed field of characteristic p, let g be a nonnegative integer, and let

$$X$$

be a projective smooth curve over k of genus g (i.e., a connected scheme that is projective and smooth over k such that the module of global sections of the relative cotangent sheaf over k is of rank g). Throughout the present paper, let us fix a positive integer

$$N.$$

Write X^F for the base change of X by (not the pth if $N \neq 1$ but) the p^Nth power Frobenius endomorphism of k, $\Phi\colon X \to X^F$ for the relative p^Nth power Frobenius morphism over k, PGL_{2,X^F} for the sheaf of groups on X^F obtained by considering automorphisms of the trivial $\mathbb{P}^1$-bundle over X^F (cf. Definition 2.1(ii)), $\mathrm{PGL}^{\infty}_{2,X^F} \subseteq \mathrm{PGL}_{2,X^F}$ for the subsheaf of PGL_{2,X^F} obtained by considering automorphisms of the trivial $\mathbb{P}^1$-bundle over X^F that

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restrict to automorphisms of the trivial $\mathbb{A}^1$-bundle over X^F (cf. Definition 2.1(ii)), and

$$\mathcal{B} \stackrel{\text{def}}{=} \Phi^{-1}\mathrm{PGL}^{\infty}_{2,X^F} \subseteq \mathcal{G} \stackrel{\text{def}}{=} \Phi^{-1}\mathrm{PGL}_{2,X^F}.$$

Write, moreover, $\mathcal{B}_{\mathrm{rtn}} \subseteq \mathcal{G}_{\mathrm{rtn}}$ for the groups obtained by forming the stalks of the sheaves $\mathcal{B} \subseteq \mathcal{G}$ of groups at the generic point of X, respectively.

A *Frobenius-affine structure of level* N on X is defined to be a subsheaf of the sheaf on X of étale morphisms to the affine line $\mathbb{A}^1_k$ over k which forms a $\mathcal{B}$-torsor with respect to the natural action of $\mathcal{B}$ on the sheaf on X of morphisms to $\mathbb{A}^1_k$ over k (cf. Definition 3.1). One finds easily that the notion of Frobenius-affine structures may be regarded as an "affine version" of the notion of Frobenius-projective structures discussed in [7] and, moreover, may be regarded as an analogue, in positive characteristic, of the notion of *complex affine structures* (cf., e.g., [1, §2]) in the classical theory of Riemann surfaces. The main result of the present paper yields a relationship between a certain rational function on X (i.e., a *Tango function*) and a certain $\mathbb{A}^1$-bundle (cf. Remark 4.3.2, and also Remark 2.1.1) equipped with a section (i.e., a *Frobenius-affine-indigenous structure*) obtained by considering Frobenius-affine structures.

A *Tango function of level* N on X is defined to be a (necessarily generically étale) morphism $f\colon X \to \mathbb{P}^1_k$ over k such that, for each closed point $x \in X$ of X, there exist an open subscheme $U \subseteq X$ of X and an element $g \in \mathcal{B}_{\mathrm{rtn}}$ such that $x \in U$, and, moreover, the restriction $g(f)|_U$ to U of the result $g(f)$ of the action of $g \in \mathcal{B}_{\mathrm{rtn}}$ on f is an étale morphism $U \to \mathbb{A}^1_k$ (cf. Definition 2.3). For instance, if $p = 2$, then every generically étale morphism to $\mathbb{P}^1_k$ over k is a Tango function of level 1 (cf. Remark 2.7.1). Moreover, we prove the following result (cf. Corollary 2.11).

THEOREM A. *It holds that X is a Tango curve (cf. Definition 2.8(ii)) if and only if X has a Tango function of level* 1.

A *Frobenius-affine-indigenous structure of level* N on X is defined to be a pair of an $\mathbb{A}^1$-bundle $A \to X^F$ over X^F and a section σ of the pullback $\Phi^*A \to X$ such that the Kodaira–Spencer section of the PD-connection ∇_{Φ^*A} on Φ^*A at σ is nowhere vanishing (cf. Definition 4.3). One may find that the notion of Frobenius-affine-indigenous structures of level 1 is closely related to the notion of *dormant Miura* GL_2*-opers* discussed in [10] (cf. Remark 5.2.3 and Proposition 5.7).

The main result of the present paper is as follows (cf. Theorem 4.10).

THEOREM B. *There exist bijective maps between the following three sets:*

(1) *the set of $\mathcal{B}_{\mathrm{rtn}}$-orbits of Tango functions of level N on X;*
(2) *the set of Frobenius-affine structures of level N on X;*
(3) *the set of isomorphism classes of Frobenius-affine-indigenous structures of level N on X.*

Note that if $(p,N) \neq (2,1)$, then the bijective maps of Theorem B are *compatible* with the bijective maps between the following three sets of [7, Th. A] (cf. Remark 4.10.1):

- the set of $\mathcal{G}_{\mathrm{rtn}}$-orbits of *pseudo-coordinates of level* N on X;
- the set of *Frobenius-projective structures of level* N on X;
- the set of isomorphism classes of *Frobenius-indigenous structures of level* N on X.

As already observed, the notion of Frobenius-affine structures may be regarded as an analogue, in positive characteristic, of the notion of *complex affine structures* in the classical theory of Riemann surfaces. Moreover, it is well-known that if a compact Riemann surface admits a complex affine structure, then the compact Riemann surface is *of genus* 1. On the other hand, one may conclude from Theorem B that there exists a projective smooth curve over k *of genus* ≥ 2 that has a *Frobenius-affine structure of level* N (cf. Remark 3.7.1).

One application of Theorem B is as follows. Suppose that $g \geq 2$. Write $\mathrm{Fr}_X \colon X \to X$ for the pth power Frobenius endomorphism of X. Then one may verify (cf. Remark 5.2.4 and Proposition 5.7) that there exists a *bijective* map between the set of Theorem B(3) and the set of $\mathbb{P}$-equivalence (cf. Definition 5.1) classes of pairs $(\mathcal{E}, \mathcal{L})$ of locally free coherent $\mathcal{O}_X$-modules $\mathcal{E}$ of rank 2 and invertible subsheaves $\mathcal{L} \subseteq \mathcal{E}$ that satisfy the following condition: if, for a nonnegative integer i, we write

$$\mathcal{L}_i \stackrel{\mathrm{def}}{=} \overbrace{\mathrm{Fr}_X^* \cdots \mathrm{Fr}_X^*}^{i} \mathcal{L} \subseteq \mathcal{E}_i \stackrel{\mathrm{def}}{=} \overbrace{\mathrm{Fr}_X^* \cdots \mathrm{Fr}_X^*}^{i} \mathcal{E},$$

then

- the locally free coherent $\mathcal{O}_X$-module $\mathcal{E}_{N-1}$, hence also $\mathcal{E}$, is *stable*, but
- there exist an invertible sheaf $\mathcal{M}$ on X of degree $\frac{p^N}{2} \cdot \deg(\mathcal{E}) + g - 1 = \frac{1}{2} \cdot \deg(\mathcal{E}_N) + g - 1$ and a locally split injective homomorphism $\mathcal{M} \hookrightarrow \mathcal{E}_N$ of $\mathcal{O}_X$-modules such that the inclusions $\mathcal{L}_N$, $\mathcal{M} \hookrightarrow \mathcal{E}_N$ determine an *isomorphism* $\mathcal{L}_N \oplus \mathcal{M} \xrightarrow{\sim} \mathcal{E}_N$ of $\mathcal{O}_X$-modules. (In particular, the locally free coherent $\mathcal{O}_X$-module $\mathcal{F}_N$ is *not semistable*.)

Thus, by applying Theorem B and some previous works, we obtain the following application in *small characteristic* cases (cf. Corollary 6.5(ii)).

Theorem C. *Suppose that $g \geq 2$, and that $p = 2$ (resp. $p = 3$). Suppose, moreover, that $N \geq 2$ whenever $p = 2$. Then the following two conditions are equivalent:*

(1) *The curve X has a Tango function of level N.*
(2) *There exist:*
   - *a (necessarily stable) locally free coherent $\mathcal{O}_X$-module $\mathcal{E}$ of rank 2,*
   - *an invertible sheaf $\mathcal{Q}$ on X of degree $(2g-2)/p^N$ (resp. $(4g-4)/p^N$),*
   - *a surjective homomorphism $\mathcal{E} \twoheadrightarrow \mathcal{Q}$ of $\mathcal{O}_X$-modules, and*
   - *an isomorphism $(\mathrm{Fr}_X)_* \mathcal{O}_X \xrightarrow{\sim} \mathcal{E}_{N-1}$ (resp. $\mathcal{B}_X \xrightarrow{\sim} \mathcal{E}_{N-1}$) of $\mathcal{O}_X$-modules,*

   where we write

$$\mathcal{B}_X \stackrel{\mathrm{def}}{=} \mathrm{Coker}\big(\mathcal{O}_X \to (\mathrm{Fr}_X)_* \mathcal{O}_X\big)$$

   for the $\mathcal{O}_X$-module obtained by forming the cokernel of the homomorphism $\mathcal{O}_X \to (\mathrm{Fr}_X)_* \mathcal{O}_X$ induced by Fr_X.

§2. Tango functions

In the present section, we introduce and discuss the notion of *Tango functions* (cf. Definition 2.3). Moreover, we also discuss a relationship between Tango functions and Tango curves studied in, for instance, [8] and [9] (cf. Theorem 2.9 and Corollary 2.11).

In the present section, let p be a prime number, let k be an algebraically closed field of characteristic p, let g be a nonnegative integer, and let

$$X$$

be a projective smooth curve over k of genus g (i.e., a connected scheme that is projective and smooth over k such that the module of global sections of the relative cotangent sheaf over k is of rank g). Throughout the present paper, let us fix a positive integer

$$N.$$

If "$(-)$" is an object over k, then we shall write "$(-)^F$" for the object over k obtained by forming the base change of "$(-)$" by (not the pth if $N \neq 1$ but) the p^Nth power Frobenius endomorphism of k. We shall write

$$W\colon X^F \longrightarrow X$$

for the morphism obtained by forming the base change of the p^Nth power Frobenius endomorphism of $\mathrm{Spec}(k)$ by the structure morphism $X \to \mathrm{Spec}(k)$. Thus, the p^Nth power Frobenius endomorphism of X factors as a composite

$$X \longrightarrow X^F \xrightarrow{W} X.$$

We shall write

$$\Phi\colon X \longrightarrow X^F$$

for the first arrow in this composite, that is, the *relative p^Nth power Frobenius morphism* over k. Note that X^F is a *projective smooth curve over k of genus g*, and Φ is a *finite flat* morphism over k *of degree p^N*.

DEFINITION 2.1. Let S be a scheme.

(i) We shall write

$$\mathbb{A}^1_S \longrightarrow S$$

for the trivial $\mathbb{A}^1$-bundle over S,

$$\mathbb{P}^1_S \longrightarrow S$$

for the trivial $\mathbb{P}^1$-bundle over S obtained by forming the smooth compactification of $\mathbb{A}^1_S \to S$, and

$$\infty_S \in \mathbb{P}^1_S(S)$$

for the section of $\mathbb{P}^1_S \to S$ obtained by considering the complement of $\mathbb{A}^1_S$ in $\mathbb{P}^1_S$. Thus, $\mathbb{A}^1_k \stackrel{\mathrm{def}}{=} \mathbb{A}^1_{\mathrm{Spec}(k)} \subseteq \mathbb{P}^1_k \stackrel{\mathrm{def}}{=} \mathbb{P}^1_{\mathrm{Spec}(k)}$ denote the affine, projective lines over k, respectively, and $\infty_k \stackrel{\mathrm{def}}{=} \infty_{\mathrm{Spec}(k)} \in \mathbb{P}^1_k(k)$ denotes the k-rational closed point of $\mathbb{P}^1_k$ obtained by considering the complement of $\mathbb{A}^1_k$ in $\mathbb{P}^1_k$.

(ii) We shall write

$$\mathrm{PGL}_{2,S}$$

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for the sheaf of groups on S that assigns, to an open subscheme $T \subseteq S$, the group $\mathrm{Aut}_T(\mathbb{P}^1_T)$ of automorphisms over T of the trivial $\mathbb{P}^1$-bundle $\mathbb{P}^1_T \to T$ and

$$\mathrm{PGL}^{\infty}_{2,S} \subseteq \mathrm{PGL}_{2,S}$$

for the sheaf of groups on S that assigns, to an open subscheme $T \subseteq S$, the subgroup of $\mathrm{Aut}_T(\mathbb{P}^1_T)$ consisting of automorphisms over T of the trivial $\mathbb{P}^1$-bundle $\mathbb{P}^1_T \to T$ that preserve the section $\infty_T \in \mathbb{P}^1_T(T)$, or, equivalently, restrict to automorphisms of the open subscheme $\mathbb{A}^1_T \subseteq \mathbb{P}^1_T$ over T (cf. Remark 2.1.1).

(iii) We shall write

$$\mathcal{B} \stackrel{\text{def}}{=} \Phi^{-1}\mathrm{PGL}^{\infty}_{2,X^F} \subseteq \mathcal{G} \stackrel{\text{def}}{=} \Phi^{-1}\mathrm{PGL}_{2,X^F}$$

and

$$\mathcal{B}_{\mathrm{rtn}} \subseteq \mathcal{G}_{\mathrm{rtn}}$$

for the groups obtained by forming the stalks of $\mathcal{B} \subseteq \mathcal{G}$ at the generic point of X, respectively.

Remark 2.1.1. One verifies easily that, in the situation of Definition 2.1, if the scheme S is *integral*, then the sheaf of groups on S that assigns, to an open subscheme $T \subseteq S$, the group $\mathrm{Aut}_T(\mathbb{A}^1_T)$ of automorphisms over T of the trivial $\mathbb{A}^1$-bundle $\mathbb{A}^1_T \to T$ may be naturally identified with the subsheaf $\mathrm{PGL}^{\infty}_{2,S} \subseteq \mathrm{PGL}_{2,S}$ of $\mathrm{PGL}_{2,S}$ of Definition 2.1(ii).

Definition 2.2.

(i) We shall write

$$\mathcal{P}$$

for the sheaf of sets on X that assigns, to an open subscheme $U \subseteq X$, the set of morphisms from U to $\mathbb{P}^1_k$ over k,

$$\mathcal{P}^{\text{gét}} \subseteq \mathcal{P}$$

for the subsheaf of $\mathcal{P}$ that assigns, to an open subscheme $U \subseteq X$, the set of generically étale morphisms from U to $\mathbb{P}^1_k$ over k, and

$$\mathcal{P}^{\text{ét}} \subseteq \mathcal{P}^{\text{gét}}$$

for the subsheaf of $\mathcal{P}^{\text{gét}}$ that assigns, to an open subscheme $U \subseteq X$, the set of étale morphisms from U to $\mathbb{P}^1_k$ over k.

(ii) We shall write

$$\mathcal{A}\ (\subseteq \mathcal{P})$$

for the sheaf of sets on X that assigns, to an open subscheme $U \subseteq X$, the set of morphisms from U to $\mathbb{A}^1_k$ over k and

$$\mathcal{A}^{\text{ét}} \stackrel{\text{def}}{=} \mathcal{A} \times_{\mathcal{P}} \mathcal{P}^{\text{ét}} \subseteq \mathcal{A}$$

for the subsheaf of $\mathcal{A}$ that assigns, to an open subscheme $U \subseteq X$, the set of étale morphisms from U to $\mathbb{A}^1_k$ over k.

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Remark 2.2.1.

(i) One verifies easily that both $\mathcal{P}$ and $\mathcal{P}^{\text{gét}}$ are (isomorphic to) *constant sheaves*.
(ii) One verifies easily that $\mathcal{P}$, $\mathcal{A}$ may be naturally identified with the sheaves of sets on X that assign, to an open subscheme $U \subseteq X$, the sets of sections of the trivial $\mathbb{P}^1$-bundle $\mathbb{P}^1_U \to U$, the trivial $\mathbb{A}^1$-bundle $\mathbb{A}^1_U \to U$, respectively.
(iii) It follows immediately from (ii) that $\mathcal{G}$, hence also $\mathcal{B}$, naturally *acts*, via Φ, on $\mathcal{P}$. Moreover, one verifies easily that the subsheaves $\mathcal{P}^{\text{ét}} \subseteq \mathcal{P}^{\text{gét}} \subseteq \mathcal{P}$ of $\mathcal{P}$ are *preserved* by this action of $\mathcal{G}$, hence also of $\mathcal{B}$, on $\mathcal{P}$.
(iv) It is immediate from (i) that the *actions* of $\mathcal{G}$ on $\mathcal{P}$, $\mathcal{P}^{\text{gét}}$ of (iii) determine *actions* of $\mathcal{G}_{\text{rtn}}$ on $\mathcal{P}(X)$, $\mathcal{P}^{\text{gét}}(X)$, respectively. In particular, the *actions* of $\mathcal{B}$ on $\mathcal{P}$, $\mathcal{P}^{\text{gét}}$ of (iii) determine *actions* of $\mathcal{B}_{\text{rtn}}$ on $\mathcal{P}(X)$, $\mathcal{P}^{\text{gét}}(X)$, respectively.
(v) It follows immediately from (ii) that $\mathcal{B}$ naturally *acts*, via Φ, on $\mathcal{A}$. In particular, it follows from (iii) that the subsheaf $\mathcal{A}^{\text{ét}} \subseteq \mathcal{A}$ of $\mathcal{A}$ is *preserved* by this action of $\mathcal{B}$ on $\mathcal{A}$.

Definition 2.3. We shall say that a generically étale morphism $f\colon X \to \mathbb{P}^1_k$ over k is a *Tango function of level N* if, for each closed point $x \in X$ of X, there exists an element $g \in \mathcal{B}_{\text{rtn}}$ such that the morphism $g(f)\colon X \to \mathbb{P}^1_k$ (cf. Remark 2.2.1(iv)) over k is étale at $x \in X$, and, moreover, $g(f)(x) \neq \infty_k$.

We shall write

$$\mathfrak{Tf}_N(X) \subseteq \mathcal{P}^{\text{gét}}(X)$$

for the subset of Tango functions of level N.

Remark 2.3.1. One verifies easily that if a global section of $\mathcal{P}^{\text{gét}}$ is a *Tango function of level N*, then every element of the $\mathcal{B}_{\text{rtn}}$-orbit ($\subseteq \mathcal{P}^{\text{gét}}(X)$) of the global section is a *Tango function of level N*.

Remark 2.3.2. It is immediate that an arbitrary *Tango function of level N* is a *pseudo-coordinate of level N* (cf. [7, Def. 2.3]). Thus, we have a commutative diagram

$$\begin{tikzcd}
\mathfrak{Tf}_N(X) \arrow[r, hook] \arrow[d] & \mathfrak{pcd}_N(X) \arrow[d] \\
\mathfrak{Tf}_N(X)/\mathcal{B}_{\text{rtn}} \arrow[r] & \mathfrak{pcd}_N(X)/\mathcal{G}_{\text{rtn}}
\end{tikzcd}$$

(cf. Remark 2.3.1, [7, Def. 2.3], and [7, Rem. 2.3.1]).

Definition 2.4. Let $f \in \mathcal{P}^{\text{gét}}(X)$ be a global section of $\mathcal{P}^{\text{gét}}$, and let $x \in X$ be a closed point of X. Let us identify $A \overset{\text{def}}{=} k[[t]]$ with the completion $\widehat{\mathcal{O}}_{X,x}$ of the local ring $\mathcal{O}_{X,x}$ by means of a fixed isomorphism $A \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}$ over k. Write $F \in \mathcal{O}_{X,x}$ for the image, via f, in $\mathcal{O}_{X,x}$ of a fixed uniformizer of the discrete valuation ring $\mathcal{O}_{\mathbb{P}^1_k,f(x)}$ and

$$F = \textstyle\sum_{i\geq 1} a_i t^i \in A$$

for the expansion of F in A. Thus, the positive integer

$$\operatorname{ind}_x(f) \overset{\text{def}}{=} \nu_A(F) = \min\{\, i \in \mathbb{Z}_{\geq 1} \mid a_i \neq 0 \,\}$$

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(where ν_A denotes the t-adic valuation on $A = k[[t]]$ that maps $t \in A$ to 1) coincides with the ramification index of the dominant morphism $f\colon X \to \mathbb{P}^1_k$ at $x \in X$. Then we shall write

$$\mathrm{ind}_x^{\notin p^N}(f) \stackrel{\mathrm{def}}{=} \min\{\, i \in \mathbb{Z}_{\geq 1} \mid a_i \neq 0 \text{ and } i \notin p^N\mathbb{Z}\} \quad \big(\geq \mathrm{ind}_x(f)\big)$$

and

$$\underline{\mathrm{ind}}_x^{\notin p^N}(f)$$

for the uniquely determined positive integer such that $1 \leq \underline{\mathrm{ind}}_x^{\notin p^N}(f) \leq p^N - 1$, and, moreover, $\mathrm{ind}_x^{\notin p^N}(f) - \underline{\mathrm{ind}}_x^{\notin p^N}(f) \in p^N\mathbb{Z}$.

Note that one verifies easily that since f is a global section of $\mathcal{P}^{\text{gét}}$, it holds that $\mathrm{ind}_x^{\notin p^N}(f) < \infty$. Moreover, one also verifies easily that both $\mathrm{ind}_x^{\notin p^N}(f)$ and $\underline{\mathrm{ind}}_x^{\notin p^N}(f)$ are *independent* of the choices of the fixed isomorphism $A \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}$ and the fixed uniformizer of $\mathcal{O}_{\mathbb{P}^1_k, f(x)}$.

LEMMA 2.5. *Let $f \in \mathcal{P}^{\text{gét}}(X)$ be a global section of $\mathcal{P}^{\text{gét}}$, and let $x \in X$ be a closed point of X. Then the following assertions hold:*

(i) *Suppose that $f(x) \neq \infty_k$. Then there exists an element $g \in \mathcal{B}_{\mathrm{rtn}}$ such that*

$$g(f)(x) \neq \infty_k, \qquad \mathrm{ind}_x\big(g(f)\big) = \underline{\mathrm{ind}}_x^{\notin p^N}(f)$$

(*which thus implies that* $\mathrm{ind}_x(g(f)) = \underline{\mathrm{ind}}_x^{\notin p^N}(g(f))$).

(ii) *Suppose that $f(x) = \infty_k$, and that $\mathrm{ind}_x(f) = \mathrm{ind}_x^{\notin p^N}(f)$. Then there exists an element $g \in \mathcal{B}_{\mathrm{rtn}}$ such that*

$$g(f)(x) \neq \infty_k, \qquad \mathrm{ind}_x\big(g(f)\big) = p^N - \underline{\mathrm{ind}}_x^{\notin p^N}(f)$$

(*which thus implies that* $\mathrm{ind}_x(g(f)) = \underline{\mathrm{ind}}_x^{\notin p^N}(g(f)) = p^N - \underline{\mathrm{ind}}_x^{\notin p^N}(g(f))$).

(iii) *Suppose that $f(x) = \infty_k$, and that $\mathrm{ind}_x(f) \neq \mathrm{ind}_x^{\notin p^N}(f)$. Then there exists an element $g \in \mathcal{B}_{\mathrm{rtn}}$ such that*

$$g(f)(x) \neq \infty_k, \qquad \mathrm{ind}_x\big(g(f)\big) = \underline{\mathrm{ind}}_x^{\notin p^N}(f)$$

(*which thus implies that* $\mathrm{ind}_x(g(f)) = \underline{\mathrm{ind}}_x^{\notin p^N}(g(f))$).

Proof. Write K_X for the function field of X. Let us identify the scheme $\mathrm{Proj}(k[u,v])$ with $\mathbb{P}^1_k$ by means of a fixed isomorphism $\mathrm{Proj}(k[u,v]) \xrightarrow{\sim} \mathbb{P}^1_k$ over k that maps the point "$(u,v) = (1,0)$" to the closed point ∞_k. Thus, the global section $f \in \mathcal{P}^{\text{gét}}(X)$ determines and is determined by an element F of $K_X \setminus K_X^p$ (i.e., the image of $u/v \in k(u/v)$ in K_X via f). Now, let us first observe that if $f(x) \neq \infty_k$, then we may assume without loss of generality, by replacing f by the composite of f and a suitable element of $\mathrm{Aut}_k(\mathbb{A}^1_k)$, that $f(x)$ is the point "$(u,v) = (0,1)$," that is, that $F \in \mathfrak{m}_x$. Let us identify $A \stackrel{\mathrm{def}}{=} k[[t]]$ with the completion $\widehat{\mathcal{O}}_{X,x}$ of the local ring $\mathcal{O}_{X,x}$ by means of a fixed isomorphism $A \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}$ over k that *maps* $t \in A$ *into* $\mathcal{O}_{X,x} \subseteq \widehat{\mathcal{O}}_{X,x}$. (Thus, it holds that $F \in tA[[t]]$ (resp. $F^{-1} \in tA[[t]]$) whenever $f(x) \neq \infty_k$ (resp. $f(x) = \infty_k$).) Write $\underline{d}_0 \stackrel{\mathrm{def}}{=} \underline{\mathrm{ind}}_x^{\notin p^N}(f)$.

Now, we verify assertion (i). Let us first observe that it follows from the definition of "$\underline{\mathrm{ind}}_x^{\notin p^N}(f)$" that there exist $a \in \mathcal{O}_{X,x}$, $u \in A^\times$, and a nonnegative integer r such that

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$F = a^{p^N} - t^{rp^N + \underline{d}_0}u$, and, moreover, either $a = 0$ or $\nu_A(a^{p^N})$ $(= p^N \nu_A(a)) < rp^N + \underline{d}_0$. Then one verifies immediately from the various definitions involved that the global section of $\mathcal{P}^{\text{gét}}$ that corresponds to the element $t^{-rp^N}(F - a^{p^N})$ of $K_X \setminus K_X^p$ is *contained* in the $\mathcal{B}_{\text{rtn}}$-orbit of f and *satisfies* the condition in the statement of assertion (i). This completes the proof of assertion (i).

Next, we verify assertions (ii) and (iii). Let us first observe that it follows from the definition of "$\underline{\text{ind}}_x^{\notin p^N}(f)$" that there exist $a \in \mathcal{O}_{X,x}$, $u \in A^\times$, and a nonnegative integer r such that $F^{-1} = a^{p^N} - t^{rp^N + \underline{d}_0}u$, and, moreover, $a = 0$ in the situation of assertion (ii) (resp. $\nu_A(a^{p^N})(= p^N \nu_A(a)) < rp^N + \underline{d}_0$ in the situation of assertion (iii)). Then one verifies immediately from the various definitions involved that if we are in the situation of assertion (ii), then the global section of $\mathcal{P}^{\text{gét}}$ that corresponds to the element $t^{(r+1)p^N}F$ of $K_X \setminus K_X^p$ is *contained* in the $\mathcal{B}_{\text{rtn}}$-orbit of f and *satisfies* the condition in the statement of assertion (ii). This completes the proof of assertion (ii).

Next, to verify assertion (iii), observe that, in the situation of assertion (iii), since $F^{-1} = a^{p^N} - t^{rp^N + \underline{d}_0}u = a^{p^N}(1 - a^{-p^N}t^{rp^N + \underline{d}_0}u)$, and $0 < rp^N + \underline{d}_0 - p^N \nu_A(a)$, it follows that

$$F = a^{-p^N}(1 + a^{-p^N}t^{rp^N + \underline{d}_0}u + a^{-2p^N}t^{2rp^N + 2\underline{d}_0}u^2 + \cdots).$$

Then one verifies immediately from the various definitions involved that the global section of $\mathcal{P}^{\text{gét}}$ that corresponds to the element $t^{-rp^N}a^{2p^N}F - t^{-rp^N}a^{p^N}$ of $K_X \setminus K_X^p$ is *contained* in the $\mathcal{B}_{\text{rtn}}$-orbit of f and *satisfies* the condition in the statement of assertion (iii). This completes the proof of assertion (iii), hence also of Lemma 2.5. $\square$

LEMMA 2.6. *Let $f \in \mathcal{P}^{\text{gét}}(X)$ be a global section of $\mathcal{P}^{\text{gét}}$, and let $x \in X$ be a closed point of X. Suppose that $f(x) \neq \infty_k$, and that $\underline{\text{ind}}_x^{\notin p^N}(f) \neq 1$. Then, for each $g \in \mathcal{B}_{\text{rtn}}$, the result $g(f) \in \mathcal{P}^{\text{gét}}(X)$ of the action of $g \in \mathcal{B}_{\text{rtn}}$ on $f \in \mathcal{P}^{\text{gét}}(X)$ either is not étale at x or maps x to ∞_k.*

Proof. Let us first observe that it follows immediately from Lemma 2.5(i) that we may assume without loss of generality, by replacing f by the result of the action of a suitable element of $\mathcal{B}_{\text{rtn}}$ on f, that

(a) $\text{ind}_x(f) = \underline{d}_0 \stackrel{\text{def}}{=} \underline{\text{ind}}_x^{\notin p^N}(f)$ $(\neq 1)$.

Let us identify $A \stackrel{\text{def}}{=} k[[t]]$ with the completion $\widehat{\mathcal{O}}_{X,x}$ of the local ring $\mathcal{O}_{X,x}$ by means of a fixed isomorphism $A \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}$ over k. Then it is immediate that, to verify Lemma 2.6, it suffices to verify that

> $(*_1)$: for each $g \in \mathcal{B}_{\text{rtn}}$, the composite of the natural morphism $\text{Spec}(A) \to X$ with $g(f)\colon X \to \mathbb{P}_k^1$ is *not formally étale* whenever this composite does *not map* the closed point of $\text{Spec}(A)$ to ∞_k.

Let g be an element of $\mathcal{B}_{\text{rtn}}$. Next, let us identify the scheme $\text{Proj}(k[u,v])$ with $\mathbb{P}_k^1$ by means of a fixed isomorphism $\text{Proj}(k[u,v]) \xrightarrow{\sim} \mathbb{P}_k^1$ over k that maps the point "$(u,v) = (1,0)$" to the closed point ∞_k. Write K for the field of fractions of A and

$$\text{Proj}(k[u,v]) \longleftarrow \text{Spec}(A); \quad (u,v) \mapsto (f_u, f_v)$$

(where f_u, $f_v \in A$) for the composite of the natural morphism $\text{Spec}(A) \to X$ with $f\colon X \to \mathbb{P}_k^1$. Thus, there exist a_g, b_g, $d_g \in k[[t^{p^N}]] = A^{p^N} \subseteq A$ (which thus implies that $\nu_A(a_g)$, $\nu_A(b_g)$,

$\nu_A(d_g) \in p^N\mathbb{Z}$) such that $a_g d_g \neq 0$, and, moreover, the composite of the natural morphism $\mathrm{Spec}(A) \to X$ with $g(f)\colon X \to \mathbb{P}^1_k$ *coincides* with the morphism determined by the composite

$$\begin{array}{ccccccc} \mathrm{Proj}(k[u,v]) & \longleftarrow & \mathrm{Proj}(K[u,v]) & \longleftarrow & \mathrm{Proj}(K[u,v]) & \longleftarrow & \mathrm{Spec}(K) \\ (u,v) & \mapsto & (u,v) & & & & \\ & & (u,v) & \mapsto & (a_g u + b_g v, d_g v) & & \\ & & & & (u,v) & \mapsto & (f_u, f_v). \end{array}$$

Next, let us observe that, to verify $(*_1)$, we may assume without loss of generality, by replacing f by the composite of f and a suitable element of $\mathrm{Aut}_k(\mathbb{A}^1_k)$, that the image of $x \in X$ via f is the point "$(u,v) = (0,1)$," that is, that (cf. (a))

(b) $\nu_A(f_u) = \underline{d}_0$, and $f_v = 1$. (Recall that $2 \leq \underline{d}_0 \leq p^N - 1$; cf. (a).)

Next, let us observe that, to verify $(*_1)$, we may assume without loss of generality, by replacing g by the product of g and a suitable element of $\mathrm{Aut}_k(\mathbb{A}^1_k)$, that the image of $x \in X$ via $g(f)$ is the point "$(u,v) = (0,1)$," that is, that (cf. (b))

(c) if we write

$$F \stackrel{\text{def}}{=} \frac{a_g f_u + b_g}{d_g} \in K,$$

then $F \in A$, and, moreover, $\nu_A(F) \geq 1$.

Thus, it is immediate that, to verify $(*_1)$, it suffices to verify that

$(*_2)$: $\nu_A(F) \neq 1$.

Next, let us observe that, to verify $(*_2)$, we may assume without loss of generality, by replacing (a_g, b_g, d_g) by $t^{-\min\{\nu_A(a_g), \nu_A(b_g), \nu_A(d_g)\}} \cdot (a_g, b_g, d_g)$, that

(d) $0 \in \{\nu_A(a_g), \nu_A(b_g), \nu_A(d_g)\}$.

Here, let us verify that

(e) $\nu_A(b_g) \geq p^N$.

Indeed, if $\nu_A(b_g) = 0$, then it follows from (b) that $\nu_A(a_g f_u + b_g) = 0$, which thus implies that $\nu_A(F) \leq 0$—in *contradiction* to (c). This completes the proof of (e).

Next, suppose that $\nu_A(d_g) = 0$. Then it follows from (b) and (e) that $\nu_A(F) = \nu_A(a_g f_u + b_g) \geq 2$, as desired. Thus, to verify $(*_2)$, we may assume without loss of generality that

(f) $\nu_A(d_g) \geq p^N$.

Thus, it follows from (d)–(f) that $\nu_A(a_g) = 0$. Then it follows from (b) and (e) that $\nu_A(a_g f_u + b_g) = \underline{d}_0$. In particular, it follows from (b) and (f) that $\nu_A(F) = \underline{d}_0 - \nu_A(d_g) < 0$—in *contradiction* to (c). This completes the proof of $(*_2)$, hence also of Lemma 2.6. ∎

Proposition 2.7. *Let $f \in \mathcal{P}^{\text{gét}}(X)$ be a global section of $\mathcal{P}^{\text{gét}}$. Then it holds that f is a Tango function of level N if and only if, for each closed point $x \in X$ of X, the equality*

$$\underline{\mathrm{ind}}_x^{\notin p^N}(f) = \begin{cases} 1, & \text{if either } f(x) \neq \infty_k \text{ or } \mathrm{ind}_x(f) \neq \mathrm{ind}_x^{\notin p^N}(f) \\ p^N - 1, & \text{if } f(x) = \infty_k \text{ and } \mathrm{ind}_x(f) = \mathrm{ind}_x^{\notin p^N}(f) \end{cases}$$

holds.

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Proof. The sufficiency follows immediately from Lemma 2.5(i)–(iii). The necessity follows from Lemma 2.5(ii) and (iii) and Lemma 2.6. ∎

REMARK 2.7.1. Suppose that $(p,N) = (2,1)$. Then one verifies easily from Proposition 2.7 that every global section of $\mathcal{P}^{\text{gét}}$ is a *Tango function of level* N:

$$\mathcal{P}^{\text{gét}}(X) = \mathfrak{Tf}_N(X).$$

Moreover, one also verifies easily that

$$\sharp\big(\mathcal{P}^{\text{gét}}(X)/\mathcal{B}_{\text{rtn}}\big) = \sharp\big(\mathfrak{Tf}_N(X)/\mathcal{B}_{\text{rtn}}\big) = 1.$$

REMARK 2.7.2. One may construct some examples of *Tango functions* by means of the well-known structure of the maximal pro-prime-to-p quotient of the abelianization of the étale fundamental group of an open subscheme of the projective line over an algebraically closed field of characteristic p as follows: Let r be a positive integer. Write $d \stackrel{\text{def}}{=} (rp^N - 1)(p^N - 1)$. Let $a_1, \ldots, a_{d-(p^N-1)} \in \mathbb{A}^1_k$ be distinct $d - (p^N - 1)$ closed points of $\mathbb{A}^1_k$. Write $a_0 \stackrel{\text{def}}{=} \infty_k \in \mathbb{P}^1_k$, P for the ring obtained by forming the pro-prime-to-p completion of the ring $\mathbb{Z}$ of rational integers, and Q for the maximal pro-prime-to-p quotient of the abelianization of the étale fundamental group of $\mathbb{A}^1_k \setminus \{a_1, \ldots, a_{d-(p^N-1)}\} = \mathbb{P}^1_k \setminus \{a_0, a_1, \ldots, a_{d-(p^N-1)}\}$. Then it is well-known that, for each $i \in \{0, 1, \ldots, d - (p^N - 1)\}$, there exists an element γ_i of Q such that:

(a) these elements of Q determine an *isomorphism* between Q and the quotient of the free P-module freely generated by the γ_i's (where $i \in \{0, 1, \ldots, d - (p^N - 1)\}$) by the P-submodule generated by $\gamma_0 + \gamma_1 + \cdots + \gamma_{d-(p^N-1)}$, and, moreover,
(b) for each $i \in \{0, 1, \ldots, d - (p^N - 1)\}$, the element γ_i topologically generates the *inertia subgroup* of Q associated with the closed point a_i of $\mathbb{P}^1_k$.

Thus, it follows from (a) that there exists a *surjective* homomorphism $Q \twoheadrightarrow \mathbb{Z}/d\mathbb{Z}$ of groups that maps the element γ_0 to $p^N - 1 \in \mathbb{Z}/d\mathbb{Z}$ and, for each $i \in \{1, \ldots, d - (p^N - 1)\}$, maps the element γ_i to $1 \in \mathbb{Z}/d\mathbb{Z}$. Write

$$f_{N,r} \colon C_{N,r} \longrightarrow \mathbb{P}^1_k$$

for the morphism over k (that is necessarily *finite* and *of degree* d) obtained by forming the smooth compactification of the finite étale Galois covering of $\mathbb{P}^1_k \setminus \{a_0, a_1, \ldots, a_{d-(p^N-1)}\}$ determined by a surjective homomorphism $Q \twoheadrightarrow \mathbb{Z}/d\mathbb{Z}$ as above. Thus, it follows from (b) and the condition imposed on the surjective homomorphism $Q \twoheadrightarrow \mathbb{Z}/d\mathbb{Z}$ that:

(c) the finite morphism $f_{N,r}$ is *étale* over the open subscheme $\mathbb{P}^1_k \setminus \{a_0, a_1, \ldots, a_{d-(p^N-1)}\}$ of $\mathbb{P}^1_k$,
(d) the fiber $f_{N,r}^{-1}(a_0)$ is of cardinality $p^N - 1$, and the equality $\text{ind}_x(f_{N,r}) = rp^N - 1$ holds for each $x \in f_{N,r}^{-1}(a_0)$, and
(e) for each $i \in \{1, \ldots, d - (p^N - 1)\}$, the fiber $f_{N,r}^{-1}(a_i)$ is of cardinality 1, and the equality $\text{ind}_x(f_{N,r}) = d$ holds for each $x \in f_{N,r}^{-1}(a_i)$.

In particular, it follows from Proposition 2.7, together with (c)–(e), that the global section $f_{N,r}$ of "$\mathcal{P}^{\text{gét}}$" for the projective smooth curve $C_{N,r}$ over k is a *Tango function of level* N.

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Note that it follows from the *Riemann–Hurwitz formula* that if one writes $g_{N,r}$ for the genus of $C_{N,r}$, then the equalities

$$2g_{N,r} - 2 = d(d - p^N - 1) = dp^N\big(r(p^N - 1) - 2\big)$$

hold. In particular, one concludes that the inequality $g_{N,r} \geq 2$ holds if and only if $(p,N,r) \notin \{(2,1,1),(2,1,2),(3,1,1)\}$.

Finally, we discuss a relationship between *Tango functions* and *Tango curves* studied in, for instance, [8] and [9].

Definition 2.8.

(i) Let $f \in \mathcal{P}^{\text{gét}}(X)$ be a global section of $\mathcal{P}^{\text{gét}}$. Then we shall write

$$\boldsymbol{n}(N;f) \stackrel{\text{def}}{=} \textstyle\sum_{x \in X:\,\text{closed}}[\nu_x(df)/p^N]$$

(where we write ν_x for the discrete valuation on the function field of X that corresponds to the closed point x and maps a uniformizer of $\mathcal{O}_{X,x}$ to 1 and "$[-]$" for the uniquely determined maximal integer less than or equal to "$(-)$"; cf. [9, Def. 9]).

(ii) We shall say that X is a *Tango curve* if there exists a global section $f \in \mathcal{P}^{\text{gét}}(X)$ of $\mathcal{P}^{\text{gét}}$ such that $\boldsymbol{n}(1;f) = (2g-2)/p$ (cf., e.g., [9] and [8, §2.1]).

Theorem 2.9. *Let $f \in \mathcal{P}^{\text{gét}}(X)$ be a global section of $\mathcal{P}^{\text{gét}}$. Then the following assertions hold:*

(i) *If f is a Tango function of level N, then the equality $\boldsymbol{n}(N;f) = (2g-2)/p^N$ holds.*

(ii) *It holds that f is a Tango function of level 1 if and only if the equality $\boldsymbol{n}(1;f) = (2g-2)/p$ holds.*

Proof. These assertions follow immediately from Proposition 2.7, together with the well-known fact that the relative cotangent sheaf of X/k is *of degree* $2g-2$. ∎

Corollary 2.10. *If X has a Tango function of level N of X, then $2g-2$ is divisible by p^N.*

Proof. This assertion is an immediate consequence of Theorem 2.9(i). ∎

Corollary 2.11. *It holds that X is a Tango curve if and only if X has a Tango function of level 1.*

Proof. This assertion is an immediate consequence of Theorem 2.9(ii). ∎

§3. Frobenius-affine structures

In the present section, we introduce and discuss the notion of *Frobenius-affine structures* (cf. Definition 3.1). Moreover, we also discuss a relationship between Frobenius-affine structures and Tango functions (cf. Proposition 3.7). In the present section, we maintain the notational conventions introduced at the beginning of §2.

Definition 3.1. We shall say that a subsheaf $\mathcal{S} \subseteq \mathcal{A}^{\text{ét}}$ of $\mathcal{A}^{\text{ét}}$ is a *Frobenius-affine structure of level N* on X if $\mathcal{S}$ is preserved by the action of $\mathcal{B}$ on $\mathcal{A}^{\text{ét}}$ (cf. Remark 2.2.1(v)), and, moreover, the sheaf $\mathcal{S}$ forms, by the resulting action of $\mathcal{B}$ on $\mathcal{S}$, a $\mathcal{B}$-torsor on X.

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We shall write

$$\mathfrak{Fas}_N(X)$$

for the set of Frobenius-affine structures of level N on X.

REMARK 3.1.1.

(i) One finds easily that the notion of *Frobenius-affine structures* may be regarded as an "affine version" of the notion of *Frobenius-projective structures* (cf. [7, Def. 3.1]) discussed in [7].
(ii) One also finds easily that the notion of *Frobenius-affine structures* may be regarded as an analogue, in positive characteristic, of the notion of *complex affine structures* (cf., e.g., [1, §2]) in the classical theory of Riemann surfaces.

LEMMA 3.2. *Let $\mathcal{S} \subseteq \mathcal{A}^{\text{ét}}$ be a Frobenius-affine structure of level N on X. Then the following assertions hold:*

(i) *Let U, $V \subseteq X$ be open subschemes of X, $f_U \in \mathcal{S}(U)$, and $f_V \in \mathcal{S}(V)$. Then the global section of $\mathcal{P}^{\text{gét}}$ determined by $f_U \in \mathcal{S}(U)$ (cf. Remark 2.2.1(i)) is contained in the $\mathcal{B}_{\text{rtn}}$-orbit of the global section of $\mathcal{P}^{\text{gét}}$ determined by $f_V \in \mathcal{S}(V)$.*
(ii) *The global section of $\mathcal{P}^{\text{gét}}$ determined by a local section of $\mathcal{S}$ is a Tango function of level N.*

Proof. Since X is *irreducible*, assertion (i) follows from the fact that $\mathcal{S}$ is a *$\mathcal{B}$-torsor*. Assertion (ii) follows from assertion (i), together with the fact that $\mathcal{S}$ is *contained* in $\mathcal{A}^{\text{ét}}$. □

DEFINITION 3.3. Let $\mathcal{S} \subseteq \mathcal{A}^{\text{ét}}$ be a Frobenius-affine structure of level N on X. Then it follows from Lemma 3.2(i) and (ii) that $\mathcal{S}$ determines a $\mathcal{B}_{\text{rtn}}$-orbit of Tango functions of level N. We shall refer to this $\mathcal{B}_{\text{rtn}}$-orbit as the *Tango-orbit of level N associated with $\mathcal{S}$*. Thus, we obtain a map

$$\mathfrak{Fas}_N(X) \longrightarrow \mathfrak{Tf}_N(X)/\mathcal{B}_{\text{rtn}}.$$

LEMMA 3.4. *Let $U \subseteq X$ be an open subscheme of X, $f \in \mathcal{A}^{\text{ét}}(U)$, and $g \in \mathcal{B}_{\text{rtn}}$. Then it holds that the result $g(f) \in \mathcal{P}^{\text{gét}}(U)$ of the action of $g \in \mathcal{B}_{\text{rtn}}$ on $f \in \mathcal{A}^{\text{ét}}(U) \subseteq \mathcal{P}^{\text{gét}}(U)$ (cf. Remark 2.2.1(i) and (iv)) is contained in the subset $\mathcal{A}^{\text{ét}}(U) \subseteq \mathcal{P}^{\text{gét}}(U)$ if and only if $g \in \mathcal{B}_{\text{rtn}}$ is contained in the subgroup $\mathcal{B}(U) \subseteq \mathcal{B}_{\text{rtn}}$.*

Proof. The sufficiency follows from Remark 2.2.1(v). To verify the necessity, suppose that $g \notin \mathcal{B}(U)$. Write K_X for the function field of X. Let $x \in X$ be a closed point of X such that $x \in U$, and, moreover, $g \notin \mathrm{PGL}_2(\mathcal{O}_{X,x})$ (if we regard g as an element of $\mathrm{PGL}_2(K_X)$). Let us identify $A \stackrel{\text{def}}{=} k[[t]]$ with the completion $\widehat{\mathcal{O}}_{X,x}$ of the local ring $\mathcal{O}_{X,x}$ by means of a fixed isomorphism $A \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}$ over k. Then it is immediate that, to verify the necessity, it suffices to verify that

> $(*_1)$: the composite of the natural morphism $\mathrm{Spec}(A) \to X$ with $g(f)\colon X \to \mathbb{P}^1_k$ is *not formally étale* whenever this composite does *not map* the closed point of $\mathrm{Spec}(A)$ to ∞_k.

Next, let us identify the scheme $\mathrm{Proj}(k[u,v])$ with $\mathbb{P}^1_k$ by means of a fixed isomorphism $\mathrm{Proj}(k[u,v]) \xrightarrow{\sim} \mathbb{P}^1_k$ over k that maps the point "$(u,v) = (1,0)$" to the closed point ∞_k. Write

K for the field of fractions of A and

$$\mathrm{Proj}(k[u,v]) \longleftarrow \mathrm{Spec}(A); \qquad (u,v) \mapsto (f_u, f_v)$$

(where f_u, $f_v \in A$) for the composite of the natural morphism $\mathrm{Spec}(A) \to X$ with $f\colon X \to \mathbb{P}^1_k$. Thus, there exist a_g, b_g, $d_g \in k[[t^{p^N}]] = A^{p^N} \subseteq A$ (which thus implies that $\nu_A(a_g)$, $\nu_A(b_g)$, $\nu_A(d_g) \in p^N\mathbb{Z}$) such that $a_g d_g \neq 0$, and, moreover, the composite of the natural morphism $\mathrm{Spec}(A) \to X$ with $g(f)\colon X \to \mathbb{P}^1_k$ *coincides* with the morphism determined by the composite

$$\begin{array}{ccccccc}
\mathrm{Proj}(k[u,v]) & \longleftarrow & \mathrm{Proj}(K[u,v]) & \longleftarrow & \mathrm{Proj}(K[u,v]) & \longleftarrow & \mathrm{Spec}(K) \\
(u,v) & \mapsto & (u,v) & & & & \\
 & & (u,v) & \mapsto & (a_g u + b_g v, d_g v) & & \\
 & & & & (u,v) & \mapsto & (f_u, f_v).
\end{array}$$

Now, let us observe that, to verify $(*_1)$, we may assume without loss of generality, by replacing (a_g, b_g, d_g) by $t^{-\min\{\nu_A(a_g), \nu_A(b_g), \nu_A(d_g)\}} \cdot (a_g, b_g, d_g)$, that

(a) $0 \in \{\nu_A(a_g), \nu_A(b_g), \nu_A(d_g)\}$.

Moreover, let us observe that since $g \notin \mathrm{PGL}_2(\mathcal{O}_{X,x})$, it holds that

(b) $\nu_A(a_g d_g) \geq p^N$.

Next, let us observe that, to verify $(*_1)$, we may assume without loss of generality, by replacing f by the composite of f and a suitable element of $\mathrm{Aut}_k(\mathbb{A}^1_k)$, that the image of $x \in X$ via f is the point "$(u,v) = (0,1)$," that is, that

(c) $\nu_A(f_u) = 1$ (cf. our assumption that $f \in \mathcal{A}^{\text{ét}}(U)$), and $f_v = 1$.

Moreover, let us observe that, to verify $(*_1)$, we may assume without loss of generality, by replacing g by the product of g and a suitable element of $\mathrm{Aut}_k(\mathbb{A}^1_k)$, that the image of $x \in X$ via $g(f)$ is the point "$(u,v) = (0,1)$," that is, that (cf. (c))

(d) if we write

$$F \stackrel{\text{def}}{=} \tfrac{a_g f_u + b_g}{d_g} \in K,$$

then $F \in A$, and, moreover, $\nu_A(F) \geq 1$.

Thus, it is immediate that, to verify $(*_1)$, it suffices to verify that

$(*_2)$: $\nu_A(F) \neq 1$.

Here, let us verify that

(e) $\nu_A(b_g) \geq p^N$.

Indeed, if $\nu_A(b_g) = 0$, then it follows from (c) that $\nu_A(a_g f_u + b_g) = 0$, which thus implies that $\nu_A(F) \leq 0$—in *contradiction* to (d). This completes the proof of (e).

Next, suppose that $\nu_A(d_g) = 0$. Then it follows from (b) that $\nu_A(a_g) \geq p^N$. In particular, it follows from (e) that $\nu_A(F) = \nu_A(a_g f_u + b_g) \geq p^N \geq 2$, as desired. Thus, to verify $(*_2)$, we may assume without loss of generality that

(f) $\nu_A(d_g) \geq p^N$.

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It follows from (a), (e), and (f) that $\nu_A(a_g) = 0$. Thus, it follows from (c) and (e) that $\nu_A(a_g f_u + b_g) = 1$. In particular, it follows from (f) that $\nu_A(F) = 1 - \nu_A(d_g) \leq -1$—in *contradiction* to (d). This completes the proof of $(*_2)$, and hence also of Lemma 3.4. ∎

LEMMA 3.5. *Let $f \in \mathcal{P}^{\text{gét}}(X)$ be a Tango function of level N. Then the following assertions hold:*

(i) *Write $\mathcal{S}_f \subseteq \mathcal{A}^{\text{ét}}$ for the subsheaf of $\mathcal{A}^{\text{ét}}$ that assigns, to an open subscheme $U \subseteq X$, the subset of $\mathcal{A}^{\text{ét}}(U)$ obtained by forming the intersection of $\mathcal{A}^{\text{ét}}(U)$ and the $\mathcal{B}_{\text{rtn}}$-orbit $(\subseteq \mathcal{P}^{\text{gét}}(U))$ of $f|_U$ (cf. Remark 2.2.1(i) and (iv)):*

$$\mathcal{S}_f(U) \stackrel{\text{def}}{=} \mathcal{A}^{\text{ét}}(U) \cap (\mathcal{B}_{\text{rtn}} \cdot f|_U).$$

Then the subsheaf $\mathcal{S}_f$ is a Frobenius-affine structure of level N on X.

(ii) *Let $g \in \mathcal{P}^{\text{gét}}(X)$ be a global section of $\mathcal{P}^{\text{gét}}$, which is contained in the $\mathcal{B}_{\text{rtn}}$-orbit of $f \in \mathcal{P}^{\text{gét}}(X)$. (So g is a Tango function of level N; cf. Remark 2.3.1.) Then $\mathcal{S}_f = \mathcal{S}_g$ (cf. (i)).*

Proof. Assertion (i) follows immediately from Lemma 3.4, together with the definition of a Tango function of level N. Assertion (ii) follows immediately from the definition of "$\mathcal{S}_f$." ∎

DEFINITION 3.6. Let $f \in \mathcal{P}^{\text{gét}}(X)$ be a Tango function of level N. Then it follows from Lemma 3.5(i) that f determines a Frobenius-affine structure of level N. We shall refer to this Frobenius-affine structure of level N as the *Frobenius-affine structure of level N associated with f*. Thus, we obtain a map

$$\mathfrak{Tf}_N(X)/\mathcal{B}_{\text{rtn}} \longrightarrow \mathfrak{Fas}_N(X)$$

(cf. Lemma 3.5, (ii)).

PROPOSITION 3.7. *The assignments of Definitions 3.3 and 3.6 determine a bijective map*

$$\mathfrak{Fas}_N(X) \xrightarrow{\sim} \mathfrak{Tf}_N(X)/\mathcal{B}_{\text{rtn}}.$$

Proof. This assertion follows immediately from the constructions of Lemmas 3.2 and 3.5. ∎

REMARK 3.7.1. As observed in Remark 3.1.1(ii), the notion of *Frobenius-affine structures* may be regarded as an analogue, in positive characteristic, of the notion of *complex affine structures* in the classical theory of Riemann surfaces. Moreover, it is well-known (cf., e.g., [1, Lem. 1]) that if a compact Riemann surface admits a complex affine structure, then the compact Riemann surface is *of genus* 1. On the other hand, one may conclude from Remark 2.7.2 and Proposition 3.7 that, for an arbitrary algebraically closed field F of positive characteristic and an arbitrary positive integer N, there exists a projective smooth curve over F *of genus* ≥ 2 that has a *Frobenius-affine structure of level N*.

LEMMA 3.8. *Let $\mathcal{S} \subseteq \mathcal{A}^{\text{ét}}$ be a Frobenius-affine structure of level N on X. Then the subsheaf $\mathcal{S}_{\mathcal{G}}$ of $\mathcal{P}^{\text{ét}}$ that assigns, to an open subscheme $U \subseteq X$, the subset of $\mathcal{P}^{\text{ét}}(U)$ obtained by forming the intersection of $\mathcal{P}^{\text{ét}}(U)$ and the union of the $\mathcal{G}_{\text{rtn}}$-orbits $(\subseteq \mathcal{P}^{\text{gét}}(U))$ of the*

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elements of $\mathcal{S}(U)$ (cf. Remark 2.2.1(i) and (iv)):

$$\mathcal{S}_{\mathcal{G}}(U) \stackrel{\text{def}}{=} \mathcal{P}^{\text{ét}}(U) \cap \big(\mathcal{G}_{\text{rtn}} \cdot \mathcal{S}(U)\big).$$

Then the subsheaf $\mathcal{S}_{\mathcal{G}}$ of $\mathcal{P}^{\text{ét}}$ is a Frobenius-projective structure of level N on X (cf. [7, Def. 3.1]).

Proof. This assertion follows—in light of Remark 2.3.2—from [7, Lem. 3.5(i)]. ∎

DEFINITION 3.9. Let $\mathcal{S} \subseteq \mathcal{A}^{\text{ét}}$ be a Frobenius-affine structure of level N on X. Then it follows from Lemma 3.8 that $\mathcal{S}$ determines a Frobenius-projective structure of level N. We shall refer to this Frobenius-projective structure as the *Frobenius-projective structure of level N associated with $\mathcal{S}$*. Thus, we obtain a map

$$\mathfrak{Fas}_N(X) \longrightarrow \mathfrak{Fps}_N(X)$$

(cf. [7, Def. 3.1]).

REMARK 3.9.1. One verifies easily from the various definitions involved that the diagram

$$\begin{tikzcd}
\mathfrak{Tf}_N(X)/\mathcal{B}_{\text{rtn}} \arrow[r] \arrow[d, "\wr"'] & \mathfrak{pcd}_N(X)/\mathcal{G}_{\text{rtn}} \arrow[d, "\wr"] \\
\mathfrak{Fas}_N(X) \arrow[r] & \mathfrak{Fps}_N(X)
\end{tikzcd}$$

(where the upper horizontal arrow is the lower horizontal arrow of the diagram of Remark 2.3.2, the lower horizontal arrow is the map of Definition 3.9, the left-hand vertical arrow is the inverse of the bijective map of Proposition 3.7, and the right-hand vertical arrow is the inverse of the bijective map of [7, Prop. 3.7]) is *commutative*.

§4. Frobenius-affine-indigenous structures

In the present section, we introduce and discuss the notion of *Frobenius-affine-indigenous structures* (cf. Definition 4.3). Moreover, we also discuss a relationship between Frobenius-affine-indigenous structures and Frobenius-affine structures (cf. Proposition 4.9).

In the present section, we maintain the notational conventions introduced at the beginning of §2. Write, moreover,

$$X^f$$

for the "X^F" in the case where $N = 1$ and

$$\phi\colon X \longrightarrow X^f$$

for the "Φ" in the case where $N = 1$. Thus, the morphism $\Phi\colon X \to X^F$ factors as a composite

$$X \xrightarrow{\phi} X^f \longrightarrow X^F.$$

We shall write

$$\Phi_{f\to F}\colon X^f \longrightarrow X^F$$

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for the second arrow in this composite (i.e., the "Φ" in the case where we take the pair "(X,N)" to be $(X^f, N-1)$).

DEFINITION 4.1. Let Z be a scheme that is smooth over X^F. Thus, the base change $\Phi^* Z \to X$ of the structure morphism $Z \to X^F$ by the morphism $\Phi\colon X \to X^F$ may be regarded as an object of the category SmSch of [5, Def. 1.7] in the case where we take the "(S,X)" of [5] to be $(\mathrm{Spec}(k), X)$. Let us recall from [6, Prop. 3.3] that the natural Frobenius-descent datum on $\Phi^* Z = \phi^*(\Phi^*_{f\to F} Z) \to X$ (i.e., the natural descent datum on $\Phi^* Z = \phi^*(\Phi^*_{f\to F} Z) \to X$ with respect to the morphism $\phi\colon X \to X^f$; cf. [6, Def. 3.2(iv)]) gives rise to an Fr-stratification on $\Phi^* Z \to X$ (cf. [4, Def. 4.6] and [6, Def. 1.8]), which thus determines (cf. [4, Lem. 4.12(i)] and [6, Prop. 1.11]) a PD-stratification on $\Phi^* Z \to X$ (cf. [4, Def. 4.6] and [5, Def. 2.5]); i.e., in the case where we take the "(S,X)" of [4]–[6] to be $(\mathrm{Spec}(k), X)$. We shall write

$$\nabla_{\Phi^* Z}$$

for the PD-*connection* on $\Phi^* Z \to X$ (cf. [4, Def. 4.1(iii)] and [5, Def. 2.5]) determined by the resulting PD-stratification on $\Phi^* Z \to X$ (cf. also [7, Def. 4.2(i) and (ii)]).

DEFINITION 4.2. Let Z be a scheme that is smooth over X, and let σ be a section of $Z \to X$. Thus, the structure morphism $Z \to X$ may be regarded as an object of the category SmSch of [5, Def. 1.7] in the case where we take the "(S,X)" of [5] to be $(\mathrm{Spec}(k), X)$. Let ∇ be a PD-connection on $Z \to X$ (cf. [4, Def. 4.1(iii)] and [5, Def. 2.5])—i.e., in the case where we take the "(S,X)" of [4] and [5] to be $(\mathrm{Spec}(k), X)$. Then, by considering the difference between the two deformations

$${}^{\mathrm{PD}}P^1 \xrightarrow{({}^{\mathrm{PD}}\mathrm{pr}_1^1)^*\sigma} ({}^{\mathrm{PD}}\mathrm{pr}_1^1)^* Z, \qquad {}^{\mathrm{PD}}P^1 \xrightarrow{({}^{\mathrm{PD}}\mathrm{pr}_2^1)^*\sigma} ({}^{\mathrm{PD}}\mathrm{pr}_2^1)^* Z \xrightarrow[\sim]{\nabla} ({}^{\mathrm{PD}}\mathrm{pr}_1^1)^* Z$$

(cf. [4, Def. 2.3(ii)] and [5, Def. 2.5]) of the section σ, we have a global section of the $\mathcal{O}_X$-module

$$\mathcal{H}om_{\mathcal{O}_X}(\sigma^*\Omega^1_{Z/X}, \Omega^1_{X/k}).$$

(Note that let us recall from elementary algebraic geometry that the set of deformations ${}^{\mathrm{PD}}P^1 \to ({}^{\mathrm{PD}}\mathrm{pr}_1^1)^* Z$ of the section $\sigma\colon X \to Z$ forms a torsor under the module $\Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\sigma^*\Omega^1_{Z/X}, \Omega^1_{X/k}))$.) We shall refer to this global section as the *Kodaira–Spencer section* of ∇ at σ (cf. also [7, Def. 4.3]).

DEFINITION 4.3. We shall say that a pair $(A \to X^F, \sigma)$ consisting of an $\mathbb{A}^1$-bundle $A \to X^F$ over X^F (cf. Remark 4.3.2, and also Remark 2.1.1) and a section σ of the pull-back $\Phi^* A \to X$ is a *Frobenius-affine-indigenous structure of level N* on X if the Kodaira–Spencer section (cf. Definition 4.2) of the PD-connection $\nabla_{\Phi^* A}$ (cf. Definition 4.1) at σ is nowhere vanishing.

For two Frobenius-affine-indigenous structures $\mathcal{I}_1 = (A_1 \to X^F, \sigma_1)$ and $\mathcal{I}_2 = (A_2 \to X^F, \sigma_2)$ of level N on X, we shall say that $\mathcal{I}_1$ is *isomorphic* to $\mathcal{I}_2$ if there exists an isomorphism $A_1 \xrightarrow{\sim} A_2$ over X^F compatible with σ_1 and σ_2.

We shall write

$$\mathfrak{Fais}_N(X)$$

for the set of isomorphism classes of Frobenius-affine-indigenous structures of level N on X.

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Remark 4.3.1. One finds easily that the notion of *Frobenius-affine-indigenous structures* may be regarded as an "affine version" of the notion of *Frobenius-indigenous structures* (cf. [7, Def. 4.4]).

Remark 4.3.2. In the present paper, an $\mathbb{A}^1$*-bundle* (resp. a $\mathbb{P}^1$*-bundle*) over a scheme S is defined to be a scheme Z over S such that, for each point $s \in S$ of S, there exist an open subscheme $U \subseteq S$ of S that contains $s \in S$ and an isomorphism of $Z|_U$ with $\mathbb{A}^1_U$ (resp. $\mathbb{P}^1_U$) over U.

Lemma 4.4. *Let $(A \to X^F, \sigma)$ be a Frobenius-affine-indigenous structure of level N on X. Write $P \to X^F$ for the $\mathbb{P}^1$-bundle over X^F obtained by forming the smooth compactification of $A \to X^F$. Then the pair of the $\mathbb{P}^1$-bundle $P \to X^F$ and the section of the $\mathbb{P}^1$-bundle $P \to X^F$ determined by σ is a Frobenius-indigenous structure of level N on X (cf. [7, Def. 4.4]).*

Proof. This assertion follows immediately from the various definitions involved. ∎

Remark 4.4.1. Let $\mathcal{I} = (A \to X^F, \sigma)$ be a *Frobenius-affine-indigenous structure of level N on X*. Then it follows from Lemma 4.4 that $\mathcal{I}$ determines a Frobenius-indigenous structure of level N. We shall refer to this Frobenius-indigenous structure of level N as the *Frobenius-indigenous structure of level N associated with $\mathcal{I}$*. Thus, we obtain a map

$$\mathfrak{Fais}_N(X) \longrightarrow \mathfrak{Fis}_N(X)$$

(cf. [7, Def. 4.4]).

Remark 4.4.2. One verifies immediately from Lemma 4.4 that giving a *Frobenius-affine-indigenous structure of level N* on X is "*equivalent*" to giving a collection $(P \to X^F, \sigma^\infty, \sigma)$ of data consisting of a $\mathbb{P}^1$-bundle $P \to X^F$ over X^F, a section σ^∞ of the $\mathbb{P}^1$-bundle $P \to X^F$, and a section σ of the pull-back $\Phi^* P \to X$ that satisfies the following two conditions:

(1) The image of $\Phi^*\sigma^\infty$ does not intersect the image of σ.
(2) The Kodaira–Spencer section (cf. Definition 4.2) of the PD-connection $\nabla_{\Phi^* P}$ (cf. Definition 4.1) at σ is nowhere vanishing.

Lemma 4.5. *Let $\mathcal{S} \subseteq \mathcal{A}^{\text{ét}}$ be a Frobenius-affine structure of level N on X. Thus, the sheaf $\Phi_*\mathcal{S}$ is a $\mathrm{PGL}^\infty_{2,X^F}$-torsor on X^F. Write $A_\mathcal{S} \to X^F$ for the $\mathbb{A}^1$-bundle associated with the $\mathrm{PGL}^\infty_{2,X^F}$-torsor $\Phi_*\mathcal{S}$ (i.e., the quotient of $\Phi_*\mathcal{S} \times_{X^F} \mathbb{A}^1_{X^F}$ by the diagonal action of $\mathrm{PGL}^\infty_{2,X^F}$). For each local section s of $\Phi_*\mathcal{S}$, write σ_s for the local section of the trivial $\mathbb{A}^1$-bundle $\mathbb{A}^1_X \to X$ that corresponds to s (cf. Remark 2.2.1(ii)). Then the pair consisting of:*

(1) *the $\mathbb{A}^1$-bundle $A_\mathcal{S} \to X^F$ over X^F and*
(2) *the section of $\Phi^* A_\mathcal{S} \to X$ determined by the various pairs "(s, σ_s)" (where "s" ranges over the local sections of $\Phi_*\mathcal{S}$)*

is a Frobenius-affine-indigenous structure of level N on X.

Proof. Write $\mathcal{S}_\mathcal{G} \subseteq \mathcal{P}^{\text{ét}}$ for the Frobenius-projective structure of level N associated with $\mathcal{S}$ (cf. Definition 3.9), $\mathcal{I} = (P \to X^F, \sigma)$ for the Frobenius-indigenous structure of level N associated with $\mathcal{S}_\mathcal{G}$ (cf. [7, Def. 4.8]), σ^∞ for the section of the $\mathbb{P}^1$-bundle $P \to X^F$ determined by the "Borel subgroup" $\mathrm{PGL}^\infty_{2,X^F} \subseteq \mathrm{PGL}_{2,X^F}$ of PGL_{2,X^F} (cf. also the construction of [7, Lem. 4.7]), and $A \to X^F$ for the $\mathbb{A}^1$-bundle obtained by forming the complement of the image of σ^∞ in P. Then one verifies immediately from the various definitions involved (cf. also the construction of [7, Lem. 4.7]) that there exists an isomorphism $A_{\mathcal{S}} \xrightarrow{\sim} A$ over X^F *compatible* with the section of (2) and σ. Thus, Lemma 4.5 is a formal consequence of [7, Lem. 4.7]. This completes the proof of Lemma 4.5. ∎

DEFINITION 4.6. Let $\mathcal{S} \subseteq \mathcal{A}^{\text{ét}}$ be a Frobenius-affine structure of level N on X. Then it follows from Lemma 4.5 that $\mathcal{S}$ determines a Frobenius-affine-indigenous structure of level N. We shall refer to this Frobenius-affine-indigenous structure of level N as the *Frobenius-affine-indigenous structure of level N associated with $\mathcal{S}$*. Thus, we obtain a map

$$\mathfrak{Fas}_N(X) \longrightarrow \mathfrak{Fais}_N(X).$$

REMARK 4.6.1. One verifies easily from the various definitions involved that the diagram

$$\begin{tikzcd} \mathfrak{Fas}_N(X) \arrow[r] \arrow[d] & \mathfrak{Fps}_N(X) \arrow[d, "\wr"] \\ \mathfrak{Fais}_N(X) \arrow[r] & \mathfrak{Fis}_N(X) \end{tikzcd}$$

(where the upper horizontal arrow is the map of Definition 3.9, the lower horizontal arrow is the map of Remark 4.4.1, the left-hand vertical arrow is the map of Definition 4.6, and the right-hand vertical arrow is the bijective map of [7, Prop. 4.11]) is *commutative*.

LEMMA 4.7. *Let $(A \to X^F, \sigma)$ be a Frobenius-affine-indigenous structure of level N on X. Write $(P \to X^F, \sigma^\infty, \sigma)$ for the collection of data discussed in Remark 4.4.2 that corresponds to the Frobenius-affine-indigenous structure $(A \to X^F, \sigma)$. Then the following assertions hold:*

(i) *Let $U \subseteq X$ be an open subscheme of X such that the restriction $A|_{U^F}$ is isomorphic to the trivial $\mathbb{A}^1$-bundle over U^F, which thus implies that there exists an isomorphism $\iota_U \colon P|_{U^F} \xrightarrow{\sim} \mathbb{P}^1_{U^F}$ over U^F compatible with the sections $\sigma^\infty|_{U^F}$ and ∞_{U^F}. Write $f_{U,\iota_U} \in \mathcal{P}(U)$ for the section of $\mathcal{P}$ obtained by forming the composite*

$$U \xrightarrow{\sigma|_U} (\Phi^* P)|_U \underset{\sim}{\xrightarrow{\Phi^*\iota_U}} \mathbb{P}^1_U = \mathbb{P}^1_k \times_k U \xrightarrow{\mathrm{pr}_1} \mathbb{P}^1_k.$$

Then $f_{U,\iota_U} \in \mathcal{P}^{\text{ét}}(U)$.

(ii) *The collection of sections $f_{U,\iota_U} \in \mathcal{P}^{\text{ét}}(U)$ (cf. (i)) (where (U, ι_U) ranges over the pairs as in (i)) determines a Frobenius-affine structure of level N on X.*

Proof. First, we verify assertion (i). Write $\tau \colon U \to \mathbb{P}^1_U$ for the section of the trivial $\mathbb{P}^1$-bundle obtained by forming the composite of the first two arrows of the displayed composite of the statement of assertion (i), that is, the composite of the section $\sigma|_U \colon U \to (\Phi^* P)|_U$ and the isomorphism $\Phi^*\iota_U \colon (\Phi^* P)|_U \xrightarrow{\sim} \mathbb{P}^1_U$. Then it is tautology (cf. the factorization discussed in [6, Lem. 1.3]) that the Kodaira–Spencer section $f^*_{U,\iota_U}\Omega^1_{\mathbb{P}^1_k/k} = \tau^*\mathrm{pr}_1^*\Omega^1_{\mathbb{P}^1_k/k} = \tau^*\Omega^1_{\mathbb{P}^1_U/U} \to \Omega^1_{U/k}$ of this section $\tau \colon U \to \mathbb{P}^1_U$ *coincides* with the homomorphism $f^*_{U,\iota_U}\Omega^1_{\mathbb{P}^1_k/k} \to \Omega^1_{U/k}$ induced by the morphism $f_{U,\iota_U} \colon U \to \mathbb{P}^1_k$. Thus, since (we have assumed that) the Kodaira–Spencer section of $\sigma|_U$, hence also the Kodaira–Spencer section of τ, is *nowhere vanishing*,

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one may conclude that $f_{U,\iota_U} \in \mathcal{P}^{\text{ét}}(U)$, as desired. This completes the proof of assertion (i). Assertion (ii) follows immediately from assertion (i). □

DEFINITION 4.8. Let $\mathcal{I}$ be a Frobenius-affine-indigenous structure of level N on X. Then it follows from Lemma 4.7(ii) that $\mathcal{I}$ determines a Frobenius-affine structure of level N. We shall refer to this Frobenius-affine structure of level N as the *Frobenius-affine structure of level N associated with $\mathcal{I}$*. Thus, we obtain a map

$$\mathfrak{Fais}_N(X) \longrightarrow \mathfrak{Fas}_N(X).$$

PROPOSITION 4.9. *The assignments of Definitions 4.6 and 4.8 determine a bijective map*

$$\mathfrak{Fas}_N(X) \xrightarrow{\sim} \mathfrak{Fais}_N(X).$$

Proof. This assertion follows immediately from the constructions of Lemmas 4.5 and 4.7. □

The main result of the present paper is as follows.

THEOREM 4.10. *There exist bijective maps*

$$\mathfrak{Tf}_N(X)/\mathcal{B}_{\mathrm{rtn}} \xrightarrow{\sim} \mathfrak{Fas}_N(X) \xrightarrow{\sim} \mathfrak{Fais}_N(X).$$

Proof. This assertion follows from Propositions 3.7 and 4.9. □

REMARK 4.10.1. If $(p,N) \neq (2,1)$, then the bijective maps of Theorem 4.10 are *compatible* with the bijective maps of [7, Th. 4.13] (cf. Remarks 3.9.1 and 4.6.1).

§5. Relationship between certain Frobenius-destabilized bundles

In the present section, we discuss a relationship between Frobenius-affine-indigenous structures and certain *Frobenius-destabilized bundles* over X^F (cf. Proposition 5.7). In the present section, we maintain the notational conventions introduced at the beginning of §4. Suppose, moreover, that

$$g \geq 2.$$

DEFINITION 5.1. Let S be a scheme. For each $i \in \{1,2\}$, let $\mathcal{E}_i$ be an $\mathcal{O}_S$-module and let $\mathcal{F}_i \subseteq \mathcal{E}_i$ be an $\mathcal{O}_S$-submodule of $\mathcal{E}_i$. Then we shall say that the pair $(\mathcal{E}_1,\mathcal{F}_1)$ is $\mathbb{P}$-*equivalent* to $(\mathcal{E}_2,\mathcal{F}_2)$ if there exist an invertible sheaf $\mathcal{L}$ on S and an isomorphism $\mathcal{E}_1 \otimes_{\mathcal{O}_S} \mathcal{L} \xrightarrow{\sim} \mathcal{E}_2$ of $\mathcal{O}_S$-modules that restricts to an isomorphism $\mathcal{F}_1 \otimes_{\mathcal{O}_S} \mathcal{L} \xrightarrow{\sim} \mathcal{F}_2$. We shall write

$$(\mathcal{E}_1,\mathcal{F}_1) \sim_{\mathbb{P}} (\mathcal{E}_2,\mathcal{F}_2)$$

if $(\mathcal{E}_1,\mathcal{F}_1)$ is $\mathbb{P}$-equivalent to $(\mathcal{E}_2,\mathcal{F}_2)$.

REMARK 5.1.1. In the situation of Definition 5.1, it is immediate that if $(\mathcal{E}_1,\mathcal{F}_1)$ is $\mathbb{P}$-*equivalent* to $(\mathcal{E}_2,\mathcal{F}_2)$ (in the sense of Definition 5.1), then $\mathcal{E}_1$ is $\mathbb{P}$-*equivalent* to $\mathcal{E}_2$ in the sense of [7, Def. 5.1].

DEFINITION 5.2. Let d be a positive integer, let $\mathcal{E}$ be a locally free coherent $\mathcal{O}_{X^F}$-module of rank 2, and let $\mathcal{L} \subseteq \mathcal{E}$ be an invertible subsheaf of $\mathcal{E}$. Then we shall say that the pair $(\mathcal{E},\mathcal{L})$ is (N,d)-*Frobenius-splitting* if the invertible sheaf $\mathcal{L}$ is of degree $\frac{1}{2}\cdot\deg(\mathcal{E}) - \frac{d}{p^N}$, the locally free coherent $\mathcal{O}_{X^f}$-module $\Phi^*_{f\to F}\mathcal{E}$ of rank 2 (hence also the locally free coherent

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$\mathcal{O}_{X^F}$-module $\mathcal{E}$ of rank 2) is stable, and, moreover, the natural inclusion $\Phi^*\mathcal{L} \hookrightarrow \Phi^*\mathcal{E}$ has a section (which thus implies that the locally free coherent $\mathcal{O}_X$-module $\Phi^*\mathcal{E} = \phi^*\Phi^*_{f\to F}\mathcal{E}$ of rank 2 is not semistable).

We shall write

$$\mathfrak{Fsp}_N(X)$$

for the set of $\mathbb{P}$-equivalence classes (cf. Remark 5.2.1) of $(N, g-1)$-Frobenius-splitting pairs on X.

Remark 5.2.1. For each $i \in \{1,2\}$, let $\mathcal{E}_i$ be a locally free coherent $\mathcal{O}_{X^F}$-module of rank 2 and let $\mathcal{L}_i \subseteq \mathcal{E}_i$ be an invertible subsheaf of $\mathcal{E}_i$. Suppose that $(\mathcal{E}_1, \mathcal{L}_1) \sim_{\mathbb{P}} (\mathcal{E}_2, \mathcal{L}_2)$. Then one verifies easily that $(\mathcal{E}_1, \mathcal{L}_1)$ is $(N, g-1)$-*Frobenius-splitting* if and only if $(\mathcal{E}_2, \mathcal{L}_2)$ is $(N, g-1)$-*Frobenius-splitting*.

Remark 5.2.2. Let $(\mathcal{E}, \mathcal{L})$ be an $(N, g-1)$-*Frobenius-splitting* pair on X. Then it is immediate that the locally free coherent $\mathcal{O}_{X^F}$-module $\mathcal{E}$ is $(N, g-1)$-*Frobenius-destabilized* (cf. [7, Def. 5.2]). In particular, we have a map

$$\mathfrak{Fsp}_N(X) \longrightarrow \mathfrak{Fds}_N(X)$$

(cf. Remark 5.1.1 and [7, Def. 5.2]).

Remark 5.2.3. Suppose that $p \neq 2$, and that $N = 1$. Then one verifies immediately from the various definitions involved (cf. also Remark 5.2.2 and the proof of [7, Lem. 5.3]) that giving an $(N, g-1)$-*Frobenius-splitting* pair on X is "*equivalent*" to giving a *dormant Miura* GL_2-*oper* (cf. [10, Def. 4.2.1] and [10, Def. 4.2.2]).

Remark 5.2.4. Write $\mathrm{Fr}_X \colon X \to X$ for the pth power Frobenius endomorphism of X. Then one verifies easily that the assignment "$(\mathcal{E}, \mathcal{L}) \mapsto (W_*\mathcal{E}, W_*\mathcal{L})$" determines a *bijective* map of the set $\mathfrak{Fsp}_N(X)$ with the set of $\mathbb{P}$-equivalence classes of pairs $(\mathcal{F}, \mathcal{G})$ of locally free coherent $\mathcal{O}_X$-modules $\mathcal{F}$ of rank 2 and invertible subsheaves $\mathcal{G} \subseteq \mathcal{F}$ that satisfy the following condition: if, for a nonnegative integer i, we write

$$\mathcal{G}_i \stackrel{\mathrm{def}}{=} \overbrace{\mathrm{Fr}_X^* \cdots \mathrm{Fr}_X^*}^{i} \mathcal{G} \subseteq \mathcal{F}_i \stackrel{\mathrm{def}}{=} \overbrace{\mathrm{Fr}_X^* \cdots \mathrm{Fr}_X^*}^{i} \mathcal{F},$$

then

- the locally free coherent $\mathcal{O}_X$-module $\mathcal{F}_{N-1}$, hence also $\mathcal{F}$, is *stable*, but
- there exist an invertible sheaf $\mathcal{M}$ on X of degree $\frac{p^N}{2} \cdot \deg(\mathcal{F}) + g - 1 = \frac{1}{2} \cdot \deg(\mathcal{F}_N) + g - 1$ and a locally split injective homomorphism $\mathcal{M} \hookrightarrow \mathcal{F}_N$ of $\mathcal{O}_X$-modules such that the inclusions $\mathcal{G}_N$, $\mathcal{M} \hookrightarrow \mathcal{F}_N$ determine an *isomorphism* $\mathcal{G}_N \oplus \mathcal{M} \xrightarrow{\sim} \mathcal{F}_N$ of $\mathcal{O}_X$-modules. (In particular, the locally free coherent $\mathcal{O}_X$-module $\mathcal{F}_N$ is *not semistable*.)

Lemma 5.3. *Let $(\mathcal{E}, \mathcal{L})$ be an $(N, g-1)$-Frobenius-splitting pair on X. Write $\mathbb{P}(\mathcal{E}) \to X^F$ for the projectivization of $\mathcal{E}$ and $\sigma^\infty(\mathcal{L})$ for the section of $\mathbb{P}(\mathcal{E}) \to X^F$ determined by $\mathcal{L} \subseteq \mathcal{E}$ (cf. Definition 5.2). Then there exists a (uniquely determined; cf. Remark 5.2.2 and [7, Lem. 4.6]) section σ of $\Phi^*\mathbb{P}(\mathcal{E}) \to X$ such that the collection $(\mathbb{P}(\mathcal{E}) \to X^F, \sigma^\infty(\mathcal{L}), \sigma)$ of data is a collection of data discussed in Remark 4.4.2 that corresponds to a Frobenius-affine-indigenous structure of level N on X.*

Proof. This assertion follows—in light of Remark 5.2.2—from [7, Lem. 5.3] (cf. also the proof of [7, Lem. 5.3]). ∎

DEFINITION 5.4. Let $(\mathcal{E},\mathcal{L})$ be an $(N,g-1)$-Frobenius-splitting pair on X. Then it follows from Lemma 5.3 that $(\mathcal{E},\mathcal{L})$ determines a Frobenius-affine-indigenous structure of level N. We shall refer to this Frobenius-affine-indigenous structure of level N as the *Frobenius-affine-indigenous structure of level N associated with* $(\mathcal{E},\mathcal{L})$. Thus, we obtain a map

$$\mathfrak{Fsp}_N(X) \longrightarrow \mathfrak{Fais}_N(X).$$

REMARK 5.4.1. One verifies easily from the various definitions involved that the diagram

$$\begin{tikzcd}
\mathfrak{Fsp}_N(X) \arrow[r] \arrow[d] & \mathfrak{Fds}_N(X) \arrow[d, "\wr"] \\
\mathfrak{Fais}_N(X) \arrow[r] & \mathfrak{Fis}_N(X)
\end{tikzcd}$$

(where the upper horizontal arrow is the map of Remark 5.2.2, the lower horizontal arrow is the map of Remark 4.4.1, the left-hand vertical arrow is the map of Definition 5.4, and the right-hand vertical arrow is the bijective map of [7, Prop. 5.7]) is *commutative*.

LEMMA 5.5. *Let $(P \to X^F, \sigma^\infty, \sigma)$ be a collection of data discussed in Remark 4.4.2 that corresponds to a Frobenius-affine-indigenous structure of level N on X and $\mathcal{E}$ a locally free coherent $\mathcal{O}_{X^F}$-module of rank 2 whose projectivization is isomorphic to P over X^F. Write $\mathcal{L} \subseteq \mathcal{E}$ for the invertible subsheaf of $\mathcal{E}$ determined by the section σ^∞. Then the pair $(\mathcal{E},\mathcal{L})$ is $(N,g-1)$-Frobenius-splitting.*

Proof. This assertion follows—in light of Remark 5.2.2—from [7, Lem. 5.5] (cf. also the proof of [7, Lem. 5.5]). ∎

DEFINITION 5.6. Let $\mathcal{I}$ be a Frobenius-affine-indigenous structure of level N on X. Then it follows from Lemma 5.5 that $\mathcal{I}$ determines a $\mathbb{P}$-equivalence class of $(N,g-1)$-Frobenius-splitting pair. We shall refer to this $\mathbb{P}$-equivalence class as the $(N,g-1)$-*Frobenius-splitting class associated with* $\mathcal{I}$. Thus, we obtain a map

$$\mathfrak{Fais}_N(X) \longrightarrow \mathfrak{Fsp}_N(X).$$

PROPOSITION 5.7. *The assignments of Definitions 5.4 and 5.6 determine a bijective map*

$$\mathfrak{Fsp}_N(X) \xrightarrow{\sim} \mathfrak{Fais}_N(X).$$

Proof. This assertion follows immediately from the constructions of Lemmas 5.3 and 5.5. ∎

COROLLARY 5.8. *Suppose that $g \geq 2$. Then there exist bijective maps*

$$\mathfrak{Tf}_N(X)/\mathcal{B}_{\mathrm{rtn}} \xrightarrow{\sim} \mathfrak{Fas}_N(X) \xrightarrow{\sim} \mathfrak{Fais}_N(X) \xrightarrow{\sim} \mathfrak{Fsp}_N(X).$$

Proof. This assertion follows from Theorem 4.10 and Proposition 5.7. ∎

REMARK 5.8.1. If $(p,N) \neq (2,1)$, then the bijective maps of Corollary 5.8 are *compatible* with the bijective maps of [7, Cor. 5.8] (cf. Remarks 4.10.1 and 5.4.1).

Remark 5.8.2. Suppose that $p \neq 2$. Then it follows from Corollary 5.8 that the existence of a *Tango function of level* N is equivalent to the existence of an $(N, g-1)$-*Frobenius-splitting* pair. In particular, by applying this equivalence to the case where $N = 1$, we conclude from Corollary 2.11 and Remark 5.2.3 that X is a *Tango curve* if and only if X has a *dormant Miura* GL_2-*oper* (cf. [10, Def. 4.2.1] and [10, Def. 4.2.2]). On the other hand, this equivalence (i.e., in the case where $N = 1$) is an immediate consequence of [10, Th. A(i)]. Thus, one concludes that Corollary 5.8 may be regarded as a "*higher level version*" of this equivalence (i.e., in the case where $N = 1$) derived from [10, Th. A(i)].

Corollary 5.9. *Suppose that* $p \neq 2$*, that* $g \geq 2$*, and that* $N = 1$*. Suppose, moreover, that there exists a projective smooth curve over* k *of genus* g *that has a Tango function of level* N *(which thus implies that* $g-1$ *is divisible by* p*; cf. Corollary 2.10). Then there exists a closed subscheme of the coarse moduli space of projective smooth curves over* k *of genus* g *of pure codimension* $(g-1)(p-2)/p$ *such that if the curve* X *is parametrized by the closed subscheme, then the four sets*

$$\mathfrak{Tf}_N(X), \qquad \mathfrak{Fas}_N(X), \qquad \mathfrak{Fais}_N(X), \qquad \mathfrak{Fsp}_N(X)$$

are nonempty

Proof. This assertion follows from [10, Th. B], together with Corollary 5.8. ∎

§6. Some results in small characteristic cases

In the present section, we prove some results related to Frobenius-affine structures in the case where $p \leq 3$. In the present section, we maintain the notational conventions introduced at the beginning of §4.

Proposition 6.1. *Suppose that* $(p, N) = (2, 1)$*. Then the following assertions hold:*

(i) *The collection of data consisting of*

- *the* $\mathbb{P}^1$*-bundle* $P \to X^F$ *over* X^F *obtained by forming the projectivization of the locally free coherent* $\mathcal{O}_{X^F}$*-module* $\Phi_* \mathcal{O}_X$ *of rank* 2*,*
- *the section of* $P \to X^F$ *determined by the invertible subsheaf* $\mathcal{O}_{X^F} \subseteq \Phi_* \mathcal{O}_X$ *obtained by forming the image of the homomorphism* $\mathcal{O}_{X^F} \to \Phi_* \mathcal{O}_X$ *of* $\mathcal{O}_{X^F}$*-modules induced by* Φ*, and*
- *the section of* $\Phi^* P \to X$ *determined by the (necessarily surjective) homomorphism* $\Phi^* \Phi_* \mathcal{O}_X \twoheadrightarrow \mathcal{O}_X$ *of* $\mathcal{O}_X$*-modules obtained by multiplication*

is a collection of data discussed in Remark 4.4.2 that corresponds to a Frobenius-affine-indigenous structure of level N *on* X*.*

(ii) *Every Frobenius-affine-indigenous structure of level* N *on* X *is isomorphic to the Frobenius-affine-indigenous structure of level* N *of (i).*

Proof. Assertion (i) follows from [7, Lem. 6.2]. Assertion (ii) follows from Remark 2.7.1 and Theorem 4.10. ∎

Corollary 6.2. *Suppose that* $(p, N) = (2, 1)$*, and that* $g \geq 2$*. Then every* $(N, g-1)$*-Frobenius-splitting pair on* X *is* $\mathbb{P}$*-equivalent to the pair* $(\Phi_* \mathcal{O}_X, \mathcal{O}_{X^F})$ *consisting of the locally free coherent* $\mathcal{O}_{X^F}$*-module* $\Phi_* \mathcal{O}_X$ *of rank* 2 *and the invertible subsheaf* $\mathcal{O}_{X^F} \subseteq \Phi_* \mathcal{O}_X$ *obtained by forming the image of the homomorphism* $\mathcal{O}_{X^F} \to \Phi_* \mathcal{O}_X$ *of* $\mathcal{O}_{X^F}$*-modules induced by* Φ*.*

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Proof. This assertion follows from Corollary 5.8 and Proposition 6.1(ii). ∎

LEMMA 6.3. *Suppose that $p=2$, that $g \geq 2$, and that $N \geq 2$. Let $\mathcal{E}$ be a locally free coherent $\mathcal{O}_{X^F}$-module of rank 2. Then the following assertions hold:*

(i) *Suppose that g is odd. Then it holds that $\mathcal{E}$ is $(N, g-1)$-Frobenius-destabilized (cf. [7, Def. 5.2]) if and only if $\Phi^*_{f \to F}\mathcal{E}$ is $\mathbb{P}$-equivalent (cf. [7, Def. 5.1]) to $\phi_*\mathcal{O}_X$.*

(ii) *Suppose that $\mathcal{E}$ is $(N, g-1)$-Frobenius-destabilized. Then the following two conditions are equivalent:*

  (1) *There exists an invertible subsheaf $\mathcal{L} \subseteq \mathcal{E}$ such that the pair $(\mathcal{E}, \mathcal{L})$ is $(N, g-1)$-Frobenius-splitting.*

  (2) *There exist invertible sheaves $\mathcal{M}$, $\mathcal{Q}$ on X^F, a surjective homomorphism $\mathcal{E} \otimes_{\mathcal{O}_{X^F}} \mathcal{M} \twoheadrightarrow \mathcal{Q}$ of $\mathcal{O}_{X^F}$-modules, and an isomorphism $\phi_*\mathcal{O}_X \xrightarrow{\sim} \Phi^*_{f \to F}(\mathcal{E} \otimes_{\mathcal{O}_{X^F}} \mathcal{M})$ of $\mathcal{O}_{X^f}$-modules such that Q is of degree $(2g-2)/p^N$.*

  If, moreover, one of conditions (1) and (2) is satisfied, then g is odd.

Proof. First, we verify assertion (i). Let us first observe that since $p=2$, and $N \geq 2$, the $\mathcal{O}_{X^f}$-module $\Phi^*_{f \to F}\mathcal{E}$ is *of even degree*. Next, let us observe that since g is *odd*, it follows from [7, Prop. 5.7], [7, Lem. 6.4(ii)], and [7, Prop. 6.6(iii)] that every $(1, g-1)$-*Frobenius-destabilized* locally free coherent $\mathcal{O}_{X^f}$-module of rank 2 *of even degree* is $\mathbb{P}$-*equivalent* to $\phi_*\mathcal{O}_X$. Thus, assertion (i) follows from [7, Rem. 5.2.2(i)]. This completes the proof of assertion (i).

Next, we verify assertion (ii). First, to verify the implication (1) ⇒ (2), let $\mathcal{L}$ be as in condition (1). Then since $N \geq 2$, it follows immediately from Corollary 2.10 and Corollary 5.8 that $g-1$ is *divisible* by 2^{N-1}, which thus implies that g is *odd*. In particular, since $\mathcal{E}$ is $(N, g-1)$-*Frobenius-destabilized*, it follows from assertion (i) that $\Phi^*_{f \to F}\mathcal{E}$ is $\mathbb{P}$-*equivalent* to $\phi_*\mathcal{O}_X$. Thus, since the locally free coherent $\mathcal{O}_{X^f}$-module $\phi_*\mathcal{O}_X$ of rank 2 is *of degree* $g-1$ (cf., e.g., [7, Lem. 6.4(ii)]), we may assume without loss of generality, by replacing $\mathcal{E}$ by the tensor product of $\mathcal{E}$ and a suitable invertible sheaf on X^F, that $\Phi^*_{f \to F}\mathcal{E}$ is *isomorphic* to $\phi_*\mathcal{O}_X$, and, moreover, the quotient $\mathcal{E}/\mathcal{L}$ of $\mathcal{E}$ by $\mathcal{L} \subseteq \mathcal{E}$ is *of degree* $(2g-2)/p^N$ (cf. Definition 5.2), as desired. This completes the proof of the implication (1) ⇒ (2).

Next, to verify the implication (2) ⇒ (1), let $\mathcal{M}$, $\mathcal{Q}$ be as in condition (2). Then since the locally free coherent $\mathcal{O}_{X^f}$-module $\phi_*\mathcal{O}_X$ of rank 2 is *of degree* $g-1$ (cf., e.g., [7, Lem. 6.4(ii)]), one verifies immediately from assertion (i), together with the proof of [7, Lem. 5.3], that if one writes $\mathcal{K} \subseteq \mathcal{E} \otimes_{\mathcal{O}_{X^F}} \mathcal{M}$ for the kernel of the surjective homomorphism $\mathcal{E} \otimes_{\mathcal{O}_{X^F}} \mathcal{M} \twoheadrightarrow \mathcal{Q}$ of condition (2), then the pair $(\mathcal{E}, \mathcal{K} \otimes_{\mathcal{O}_{X^F}} \mathcal{M}^{-1})$ is $(N, g-1)$-*Frobenius-splitting*, as desired. This completes the proof of the implication (2) ⇒ (1).

Finally, since $N \geq 2$, the final assertion follows from Corollaries 2.10 and 5.8. This completes the proof of assertion (ii), hence also of Lemma 6.3. ∎

LEMMA 6.4. *Suppose that $p=3$, and that $g \geq 2$. Write*

$$\mathcal{B}_{X^f} \stackrel{\text{def}}{=} \operatorname{Coker}(\mathcal{O}_{X^f} \to \phi_*\mathcal{O}_X)$$

for the $\mathcal{O}_{X^f}$-module obtained by forming the cokernel of the homomorphism $\mathcal{O}_{X^f} \to \phi_*\mathcal{O}_X$ of $\mathcal{O}_{X^f}$-modules induced by ϕ. Let $\mathcal{E}$ be a locally free coherent $\mathcal{O}_{X^F}$-module of rank 2. Then the following assertions hold:

(i) *It holds that $\mathcal{E}$ is $(N, g-1)$-Frobenius-destabilized if and only if $\Phi^*_{f\to F}\mathcal{E}$ is $\mathbb{P}$-equivalent to $\mathcal{B}_{X^f}$.*

(ii) *Suppose that $\mathcal{E}$ is $(N, g-1)$-Frobenius-destabilized. Then the following two conditions are equivalent:*

  (1) *There exists an invertible subsheaf $\mathcal{L} \subseteq \mathcal{E}$ such that the pair $(\mathcal{E}, \mathcal{L})$ is $(N, g-1)$-Frobenius-splitting.*

  (2) *There exist invertible sheaves $\mathcal{M}$, $\mathcal{Q}$ on X^F, a surjective homomorphism $\mathcal{E} \otimes_{\mathcal{O}_{X^F}} \mathcal{M} \twoheadrightarrow \mathcal{Q}$ of $\mathcal{O}_{X^F}$-modules, and an isomorphism $\mathcal{B}_{X^f} \xrightarrow{\sim} \Phi^*_{f\to F}(\mathcal{E} \otimes_{\mathcal{O}_{X^F}} \mathcal{M})$ of $\mathcal{O}_{X^f}$-modules such that Q is of degree $(4g-4)/p^N$.*

Proof. First, we verify assertion (i). Let us first observe that it follows from [7, Cor. 5.8] and [2, Th. A] (cf. also [3, §1]), together with [7, Rem. 4.4.1(ii)] and the construction of [7, Lem. 5.5], that every $(1, g-1)$-*Frobenius-destabilized* locally free coherent $\mathcal{O}_{X^f}$-module of rank 2 is $\mathbb{P}$-*equivalent* to $\mathcal{B}_{X^f}$. Thus, assertion (i) follows from [7, Rem. 5.2.2(i)]. This completes the proof of assertion (i).

Next, we verify assertion (ii). First, to verify the implication (1) ⇒ (2), let $\mathcal{L}$ be as in condition (1). Then since $\mathcal{E}$ is $(N, g-1)$-*Frobenius-destabilized*, it follows from assertion (i) that $\Phi^*_{f\to F}\mathcal{E}$ is $\mathbb{P}$-*equivalent* to $\mathcal{B}_{X^f}$. Now, let us observe that it follows from Corollaries 2.10 and 5.8 that $2g-2$ is *divisible* by p^N. Thus, since the locally free coherent $\mathcal{O}_{X^f}$-module $\mathcal{B}_{X^f}$ of rank 2 is *of degree* $2g-2$ (cf., e.g., [2, Lem. 1.2]), we may assume without loss of generality, by replacing $\mathcal{E}$ by the tensor product of $\mathcal{E}$ and a suitable invertible sheaf on X^F, that $\Phi^*_{f\to F}\mathcal{E}$ is *isomorphic* to $\mathcal{B}_{X^f}$, and, moreover, the quotient $\mathcal{E}/\mathcal{L}$ of $\mathcal{E}$ by $\mathcal{L} \subseteq \mathcal{E}$ is *of degree* $(4g-4)/p^N$ (cf. Definition 5.2), as desired. This completes the proof of the implication (1) ⇒ (2).

Next, to verify the implication (2) ⇒ (1), let $\mathcal{M}$, $\mathcal{Q}$ be as in condition (2). Then since the locally free coherent $\mathcal{O}_{X^f}$-module $\mathcal{B}_{X^f}$ of rank 2 is *of degree* $2g-2$ (cf., e.g., [2, Lem. 1.2]), one verifies immediately from assertion (i), together with the proof of [7, Lem. 5.3], that if one writes $\mathcal{K} \subseteq \mathcal{E} \otimes_{\mathcal{O}_{X^F}} \mathcal{M}$ for the kernel of the surjective homomorphism $\mathcal{E} \otimes_{\mathcal{O}_{X^F}} \mathcal{M} \twoheadrightarrow \mathcal{Q}$ of condition (2), then the pair $(\mathcal{E}, \mathcal{K} \otimes_{\mathcal{O}_{X^F}} \mathcal{M}^{-1})$ is $(N, g-1)$-*Frobenius-splitting*, as desired. This completes the proof of the implication (2) ⇒ (1), hence also of assertion (ii). ∎

Corollary 6.5. *Suppose that $g \geq 2$. Suppose, moreover, that $N \geq 2$ whenever $p = 2$. Write*

$$\mathcal{B}_{X^f} \overset{\text{def}}{=} \mathrm{Coker}(\mathcal{O}_{X^f} \to \phi_*\mathcal{O}_X)$$

for the $\mathcal{O}_{X^f}$-module obtained by forming the cokernel of the homomorphism $\mathcal{O}_{X^f} \to \phi_*\mathcal{O}_X$ of $\mathcal{O}_{X^f}$-modules induced by ϕ. Then the following assertions hold:

(i) *Suppose that $p = 2$ (resp. $p = 3$). Suppose, moreover, that g is odd whenever $p = 2$. Then it holds that X has a pseudo-coordinate of level N (cf. [7, Def. 2.3]) if and only if there exists a locally free coherent $\mathcal{O}_{X^F}$-module $\mathcal{E}$ of rank 2 such that $\phi_*\mathcal{O}_X$ (resp. $\mathcal{B}_{X^f}$) is $\mathbb{P}$-equivalent to $\Phi^*_{f\to F}\mathcal{E}$.*

(ii) *Suppose that $p = 2$ (resp. $p = 3$). Then it holds that X has a Tango function of level N if and only if there exist a locally free coherent $\mathcal{O}_{X^F}$-module $\mathcal{E}$ of rank 2, an invertible*

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sheaf $\mathcal{Q}$ *on* X^F *of degree* $(2g-2)/p^N$ *(resp.* $(4g-4)/p^N$*), a surjective homomorphism* $\mathcal{E} \twoheadrightarrow \mathcal{Q}$ *of* $\mathcal{O}_{X^F}$*-modules, and an isomorphism* $\phi_* \mathcal{O}_X \xrightarrow{\sim} \Phi^*_{f \to F}\mathcal{E}$ *(resp.* $\mathcal{B}_{X^f} \xrightarrow{\sim} \Phi^*_{f \to F}\mathcal{E}$*) of* $\mathcal{O}_{X^f}$*-modules.*

Proof. First, we verify assertion (i). Let us first observe that it follows from [7, Cor. 5.8] that X has a *pseudo-coordinate of level* N if and only if there exists an $(N, g-1)$-*Frobenius-destabilized* locally free coherent $\mathcal{O}_{X^F}$-module of rank 2. Thus, assertion (i) follows from Lemmas 6.3(i) and 6.4(i). This completes the proof of assertion (i).

Next, we verify assertion (ii). Let us first observe that it follows from Corollary 5.8 that X has a *Tango function of level* N if and only if there exists an $(N, g-1)$-*Frobenius-splitting* pair on X. Thus, assertion (ii) follows from Lemmas 6.3(ii) and 6.4(ii). This completes the proof of assertion (ii), hence also of Corollary 6.5. ∎

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