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Symbolic factors of $\mathcal{S}$-adic subshifts of finite alphabet rank

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Abstract. This paper studies several aspects of symbolic (i.e. subshift) factors of $\mathcal{S}$-adic subshifts of finite alphabet rank. First, we address a problem raised by Donoso *et al* [Interplay between finite topological rank minimal Cantor systems, S-adic subshifts and their complexity. *Trans. Amer. Math. Soc.* **374**(5) (2021), 3453–3489] about the topological rank of symbolic factors of $\mathcal{S}$-adic subshifts and prove that this rank is at most the one of the extension system, improving on the previous results [B. Espinoza. On symbolic factors of S-adic subshifts of finite alphabet rank. *Preprint*, 2022, arXiv:2008.13689v2; N. Golestani and M. Hosseini. On topological rank of factors of Cantor minimal systems. *Ergod. Th. & Dynam. Sys.* doi:10.1017/etds.2021.62. Published online 8 June 2021]. As a consequence of our methods, we prove that finite topological rank systems are coalescent. Second, we investigate the structure of fibers $\pi^{-1}(y)$ of factor maps $\pi : (X, T) \to (Y, S)$ between minimal $\mathcal{S}$-adic subshifts of finite alphabet rank and show that they have the same finite cardinality for all y in a residual subset of Y. Finally, we prove that the number of symbolic factors (up to conjugacy) of a fixed subshift of finite topological rank is finite, thus extending Durand's similar theorem on linearly recurrent subshifts [F. Durand. Linearly recurrent subshifts have a finite number of non-periodic subshift factors. *Ergod. Th. & Dynam. Sys.* **20**(4) (2000), 1061–1078].

Key words: topological rank, S-adic representations, substitutional systems
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1. *Introduction*

An ordered Bratteli diagram is an infinite directed graph $B = (V, E, \leq)$ such that the vertex set V and the edge set E are partitioned into levels $V = V_0 \cup V_1 \cup \cdots$, $E = E_0 \cup \cdots$ so that E_n are edges from V_{n+1} to V_n, V_0 is a singleton, each V_n is finite and $\leq$ is a partial order on E such that two edges are comparable if and only if they start at the same

vertex. The order $\leq$ can be extended to the set X_B of all infinite paths in B, and the Vershik action V_B on X_B is defined when B has unique minimal and maximal infinite paths with respect to $\leq$. We say that (X_B, V_B) is a Bratteli–Vershik (BV) representation of the Cantor system (X, T) if both are conjugate. Bratteli diagrams are a tool coming from C^*-algebras that, at the beginning of the 1990's, Herman *et al* [**HPS92**] used to study minimal Cantor systems. Their success at characterizing the strong and weak orbit equivalence for systems of this kind marked a milestone in the theory that motivated many posterior works. Some of these works focused on using Bratteli diagrams to study specific classes of systems and, as a consequence, many of the classical minimal systems have been characterized as BV systems with a specific structure. Some examples include odometers as those systems that have a BV representation with one vertex per level, substitutive subshifts as *stationary* BV (all levels are the same) [**DHS99**], certain Toeplitz sequences as 'equal row-sum' BV [**GJ00**], and (codings of) interval exchanges as BV where the diagram codifies a path in a Rauzy graph [**GJ02**]. Now, almost all of these examples share certain coarse dynamical behavior: they have finitely many ergodic measures, are not strongly mixing, have zero entropy, are subshifts, and their BV representations have a bounded number of vertices per level, among many others. It turns out that just having a BV representation with a bounded number of vertices per level (or, from now on, having *finite topological rank*) implies the previous properties (see, for example, [**BKMS13**, **DM08**]). Hence, the finite topological rank class arises as a possible framework for studying minimal subshifts and proving general theorems.

This idea has been exploited in many works: Durand *et al*, in a series of papers (with [**DFM19**] being the last one), developed techniques from the well-known substitutive case and obtained criteria for any BV of finite topological rank to decide if a given complex number is a continuous or measurable eigenvalue; Bezuglyi *et al* described in [**BKMS13**] the simplex of invariant measures together with natural conditions for being uniquely ergodic; Giordano *et al* bounded the rational rank of the dimension group by the topological rank [**HPS92**, **GHH18**], among other works. It is important to remark that these works were inspired by or first proved in the substitutive case.

Now, since Bratteli–Vershik systems with finite topological rank at least two are conjugate to a subshift [**DM08**], it is interesting to try to define them directly as a subshift. This can be done by codifying the levels of the Bratteli diagram as substitutions and then iterate them to obtain a sequence of symbols defining a subshift conjugate to the initial BV system. This procedure also makes sense for arbitrary nested sequences of substitutions (called *directive sequences*), independently from the Bratteli diagram and the various additional properties of its codifying substitutions. Subshifts obtained in this way are called $\mathcal{S}$-adic (substitution-adic) and may be non-minimal (see, for example, [**BSTY19**]).

Although there are some open problems about finite topological rank systems depending directly on the combinatorics of the underlying Bratteli diagrams, others are more naturally stated in the $\mathcal{S}$-adic setting (e.g. when dealing with endomorphisms, it is useful to have the Curtisâ–Hedlundâ–Lyndon theorem) and, hence, there exists an interplay between $\mathcal{S}$-adic subshifts and finite topological rank systems in which theorems and techniques obtained for one of these classes can sometimes be transferred to the other. The question about which

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is the exact relation between these classes has been recently addressed in [**DDMP21**] and, in particular, the authors proved the following.

THEOREM 1.1. [**DDMP21**] *A minimal subshift (X, T) has topological rank at most K if and only if it is generated by a proper, primitive, and recognizable $\mathcal{S}$-adic sequence of alphabet rank at most K.*

In this context, a fundamental question is the following.

Question 1.2. Are subshift factors of finite topological rank systems of finite topological rank?

Indeed, the topological rank controls various coarse dynamical properties (number of ergodic measures, rational rank of dimension group, among others) which cannot increase after a factor map, and we also know that big subclasses of the finite topological rank class are stable under symbolic factors, such as the linearly recurrent and the non-superlineal complexity classes [**DDMP21**], so it is expected that this question has an affirmative answer. However, when trying to prove this using Theorem 1.1, we realize that the naturally inherited $\mathcal{S}$-adic structure of finite alphabet rank that a symbolic factor has is never recognizable. Moreover, this last property is crucial for many of the currently known techniques to handle finite topological rank systems (even in the substitutive case, it is a deep and fundamental theorem of Mossé), so it is not clear why it would be always possible to obtain this property while keeping the alphabet rank bounded or why recognizability is not connected with a dynamical property of the system. Thus, an answer to this question seems to be fundamental to the understanding of the finite topological rank class.

This question has been recently addressed, first in [**Esp20**] by purely combinatorial methods, and then also in [**GH21**] in the BV formulation by using an abstract construction from [**AEG15**]. In this work, we refine both approaches and obtain, as a first consequence, the optimal answer to Question 1.2 in a more general, non-minimal context.

THEOREM 1.3. *Let (X, T) be an $\mathcal{S}$-adic subshift generated by an everywhere growing and proper directive sequence of alphabet rank equal to K, and $\pi : (X, T) \to (Y, S)$ be an aperiodic subshift factor. Then, (Y, S) is an $\mathcal{S}$-adic subshift generated by an everywhere growing, proper, and recognizable directive sequence of alphabet rank at most K.*

Here, a directive sequence $\boldsymbol{\sigma} = (\sigma_n : \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\in\mathbb{N}}$ is *everywhere growing* if $\lim_{n\to\infty} \min_{a\in\mathcal{A}_n} |\sigma_0 \dots \sigma_{n-1}(a)| = \infty$, and a system (X, T) is *aperiodic* if every orbit $\{T^n x : n \in \mathbb{Z}\}$ is infinite. Theorem 1.3 implies that the topological rank cannot increase after a factor map (Corollary 4.8). Theorem 1.3 implies the following sufficient condition for a system to be of finite topological rank.

COROLLARY 1.4. *Let (X, T) be an aperiodic minimal $\mathcal{S}$-adic subshift generated by an everywhere growing directive sequence of finite alphabet rank. Then, the topological rank of (X, T) is finite.*

An interesting corollary of the underlying construction of the proof of Theorem 1.3 is the coalescence property for this kind of system, in the following stronger form.

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COROLLARY 1.5. *Let (X, T) be an $\mathcal{S}$-adic subshift generated by an everywhere growing and proper directive sequence of alphabet rank equal to K, and $(X, T) \xrightarrow{\pi_1} (X_1, T_1) \xrightarrow{\pi_2} \dots \xrightarrow{\pi_L} (X_L, T_L)$ be a chain of aperiodic subshift factors. If $L > \log_2 K$, then at least one π_j is a conjugacy.*

One of the results in [**Dur00**] is that factor maps between aperiodic linearly recurrent subshifts are finite-to-one. In particular, they are *almost k-to-1* for some finite k. For finite topological rank subshifts, we prove the following.

THEOREM 1.6. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between aperiodic minimal subshifts. Suppose that (X, T) has topological rank equal to K. Then π is almost k-to-1 for some $k \leq K$.*

We use this theorem, in Corollary 4.12, to prove that Cantor factors of finite topological rank subshifts are either odometers or subshifts.

In [**Dur00**], the author proved that *linearly recurrent* subshifts have finite topological rank, and that this kind of system has finitely many aperiodic subshift factors up to conjugacy. Inspired by this result, we use ideas from the proof of Theorem 1.3 to obtain the following.

THEOREM 1.7. *Let (X, T) be a minimal subshift of topological rank K. Then, (X, T) has at most $(3K)^{32K}$ aperiodic subshift factors up to conjugacy.*

Altogether, these results give a rough picture of the set of totally disconnected factors of a given finite topological rank system: they are either equicontinuous or subshifts satisfying the properties in Theorems 1.3, 1.5, 1.7, and 1.6. Now, in a topological sense, totally disconnected factors of a given system (X, T) are 'maximal,' so, the natural next step in the study of finite topological rank systems is asking about the connected factors. As we have seen, the finite topological rank condition is a rigidity condition. By this reason, we think that the following question has an affirmative answer.

Question 1.8. Let (X, T) be a minimal system of finite topological rank and $\pi : (X, T) \to (Y, S)$ be a factor map. Suppose that Y is connected. Is (Y, S) an equicontinuous system?

We remark that the finite topological rank class contains all minimal subshifts of non-superlinear complexity [**DDMP21**], but even for the much smaller class of linear complexity subshifts, the author is not aware of results concerning Question 1.8.

1.1. *Organization.* In the next section, we give the basic background in topological and symbolic dynamics needed in this article. Section 3 is devoted to prove some combinatorial lemmas. The main results about the topological rank of factors are stated and proved in §4. Next, in §5, we prove Theorem 1.6, which is mainly a consequence of the so-called critical factorization theorem. Finally, in §6, we study the problem about the number of symbolic factors and prove Theorem 1.7.

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2. *Preliminaries*

For us, the set of natural numbers starts with zero, i.e. $\mathbb{N} = \{0, 1, 2, \ldots\}$.

2.1. *Basics in topological dynamics.*

A *topological dynamical system* (or just a system) is a pair (X, T), where X is a compact metric space and $T : X \to X$ is a *homeomorphism* of X. We denote by $\mathrm{Orb}_T(x)$ the orbit $\{T^n x : n \in \mathbb{Z}\}$ of $x \in X$. A point $x \in X$ is *p-periodic* if $T^p x = x$, *periodic* if it is p-periodic for some $p \geq 1$, and *aperiodic* otherwise. A topological dynamical system is *aperiodic* if any point $x \in X$ is aperiodic, is *minimal* if the orbit of every point is dense in X, and is Cantor if X is a Cantor space (i.e. X is totally disconnected and does not have isolated points). We use the letter T to denote the action of a topological dynamical system independently of the base set X. The *hyperspace* of (X, T) is the system $(2^X, T)$, where 2^X is the set of all closed subsets of X with the topology generated by the Hausdorff metric $d_H(A, B) = \max(\sup_{x\in A} d(x, A), \sup_{y\in B} d(y, A))$, and T the action $A \mapsto T(A)$.

A *factor* between the topological dynamical systems (X, T) and (Y, T) is a continuous function π from X onto Y such that $\pi \circ T = T \circ \pi$. We use the notation $\pi : (X, T) \to (Y, T)$ to indicate the factor. A factor map $\pi : (X, T) \to (Y, T)$ is *almost K-to-1* if $\#\pi^{-1}(y) = K$ for all y in a residual subset of Y. We say that π is *distal* if whenever $\pi(x) = \pi(x')$ and $x \neq x'$, we have $\inf_{k\in\mathbb{Z}} \mathrm{dist}(T^k x, T^k x') > 0$.

Given a system (X, T), the *Ellis semigroup* $E(X, T)$ associated with (X, T) is defined as the closure of $\{x \mapsto T^n x : n \in \mathbb{Z}\} \subseteq X^X$ in the product topology, where the semi-group operation is given by the composition of functions. On X, we may consider the $E(X, T)$-action given by $x \mapsto ux$. Then, the closure of the orbit under T of a point $x \in X$ is equal to the orbit of x under $E(X, T)$. If $\pi : (X, T) \to (Y, T)$ is a factor between minimal systems, then π induces a surjective map $\pi^* : E(X, T) \to E(Y, T)$, which is characterized by the formula

$$\pi(ux) = \pi^*(u)\pi(x) \quad \text{for all } u \in E(X, T) \text{ and } x \in X.$$

If the context is clear, we will not distinguish between u and $\pi^*(u)$. When $u \in E(2^X, T)$, we write $u \circ A$ instead of uA, the last symbol being reserved to mean $uA = \{ux : x \in A\}$. We can describe more explicitly $u \circ A$ as follows: it is the set of all $x \in X$ for which we can find nets $x_\lambda \in A$ and $m_\lambda \in \mathbb{Z}$ such that $\lim_\lambda T^{m_\lambda} x_\lambda = x$ and $\lim_\lambda T^{m_\lambda} = u$. Finally, we identify X with $\{\{x\} \subseteq 2^X : x \in X\}$, so that the restriction map $E(2^X, T) \to E(X, T)$, which sends $u \in E(2^X, T)$ to the restriction $u|_X : X \to X$, is an onto morphism of semigroups. As above, we will not distinguish between $u \in 2^X$ and $u|_X$.

2.2. *Basics in symbolic dynamics.*

2.2.1. *Words and subshifts.* Let $\mathcal{A}$ be an *alphabet* i.e. a finite set. Elements in $\mathcal{A}$ are called *letters* and concatenations $w = a_1 \ldots a_\ell$ of them are called words. The number ℓ is the length of w and it is denoted by $|w|$, the set of all words in $\mathcal{A}$ of length ℓ is $\mathcal{A}^\ell$, and $\mathcal{A}^+ = \bigcup_{\ell\geq 1} \mathcal{A}^\ell$. The word $w \in \mathcal{A}^+$ is $|u|$-*periodic*, with $u \in \mathcal{A}^+$, if w occurs in a word of the form $uu \ldots u$. We define $\mathrm{per}(w)$ as the smallest p for which w is p-periodic. We

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will use notation analogous to that introduced in this paragraph when dealing with infinite words $x \in \mathcal{A}^{\mathbb{N}}$ and bi-infinite words $x \in \mathcal{A}^{\mathbb{Z}}$. The set $\mathcal{A}^+$ equipped with the operation of concatenation can be viewed as the free semigroup on $\mathcal{A}$. It is convenient to introduce the empty word 1, which has length 0 and is a neutral element for the concatenation. In particular, $\mathcal{A}^+ \cup \{1\}$ is the free monoid in $\mathcal{A}$. Finally, for $\mathcal{W} \subseteq \mathcal{A}^+$, we write $\langle \mathcal{W} \rangle := \min_{w \in \mathcal{W}} |w|$ and $|W| := \max_{w \in \mathcal{W}} |w|$.

The *shift map* $T : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by $T((x_n)_{n \in \mathbb{Z}}) = (x_{n+1})_{n \in \mathbb{Z}}$. For $x \in \mathcal{A}^{\mathbb{Z}}$ and integers $i < j$, we denote by $x_{[i,j)}$ the word $x_i x_{i+1} \dots x_j$. Analogous notation will be used when dealing with intervals of the form $[i, \infty)$, (i, ∞), $(-\infty, i]$, and $(-\infty, i)$. A *subshift* is a topological dynamical system (X, T), where X is a closed and T-invariant subset of $\mathcal{A}^{\mathbb{Z}}$ (we consider the product topology in $\mathcal{A}^{\mathbb{Z}}$) and T is the shift map. Classically one identifies (X, T) with X, so one says that X itself is a subshift. When we say that a sequence in a subshift is periodic (respectively aperiodic), we implicitly mean that this sequence is periodic (respectively aperiodic) for the action of the shift. Therefore, if $x \in \mathcal{A}^{\mathbb{Z}}$ is periodic, then $\mathrm{per}(x)$ is equal to the size of the orbit of x. The *language* of a subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is the set $\mathcal{L}(X)$ of all words $w \in \mathcal{A}^+$ that occur in some $x \in X$.

The pair $(x, \tilde{x}) \in \mathcal{A}^{\mathbb{Z}} \times \mathcal{A}^{\mathbb{Z}}$ is *right asymptotic* if there exist $k \in \mathbb{Z}$ satisfying $x_{(k,\infty)} = \tilde{x}_{(k,\infty)}$ and $x_k \neq \tilde{x}_k$. If moreover $k = 0$, $(x, \tilde{x})$ is a *centered right asymptotic*. A *right asymptotic tail* is an element $x_{(0,\infty)}$, where $(x, \tilde{x})$ is a centered right asymptotic pair. We make similar definitions for left asymptotic pairs and tails.

2.2.2. *Morphisms and substitutions.* Let $\mathcal{A}$ and $\mathcal{B}$ be finite alphabets and $\tau : \mathcal{A}^+ \to \mathcal{B}^+$ be a morphism between the free semigroups that they define. Then, τ extends naturally to maps from $\mathcal{A}^{\mathbb{N}}$ to itself and from $\mathcal{A}^{\mathbb{Z}}$ to itself in the obvious way by concatenation (in the case of a two-sided sequence, we apply τ to positive and negative coordinates separately and we concatenate the results at coordinate zero). We say that τ is *positive* if for every $a \in \mathcal{A}$, all letters $b \in \mathcal{B}$ occur in $\tau(a)$, is r-proper, with $r \geq 1$, if there exist $u, v \in \mathcal{B}^r$ such that $\tau(a)$ starts with u and ends with v for any $a \in \mathcal{A}$, is *proper* when is 1-proper, and is *letter-onto* if for every $b \in \mathcal{B}$ there exists $a \in \mathcal{A}$ such that b occurs in a. The minimum and maximum lengths of τ are respectively the numbers $\langle \tau \rangle := \langle \tau(\mathcal{A}) \rangle = \min_{a \in \mathcal{A}} |\tau(a)|$ and $|\tau| := |\tau(\mathcal{A})| = \max_{a \in \mathcal{A}} |\tau(a)|$.

We observe that any map $\tau : \mathcal{A} \to \mathcal{B}^+$ can be naturally extended to a morphism (that we also denote by τ) from $\mathcal{A}^+$ to $\mathcal{B}^+$ by concatenation, and any morphism $\tau : \mathcal{A}^+ \to \mathcal{B}^+$ is uniquely determined by its restriction to $\mathcal{A}$. From now on, we will use the same notation for denoting a map $\tau : \mathcal{A} \to \mathcal{B}^+$ and its extension to a morphism $\tau : \mathcal{A}^+ \to \mathcal{B}^+$.

Definition 2.1. Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a subshift and $\sigma : \mathcal{A}^+ \to \mathcal{B}^+$ be a morphism. We say that $(k, x) \in \mathbb{Z} \times X$ is a σ-factorization of $y \in \mathcal{B}^{\mathbb{Z}}$ *in* X if $y = T^k \sigma(x)$. If moreover $k \in [0, |\sigma(x_0)|)$, then (k, x) is a *centered* σ*-factorization in* X.

The pair (X, σ) is *recognizable* if every point $y \in \mathcal{B}^{\mathbb{Z}}$ has at most one centered σ-factorization in X, and *recognizable with constant* $r \in \mathbb{N}$ if whenever $y_{[-r,r]} = y'_{[-r,r]}$ and (k, x), (k', x') are centered σ-factorizations of y, $y' \in \mathcal{B}^{\mathbb{Z}}$ in X, respectively, we have $(k, x_0) = (k', x'_0)$.

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The *cuts* of (k, x) are defined by

$$c_{\sigma,j}(k,x) = \begin{cases} -k + |\sigma(x_{[0,j)})| & \text{if } j \geq 0, \\ -k - |\sigma(x_{[j,0)})| & \text{if } j < 0. \end{cases}$$

We write $C_\sigma(k,x) = \{c_{\sigma,j}(k,x) : j \in \mathbb{Z}\}$.

Remark 2.2. In the context of the previous definition we have the following.

(i) The point $y \in \mathcal{B}^{\mathbb{Z}}$ has a (centered) σ-factorization in X if and only if y belongs to the subshift $Y := \bigcup_{n\in\mathbb{Z}} T^n\sigma(X)$. Hence, (X,σ) is recognizable if and only if every $y \in Y$ has a exactly *one* centered σ-factorization in X.

(ii) If (k,x) is a σ-factorization of $y \in \mathcal{B}^{\mathbb{Z}}$ in X, then $(c_{\sigma,j}(k,x), T^j x)$ is a σ-factorization of y in X for any $j \in \mathbb{Z}$. There is exactly one factorization in this class that is centered.

(iii) If (X,σ) is recognizable, then it is recognizable with constant r for some $r \in \mathbb{N}$ [**DDMP21**].

The behavior of recognizability under composition of morphisms is given by the following lemma.

LEMMA 2.3. [**BSTY19**, Lemma 3.5] *Let $\sigma : \mathcal{A}^+ \to \mathcal{B}^+$ and $\tau : \mathcal{B}^+ \to \mathcal{C}^+$ be morphisms, $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a subshift, and $Y = \bigcup_{k\in\mathbb{Z}} T^k\sigma(X)$. Then, $(X, \tau\sigma)$ is recognizable if and only if (X,σ) and (Y,τ) are recognizable.*

Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ and $Z \subseteq \mathcal{C}^{\mathbb{Z}}$ be subshifts and $\pi : (X,T) \to (Z,T)$ a factor map. The classic Curtisâ–Hedlundâ–Lyndon theorem asserts that π has a *local code*, this is, a function $\psi : \mathcal{A}^{2r+1} \to \mathcal{C}$, where $r \in \mathbb{N}$, such that $\pi(x) = (\psi(x_{[i-r,i+r]}))_{i\in\mathbb{Z}}$ for all $x \in X$. The integer r is called the a radius of π. The following lemma relates the local code of a factor map to proper morphisms.

LEMMA 2.4. *Let $\sigma : \mathcal{A}^+ \to \mathcal{B}^+$ be a morphism, $X \subseteq \mathcal{A}^{\mathbb{Z}}$ and $Z \subseteq \mathcal{C}^{\mathbb{Z}}$ be subshifts, and $Y = \bigcup_{k\in\mathbb{Z}} T^k\sigma(X)$. Suppose that $\pi : (Y,T) \to (Z,T)$ is a factor map of radius r and that σ is r-proper. Then, there exists a proper morphism $\tau : \mathcal{A}^+ \to \mathcal{C}^+$ such that $|\tau(a)| = |\sigma(a)|$ for any $a \in \mathcal{A}$, $Z = \bigcup_{k\in\mathbb{Z}} T^k\tau(X)$ and the following diagram commutes:*

$$\begin{array}{ccc} X & & \\ {\scriptstyle\sigma}\downarrow & \searrow^{\tau} & \\ Y & \xrightarrow{\pi} & Z \end{array} \tag{1}$$

Proof. Let $\psi : \mathcal{A}^{2r+1} \to \mathcal{B}$ be a local code of radius r for π and $u, v \in \mathcal{B}^r$ be such that $\sigma(a)$ starts with u and ends with v for all $a \in \mathcal{A}$. We define $\tau : \mathcal{A} \to \mathcal{C}^+$ by $\tau(a) = \psi(v\sigma(a)u)$. Then, since σ is r-proper, τ is proper and we have $\pi(\sigma(x)) = \tau(x)$ for all $x \in X$ (that is, Diagram (1) commutes). In particular,

$$\bigcup_{k\in\mathbb{Z}} T^k\tau(X) = \bigcup_{k\in\mathbb{Z}} T^k\pi(\sigma(X)) = \pi(Y) = Z. \qquad \square$$

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2.2.3. *$\mathcal{S}$-adic subshifts.* We recall the definition of an *$\mathcal{S}$-adic subshift* as stated in [BSTY19]. A *directive sequence* $\boldsymbol{\sigma} = (\sigma_n : \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\in\mathbb{N}}$ is a sequence of morphisms. For $0 \le n < N$, we denote by $\sigma_{[n,N)}$ the morphism $\sigma_n \circ \sigma_{n+1} \circ \cdots \circ \sigma_{N-1}$. We say that $\boldsymbol{\sigma}$ is *everywhere growing* if

$$\lim_{N\to+\infty} \langle \sigma_{[0,N)} \rangle = +\infty, \tag{2}$$

and *primitive* if for any $n \in \mathbb{N}$, there exists $N > n$ such that $\sigma_{[n,N)}$ is positive. We remark that this notion is slightly different from the usual one used in the context of substitutional dynamical systems. Observe that $\boldsymbol{\sigma}$ is everywhere growing if $\boldsymbol{\sigma}$ is primitive. Let $\mathcal{P}$ be a property for morphisms (e.g. proper, letter-onto, etc). We say that $\boldsymbol{\sigma}$ has property $\mathcal{P}$ if σ_n has property $\mathcal{P}$ for every $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, we define

$$X_{\boldsymbol{\sigma}}^{(n)} = \{x \in \mathcal{A}_n^{\mathbb{Z}} : \text{ for all } \ell \in \mathbb{N}, x_{[-\ell,\ell]} \text{ occurs in } \sigma_{[n,N)}(a) \text{ for some } N > n, a \in \mathcal{A}_N\}.$$

This set clearly defines a subshift that we call the *nth level of the* $\mathcal{S}$-adic subshift generated by $\boldsymbol{\sigma}$. We set $X_{\boldsymbol{\sigma}} = X_{\boldsymbol{\sigma}}^{(0)}$ and simply call it the *$\mathcal{S}$-adic subshift* generated by $\boldsymbol{\sigma}$. If $\boldsymbol{\sigma}$ is everywhere growing, then every $X_{\boldsymbol{\sigma}}^{(n)}$, $n \in \mathbb{N}$, is non-empty; if $\boldsymbol{\sigma}$ is primitive, then $X_{\boldsymbol{\sigma}}^{(n)}$ is minimal for every $n \in \mathbb{N}$. There are non-everywhere growing directive sequences that generate minimal subshifts.

The relation between levels of an $\mathcal{S}$-adic subshift is given by the following lemma.

LEMMA 2.5. [BSTY19, Lemma 4.2] *Let $\boldsymbol{\sigma} = (\sigma_n : \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\in\mathbb{N}}$ be a directive sequence of morphisms. If $0 \le n < N$ and $x \in X_{\boldsymbol{\sigma}}^{(n)}$, then there exists a (centered) $\sigma_{[n,N)}$-factorization in $X_{\boldsymbol{\sigma}}^{(N)}$. In particular, $X_{\boldsymbol{\sigma}}^{(n)} = \bigcup_{k\in\mathbb{Z}} T^k \sigma_{[n,N)}(X_{\boldsymbol{\sigma}}^{(N)})$.*

The levels $X_{\boldsymbol{\sigma}}^{(n)}$ can be described in an alternative way if $\boldsymbol{\sigma}$ satisfies the correct hypothesis.

LEMMA 2.6. *Let $\boldsymbol{\sigma} = (\sigma_n : \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\in\mathbb{N}}$ be an everywhere growing and proper directive sequence. Then, for every $n \in \mathbb{N}$,*

$$X_{\boldsymbol{\sigma}}^{(n)} = \bigcap_{N>n} \bigcup_{k\in\mathbb{Z}} T^k \sigma_{[n,N)}(\mathcal{A}_N^{\mathbb{Z}}). \tag{3}$$

Proof. Let Z be the set in the right-hand side of (3). Since, by Lemma 2.5, $X_{\boldsymbol{\sigma}}^{(n)} = \bigcup_{k\in\mathbb{Z}} T^k \sigma_{[n,N)}(X_{\boldsymbol{\sigma}}^{(N)})$ for any $N > n$, we have that $X_{\boldsymbol{\sigma}}^{(n)}$ included in Z.

Conversely, let $x \in Z$ and $\ell \in \mathbb{N}$. We have to show that $x_{[-\ell,\ell)}$ occurs in $\sigma_{[n,N)}(a)$ for some $N > n$ and $a \in \mathcal{A}_N$. Let $N > n$ be big enough so that $\sigma_{[n,N)}$ is ℓ-proper. Then, by the definition of Z, there exists $y \in \mathcal{A}_N^{\mathbb{Z}}$ such that $x_{[-\ell,\ell)}$ occurs in $\sigma_{[n,N)}(y)$. Since $\langle \sigma_{[n,N)} \rangle \ge \ell$ (as $\sigma_{[n,N)}$ is ℓ-proper), we deduce that

$$x_{[-\ell,\ell)} \text{ occurs in } \sigma_{[n,N)}(ab) \text{ for some word } ab \text{ of length 2 occurring in } y. \tag{4}$$

Hence, by denoting by u and v the suffix and prefix of length ℓ of $\tau_{[n,N)}(a)$ and $\tau_{[n,N)}(b)$, respectively, we have that $x_{[-\ell,\ell)}$ occurs in $\sigma_{[n,N)}(a)$, in $\tau_{[n,N)}(b)$, or in uv. In the first two cases, we are done. In the last case, we observe that since $\sigma_{[n,N)}$ is ℓ-proper, the following

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is true: for every $M > N$ such that $\langle \sigma_{[N,M)} \rangle \geq 2$, $vu \sqsubseteq \sigma_{[n,M)}(c)$ for any $c \in \mathcal{A}_M$. In particular, $x_{[-\ell,\ell)} \sqsubseteq \tau_{[n,M)}(c)$ for such M and c. We have proved that $x \in X_{\boldsymbol{\sigma}}^{(n)}$. □

We define the *alphabet rank* of a directive sequence $\boldsymbol{\tau}$ as

$$AR(\boldsymbol{\tau}) = \liminf_{n \to +\infty} \#\mathcal{A}_n.$$

A *contraction* of $\boldsymbol{\tau}$ is a sequence $\tilde{\boldsymbol{\tau}} = (\tau_{[n_k,n_{k+1})} \colon \mathcal{A}_{n_{k+1}}^+ \to \mathcal{A}_{n_k}^+)_{k \in \mathbb{N}}$, where $0 = n_0 < n_1 < n_2 < \dots$. Observe that any contraction of $\boldsymbol{\tau}$ generates the same $\mathcal{S}$-adic subshift $X_{\boldsymbol{\tau}}$. When the context is clear, we will use the same notation to refer to $\boldsymbol{\tau}$ and its contractions. If $\boldsymbol{\tau}$ has finite alphabet rank, then there exists a contraction $\tilde{\boldsymbol{\tau}} = (\tau_{[n_k,n_{k+1})} \colon \mathcal{A}_{n_{k+1}}^+ \to \mathcal{A}_{n_k}^+)_{k \in \mathbb{N}}$ of $\boldsymbol{\tau}$ in which $\mathcal{A}_{n_k}$ has cardinality $AR(\boldsymbol{\tau})$ for every $k \geq 1$.

Finite alphabet rank $\mathcal{S}$-adic subshifts are *eventually recognizable*.

THEOREM 2.7. [**DDMP21**, Theorem 3.7] *Let $\boldsymbol{\sigma}$ be an everywhere growing directive sequence of alphabet rank equal to K. Suppose that $X_{\boldsymbol{\sigma}}$ is aperiodic. Then, at most $\log_2 K$ levels $(X_{\boldsymbol{\sigma}}^{(n)}, \sigma_n)$ are not recognizable.*

We will also need the following property.

THEOREM 2.8. [**EM21**, Theorem 3.3] *Let (X, T) be an $\mathcal{S}$-adic subshift generated by an everywhere growing directive sequence of alphabet rank K. Then, X has at most $144K^7$ right (respectively left) asymptotic tails.*

Proof. In the proof of theorem 3.3 in [**EM21**], the authors show the following: the set consisting of pairs $(x, y) \in X \times X$ such that $x_{(-\infty,0)} = y_{(-\infty,0)}$ and $x_0 \neq y_0$ has at most $144K^7$ elements. In our language, this is equivalent to saying that X has at most $144K^7$ left asymptotic tails. Since this is valid for any $\mathcal{S}$-adic subshift generated by an everywhere growing directive sequence of alphabet rank K, $144K^7$ is also an upper bound for right asymptotic tails. □

3. *Combinatorics on words lemmas*

In this section, we present several combinatorial lemmas that will be used throughout the article.

3.1. *Lowering the rank.*

Let $\sigma \colon \mathcal{A}^+ \to \mathcal{B}^+$ be a morphism. Following ideas from [**RS97**], we define the *rank* of σ as the least cardinality of a set of words $\mathcal{D} \subseteq \mathcal{B}^+$ such that $\sigma(\mathcal{A}^+) \subseteq \mathcal{D}^+$. Equivalently, the rank is the minimum cardinality of an alphabet $\mathcal{C}$ in a decomposition into morphisms $\mathcal{A}^+ \xrightarrow{q} \mathcal{C}^+ \xrightarrow{p} \mathcal{B}^+$ such that $\sigma = pq$. In this subsection, we prove Lemma 3.6, which states that in certain technical situations, the rank of the morphism σ under consideration is small and its decomposition $\sigma = pq$ satisfies additional properties.

We start by defining some morphisms that will be used in the proofs of this subsection. If $a \neq b \in \mathcal{A}$ are different letters and $\tilde{a}$ is a letter not in $\mathcal{A}$, then we define $\phi_{a,b} \colon \mathcal{A}^+ \to$

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$(\mathcal{A}\setminus\{b\})^+$, $\psi_{a,b}\colon \mathcal{A}^+\to\mathcal{A}^+$, and $\theta_{a,\tilde{a}}\colon \mathcal{A}^+\to(\mathcal{A}\cup\{\tilde{a}\})^+$ by

$$\phi_{a,b}(c)=\begin{cases}c & \text{if } c\neq b,\\ a & \text{if } c=b.\end{cases}\qquad \psi_{a,b}(c)=\begin{cases}c & \text{if } c\neq b,\\ ab & \text{if } c=b.\end{cases}\qquad \theta_{a,\tilde{a}}(c)=\begin{cases}c & \text{if } c\neq a,\\ \tilde{a}a & \text{if } c=a.\end{cases}$$

Observe that these morphisms are letter-onto. Before stating the basic properties of these morphisms, we need one more set of definitions.

For a morphism $\sigma\colon\mathcal{A}^+\to\mathcal{B}^+$, we define $|\sigma|_1=\sum_{a\in\mathcal{A}}|\sigma(a)|$. When $u,v,w\in\mathcal{A}^+$ satisfy $w=uv$, we say that u is a prefix of w and that v a suffix of w. Recall that 1 stands for the empty word.

LEMMA 3.1. *Let $\sigma\colon\mathcal{A}^+\to\mathcal{B}^+$ be a morphism.*

(i) *If $\sigma(a)=\sigma(b)$ for some $a\neq b\in\mathcal{A}$, then $\sigma=\sigma'\phi_{a,b}$, where $\sigma'\colon(\mathcal{A}\setminus\{b\})^+\to\mathcal{B}^+$ is the restriction of σ to $(\mathcal{A}\setminus\{b\})^+$.*

(ii) *If $\sigma(a)$ is a prefix of $\sigma(b)$ and $\sigma(b)=\sigma(a)t$ for some non-empty $t\in\mathcal{B}^+$, then $\sigma=\sigma'\psi_{a,b}$, where $\sigma'\colon\mathcal{A}^+\to\mathcal{B}^+$ is defined by*

$$\sigma'(c)=\begin{cases}\sigma(c) & \text{if } c\neq b,\\ t & \text{if } c=b.\end{cases}\tag{5}$$

(iii) *If $\sigma(a)=st$ for some $s,t\in\mathcal{B}^+$ and $a\in\mathcal{A}$, then $\sigma=\sigma'\theta_{a,\tilde{a}}$, where $\sigma'\colon(\mathcal{A}\cup\{\tilde{a}\})^+\to\mathcal{B}^+$ is defined by*

$$\sigma'(c)=\begin{cases}\sigma(c) & \text{if } c\neq a,\tilde{a},\\ s & \text{if } c=\tilde{a},\\ t & \text{if } c=a.\end{cases}\tag{6}$$

Proof. The lemma follows from unraveling the definitions. For instance, in case (ii), we have $\sigma'(\psi_{a,b}(a))=\sigma'(a)=\sigma(a)$, $\sigma'(\psi_{a,b}(b))=\sigma'(ab)=\sigma(a)t=\sigma(b)$, and $\sigma'(\psi_{a,b}(c))=\sigma'(c)=\sigma(c)$ for all $c\neq a,b$, which shows that $\sigma'\psi_{a,b}=\sigma$. □

LEMMA 3.2. *Let $\{\sigma_j\colon\mathcal{A}^+\to\mathcal{B}_j^+\}_{j\in J}$ be a set of morphisms such that*

$$\text{for every fixed } a\in\mathcal{A},\ \ell_a:=|\sigma_j(a)| \text{ is constant for any chosen } j\in J,\tag{7}$$

and $u,v\in\mathcal{A}^+$, with u of length at least $\ell:=\sum_{a\in\mathcal{A}}\ell_a$. Assume that u and v start with different letters and that $\sigma_j(u)$ is a prefix of $\sigma_j(v)$ for every $j\in J$.

Then, there exists a letter-onto morphism $q\colon\mathcal{A}^+\to\mathcal{C}^+$, with $\#\mathcal{C}<\#\mathcal{A}$, and morphisms $\{p_j\colon\mathcal{C}^+\to\mathcal{B}_j^+\}_{j\in J}$ satisfying a condition analogous to (7) and such that $\sigma_j=p_jq$.

Remark 3.3. If in the previous lemma we change the last hypothesis to '*u* and *v end* with different letters and $\sigma_j(u)$ is a *suffix* of $\sigma_j(v)$ for every $j\in J$', then the same conclusion holds. This observation will be used in the proof of Lemma 6.7.

Proof of Lemma 3.2. By contradiction, we assume that u, v, and $\{\sigma_j\}_{j\in J}$ are counterexamples for the lemma. Moreover, we suppose that ℓ is as small as possible.

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Let us write $u = au'$ and $v = bv'$, where $a, b \in \mathcal{A}$. Since $\sigma_j(u)$ is a prefix of $\sigma_j(v)$, we have that for every $j \in J$,

$$\text{one of the words in } \{\sigma_j(a), \sigma_j(b)\} \text{ is a prefix of the other.} \tag{8}$$

We consider two cases. First, we suppose that $\ell_a = \ell_b$. In this case, (8) implies that $\sigma_j(a) = \sigma_j(b)$ for every $j \in J$. Hence, we can use (i) of Lemma 3.1 to decompose each σ_j as $\sigma'_j\phi_{a,b}$, where σ'_j is the restriction of σ_j to $(\mathcal{A} \setminus \{b\})^+$. Since $\phi_{a,b}$ is letter-onto and $\ell_c = |\sigma'_j(c)|$ for every $j \in J$, $c \in \mathcal{A} \setminus \{b\}$, the conclusion of the lemma holds, contrary to our assumptions.

This leaves to consider the case in which $\ell_a \neq \ell_b$. We only do the case $\ell_a < \ell_b$ as the other is similar. Then, by (8), for every $j \in J$, there exists a non-empty word $t_j \in \mathcal{B}_j^{\ell_b - \ell_a}$ of length $\ell_b - \ell_a$ such that $\sigma_j(b) = \sigma_j(a)t_j$. Thus, we can use (ii) of Lemma 3.1 to write, for any $j \in J$, $\sigma_j = \sigma'_j\psi_{a,b}$, where σ'_j is defined as in (5).

Let $\tilde{u} = \psi_{a,b}(u')$ and $\tilde{v} = b\psi_{a,b}(v')$. We want now to prove that $\tilde{u}$, $\tilde{v}$, and $\{\sigma'_j : j \in J\}$ satisfy the hypothesis of the lemma. First, we observe that for every $j \in J$,

$$\text{if } c \neq b, \text{ then } |\sigma'_j(c)| = \ell_c, \text{ and } |\sigma'_j(b)| = |t_j| = \ell_b - \ell_a. \tag{9}$$

Therefore, $\{\sigma'_j\}_{j\in J}$ satisfy (7). Also, since $\psi_{a,b}(c)$ never starts with b, we have that

$$\tilde{u}, \tilde{v} \text{ start with different letters.} \tag{10}$$

Furthermore, by using the symbol $\leq_p$ to denote the prefix relation, we can compute:

$$\sigma_j(a)\sigma'_j(\tilde{u}) = \sigma_j(a)\sigma_j(u') = \sigma_j(u) \leq_p \sigma_j(v) = \sigma'_j(\psi_{a,b}(v)) = \sigma'_j(a)\sigma'_j(\tilde{v}).$$

This and the fact that $\sigma_j(a)$ is equal to $\sigma'_j(a)$ imply that

$$\sigma'_j(\tilde{u}) \text{ is a prefix of } \sigma'_j(\tilde{v}) \text{ for every } j \in J. \tag{11}$$

Finally, we note

$$|\tilde{u}| \geq |u| - 1 \geq \sum_{c\in\mathcal{A}} \ell_c - \ell_a =: \ell'. \tag{12}$$

We conclude from (9), (10), (11), and (12) that $\tilde{u}$, $\tilde{v}$, and $\{\sigma'_j : j \in J\}$ satisfy the hypothesis of this lemma. Since $\ell' < \ell$, the minimality of ℓ implies that there exist a letter-onto morphism $q' : \mathcal{A}^+ \to C^+$, with $\#C < \#\mathcal{A}$, and morphisms $\{p_j : C^+ \to \mathcal{B}_j^+\}_{j\in J}$ satisfying $\sigma'_j = p_j q'$ and a property analogous to (7). However, then $q := q'\psi_{a,b}$ is also letter-onto and the morphisms $\{p_j\}_{j\in J}$ satisfy $\sigma_j = p_j q$ and a property analogous to (7). Thus, the conclusion of the lemma holds for $\{\sigma_j\}_{j\in J}$, contrary to our assumptions. □

LEMMA 3.4. *Let $\sigma : \mathcal{A}^+ \to \mathcal{B}^+$ be a morphism, $u, v \in \mathcal{A}^+$, a, b be the first letters of u, v respectively, and $\sigma(a) = st$ be a decomposition of $\sigma(a)$ in which t is non-empty. Assume that $\sigma(u)$ is a prefix of $s\sigma(v)$, $|u| \geq |\sigma|_1 + |s|$, and either that $s = 1$ and $a \neq b$ or that $s \neq 1$.*

Then, there exist morphisms $q : \mathcal{A}^+ \to C^+$ and $p : C^+ \to \mathcal{B}^+$ such that $\#C \leq \#\mathcal{A}$, q is letter-onto, $|p|_1 < |\sigma|_1$, and $\sigma = pq$.

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Remark 3.5. As in Lemma 3.2, there are symmetric hypotheses for the previous lemma that involve suffixes instead of prefixes and which give the same conclusion. We will use this in the proof of Lemma 3.6.

Proof of Lemma 3.4. Let us write $u = au'$ and $v = bv'$. We first consider the case in which $s = 1$. In this situation, u and v start with different letters, so Lemma 3.2 can be applied (with the index set J chosen as a singleton) to obtain a decomposition $\mathcal{A}^+ \xrightarrow{q} C^+ \xrightarrow{p} \mathcal{B}^+$ such that q is letter-onto, $\#C < \#\mathcal{A}$, and $\sigma = pq$. Since C has strictly fewer elements than $\mathcal{A}$, we have $|p|_1 < |\sigma|_1$. Hence, the conclusion of the lemma holds in this case.

We now assume that $s \neq 1$. In this case, t and s are non-empty, so we can use (iii) of Lemma 3.1 to factorize $\sigma = \sigma'\theta_{a,\tilde{a}}$, where $\tilde{a}$ is a letter not in $\mathcal{A}$ and σ' is defined as in (6). We set $\tilde{u} = a\theta_{a,\tilde{a}}(u')$ and $\tilde{v} = \theta_{a,\tilde{a}}(v)$. Our plan is to use Lemma 3.2 with $\tilde{u}$, $\tilde{v}$ and σ'.

Observe that $\theta_{a,\tilde{a}}(c)$ never starts with a, so

$$\tilde{u}, \tilde{v} \text{ start with different letters.} \tag{13}$$

Also, by using, as in the previous proof, the symbol $\leq_p$ to denote the prefix relation,

$$s\sigma'(\tilde{u}) = s\sigma'(a)\sigma'(\theta_{a,\tilde{a}}(u')) = st\sigma(u') = \sigma(u) \leq_p s\sigma(v) = s\sigma'(\theta_{a,\tilde{a}}(v)) = s\sigma'(\tilde{v}),$$

which implies that

$$\sigma'(\tilde{u}) \text{ is a prefix of } \sigma'(\tilde{v}). \tag{14}$$

Finally, we use (6) to compute:

$$|\tilde{u}| \geq |u| - 1 \geq |\sigma|_1 + |s| - 1 \geq |\sigma|_1 = |\sigma'|_1. \tag{15}$$

We conclude, by (13), (14), and (15) that Lemma 3.2 can be applied with $\tilde{u}$, $\tilde{v}$, and σ' (and J as a singleton). Thus, there exist morphisms $q' \colon (\mathcal{A} \cup \{\tilde{a}\})^+ \to C^+$ and $p \colon C^+ \to \mathcal{B}^+$ such that $\#C < \#(\mathcal{A} \cup \{\tilde{a}\})$, q' is letter-onto, and $\sigma' = pq'$. Then, $\#C \leq \#\mathcal{A}$, $q := q'\theta_{a,\tilde{a}}$ is letter-onto, and $\sigma = pq'\theta_{a,\tilde{a}} = pq$. Moreover, since $\theta_{a,\tilde{a}}$ is not the identity function, we have $|p|_1 < |\sigma|_1$. □

The next lemma is the main result of this subsection. To state it, we introduce additional notation. For an alphabet $\mathcal{A}$, let $\mathcal{A}^{++}$ be the set of words $w \in \mathcal{A}^+$ in which all letters occur. Observe that $\sigma \colon \mathcal{A}^+ \to \mathcal{B}^+$ is letter-onto if and only if $\sigma(\mathcal{A}^{++}) \subseteq \mathcal{B}^{++}$.

LEMMA 3.6. *Let $\phi \colon \mathcal{A}^+ \to C^+$, $\tau \colon \mathcal{B}^+ \to C^+$ be morphisms such that τ is ℓ-proper, with $\ell \geq |\phi|_1^4$, and $\phi(\mathcal{A}^+) \cap \tau(\mathcal{B}^{++}) \neq \emptyset$. Then, there exist $\mathcal{B}^+ \xrightarrow{q} \mathcal{D}^+ \xrightarrow{p} C^+$ such that*

$$\textit{(i) } \#\mathcal{D} \leq \#\mathcal{A}, \quad \textit{(ii) } \tau = pq, \quad \textit{(iii) } q \textit{ is letter-onto and proper.}$$

Proof. By contradiction, we suppose that the lemma does not hold for ϕ and τ and, moreover, that $|\phi|_1$ as small as possible.

That $\phi(\mathcal{A})^+ \cap \tau(\mathcal{B}^{++})$ is non-empty means that there exist $u = u_1 \dots u_n \in \mathcal{A}^+$ and $w = w_1 \dots w_m \in \mathcal{B}^{++}$ with $\phi(u) = \tau(w)$. If $m = 1$, then, since $w \in \mathcal{B}^{++}$, we have $\#\mathcal{B} = \{v_1\}$ and the conclusion of the lemma trivially holds for $\mathcal{D} = \{a \in C : a \text{ occurs in } \tau(w_1)\}$, $q \colon \mathcal{B}^+ \to \mathcal{D}^+$, $w_1 \mapsto \tau(w_1)$, and $p \colon \mathcal{D}^+ \to C^+$ the inclusion map, contradicting our initial assumption. Therefore, $m \geq 2$ and $\{1, \dots, m-1\}$ is non-empty.

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Let $k \in \{1, \dots, m-1\}$. We define i_k as the smallest number in $\{1, \dots, n\}$ for which $|\tau(w_1 \dots w_k)| < |\phi(u_1 \dots u_{i_k})|$ holds. Since $|\phi(u_1)| \le |\phi|_1 \le \ell \le |\tau(w_1 \dots w_k)|$, i_k is at least 2 and, thus, $|\phi(u_1 \dots u_{i_k-1})| \le |\tau(w_1 \dots w_k)|$ by minimality of i_k. Hence, there exists a decomposition $\phi(u_{i_k}) = s_k t_k$ such that t_k is non-empty and

$$t_k \phi(u_{i_k+1} \dots u_n) = \tau(w_{k+1} \dots w_m). \tag{16}$$

Our next objective is to use Lemma 3.4 to prove that s_k and u_k have a very particular form.

CLAIM 3.6.1. *For every $k \in \{1, \dots, m-1\}$, $s_k = 1$ and $u_1 = u_{i_k}$.*

Proof. To prove this, we suppose that it is not true, that is, there exists $k \in \{1, \dots, m-1\}$ such that

$$s_k \neq 1 \quad \text{or} \quad u_1 \neq u_{i_k}. \tag{17}$$

Let $\tilde{u} := u_{i_k} \dots u_{i_k+|\phi|_1^2-1}$ and $\tilde{v} := u_1 \dots u_{|\phi|_1^3}$. We are going to check the hypothesis of Lemma 3.4 for $\tilde{u}$, $\tilde{v}$ and ϕ.

First, we observe that, since $\phi(u) = \tau(v)$, we have that $\phi(\tilde{v})$ is a prefix of $\tau(v)$. Moreover, given that $|\phi(\tilde{v})| \le |\phi|_1^4 \le \ell$ and that τ is ℓ-proper, $\phi(\tilde{v})$ is a prefix of $\tau(b)$ for every $b \in \mathcal{B}$. In particular,

$$\phi(\tilde{v}) \text{ is a prefix of } \tau(w_k). \tag{18}$$

Second, from (16) and the inequalities $|t_k \phi(u_{i_k+1} \dots u_{i_k+|\phi|_1^2-1})| \le |\phi|_1^3 \le \ell \le |\tau(w_k)|$, we deduce that $t_k \phi(u_{i_k+1} \dots u_{i_k+|\phi|_1^2-1})$ is a prefix of $\tau(w_k)$. Therefore,

$$\phi(\tilde{u}) = s_k t_k \phi(u_{i_k+1} \dots u_{i_k+|\phi|_1^2-1}) \text{ is a prefix of } s_k \tau(w_k). \tag{19}$$

We conclude from (18), (19), and the inequality $|\phi(\tilde{u})| \le |\phi|_1^3 = |\tilde{v}| \le |s_k \phi(\tilde{v})|$ that

$$\phi(\tilde{u}) \text{ is a prefix of } s_k \phi(\tilde{v}).$$

This, the inequality $|\tilde{u}| \ge |\phi|_1 + |s_k|$ and (17) allow us to use Lemma 3.4 with $\tilde{u}$, $\tilde{v}$, and ϕ and obtain morphisms $\mathcal{A}^+ \xrightarrow{\tilde{q}} \tilde{\mathcal{A}}^+ \xrightarrow{\tilde{\phi}} \mathcal{C}^+$ such that $\#\tilde{\mathcal{A}} \le \#\mathcal{A}$, $\phi = \tilde{\phi}\tilde{q}$ and $|\tilde{\phi}|_1 < |\phi|_1$. Then, $\ell \ge |\phi|_1^4 > |\tilde{\phi}|_1^4$ and $\tilde{\phi}(\tilde{\mathcal{A}}^+) \cap \tau(\mathcal{B}^{++})$ contains the element $\tilde{\phi}(\tilde{q}(u)) = \tau(w)$, and so τ and $\tilde{\phi}$ satisfy the hypothesis of this lemma. Therefore, by the minimality of $|\phi|_1$, there exists a decomposition $\mathcal{B}^+ \xrightarrow{q} \mathcal{D}^+ \xrightarrow{p} \mathcal{C}^+$ of τ satisfying conditions (i)–(iii) of this lemma, contrary to our assumptions. □

An argument similar to the one used in the proof of the previous claim gives us that

$$u_n = u_{i_k-1} \quad \text{for every } k \in \{1, \dots, m-1\}. \tag{20}$$

We refer the reader to Remark 3.5 for further details.

Now we can finish the proof. First, from (16) and the first part of the claim, we get that $\tau(w_k) = \phi(u_{i_{k-1}} \dots u_{i_k-1})$ for $k \in \{2, \dots, m-1\}$, $\tau(w_1) = \phi(u_1 \dots u_{i_1-1})$ and $\tau(w_m) = \phi(u_{i_{m-1}} \dots u_n)$. Being $w \in \mathcal{B}^{++}$, these equations imply that each $\tau(b)$, $b \in \mathcal{B}$, can be written as a concatenation $x_1 \dots x_N$, with $x_j \in \phi(\mathcal{A})$. Moreover, by the second part

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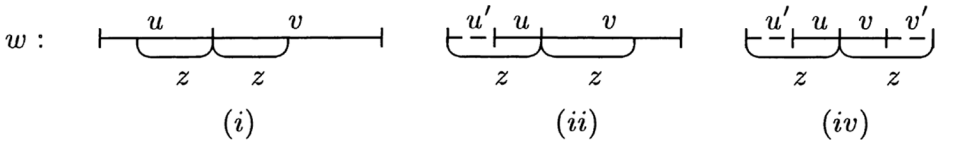

FIGURE 1. Illustration of a local period.

of the claim and (20), we can choose this decomposition so that $x_1 = u_1$ and $x_N = u_n$. This defines (maybe non-unique) morphisms $\mathcal{B}^+ \xrightarrow{q} \mathcal{D}_1^+ \xrightarrow{p_1} \mathcal{C}^+$ such that $\tau = p_1 q$, $\#\mathcal{D}_1 \leq \#\{\phi(u_1), \ldots, \phi(u_n)\} \leq \#\mathcal{A}$, and q is proper. If we define $\mathcal{D}$ as the set of letters $d \in \mathcal{D}_1$ that occur in some $w \in q(\mathcal{B})$, and p as the restriction of p_1 to $\mathcal{D}$, then we obtain a decomposition $\mathcal{B}^+ \xrightarrow{q} \mathcal{D}^+ \xrightarrow{p} \mathcal{C}^+$ that still satisfies the previous properties, but in which q is letter-onto. Hence, p and q met conditions (i), (ii), and (iii). □

3.2. *Periodicity lemmas.* We will also need classic results from combinatorics on words. We follow the presentation of [**RS97**, Ch. 6].

Let $w \in \mathcal{A}^*$ be a non-empty word. We say that p is a *local period* of w at the position $|u|$ if $w = uv$, with $u, v \neq 1$, and there exists a word z, with $|z| = p$, such that one of the following conditions holds for some words u' and v':

$$\begin{cases} \text{(i)} & u = u'z \text{ and } v = zv'; \\ \text{(ii)} & z = u'u \text{ and } v = zv'; \\ \text{(iii)} & u = u'z \text{ and } z = vv'; \\ \text{(iv)} & z = u'u = vv'. \end{cases} \tag{21}$$

Further, the *local period of w at the position $|u|$*, in symbols $\mathrm{per}(w, u)$, is defined as the smallest local period of w at the position u (see Fig. 1). It follows directly from (21) that $\mathrm{per}(w, u) \leq \mathrm{per}(w)$.

The following result is known as the critical factorization theorem.

THEOREM 3.7. (Theorem 6.2 and Ch. 6, [**RS97**]) *Each non-empty word $w \in \mathcal{A}^*$, with $|w| \geq 2$, possesses at least one factorization $w = uv$, with $u, v \neq 1$, which is critical, i.e.* $\mathrm{per}(w) = \mathrm{per}(w, u)$.

4. *Rank of symbolic factors*

In this section, we prove Theorem 1.3. We start by introducing the concept of *factor* between directive sequences and, in Proposition 4.4, its relation with factor maps between $\mathcal{S}$-adic subshifts. These ideas are the $\mathcal{S}$-adic analogs of the concept of *premorphism* between ordered Bratteli diagrams from [**AEG15**] and their proposition 4.6. Although Proposition 4.4 can be deduced from proposition 4.6 in [**AEG15**] by passing from directive sequences to ordered Bratteli diagrams and backwards, we consider this a little bit artificial since it is possible to provide a direct combinatorial proof; this is done in the Appendix. It is interesting to note that our proof is constructive (in contrast of the existential proof in

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[**AEG15**]) and shows some additional features that are a consequence of the combinatorics on words analysis made.

Next, we use ideas from [**Esp20**, **GH21**] to prove Theorem 1.3. In particular, this improves the previous bounds in [**Esp20**, **GH21**] to the best possible one. We apply these results, in Corollary 4.8, to answer affirmatively Question 1.2 and, in Theorem 1.5, to prove a strong coalescence property for the class of systems considered in Theorem 1.3. It is worth noting that this last result is only possible due to the bound in Theorem 1.3 being optimal. We end this section by proving that Cantor factors of finite topological rank systems are either subshifts or odometers.

4.1. *Rank of factors of directive sequences.* The following is the $\mathcal{S}$-adic analog of the notion of *premorphism* between ordered Bratteli diagrams in [**AEG15**].

Definition 4.1. Let $\boldsymbol{\sigma} = (\mathcal{A}_{n+1}^{+} \to \mathcal{A}_n^{+})_{n\in\mathbb{N}}$, $\boldsymbol{\tau} = (\mathcal{B}_{n+1}^{+} \to \mathcal{B}_n^{+})_{n\in\mathbb{N}}$ be directive sequences. A *factor* $\boldsymbol{\phi}\colon \boldsymbol{\sigma} \to \boldsymbol{\tau}$ is a sequence of morphisms $\boldsymbol{\phi} = (\phi_n)_{n\in\mathbb{N}}$, where $\phi_0\colon \mathcal{A}_1^{+} \to \mathcal{B}_0^{+}$ and $\phi_n\colon \mathcal{A}_n^{+} \to \mathcal{B}_n^{+}$ for $n \geq 1$, such that $\phi_0 = \tau_0\phi_1$ and $\phi_n\sigma_n = \tau_n\phi_{n+1}$ for every $n \geq 1$.

We say that $\boldsymbol{\phi}$ is *proper* (respectively *letter-onto*) if ϕ_n is proper (respectively *letter-onto*) for every $n \in \mathbb{N}$.

Remark 4.2. Factors are not affected by contractions. More precisely, if $0 = n_0 < n_1 < n_2 < \ldots$, then $\boldsymbol{\phi}' = (\phi_{n_k})_{k\in\mathbb{N}}$ is a factor from $\boldsymbol{\sigma}' = (\sigma_{[n_k,n_{k+1})})_{k\in\mathbb{N}}$ to $\boldsymbol{\tau}' = (\tau_{[n_k,n_{k+1})})_{k\in\mathbb{N}}$.

The next lemma will be needed at the end of this section.

LEMMA 4.3. *Let $\boldsymbol{\phi} = (\phi_n)_{n\geq 1}\colon \boldsymbol{\sigma} \to \boldsymbol{\tau}$ be a factor. Assume that $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are everywhere growing and proper and that $\boldsymbol{\phi}$ is letter-onto. Then, $X_{\boldsymbol{\tau}} = \bigcup_{k\in\mathbb{Z}} T^k\phi_0(X_{\boldsymbol{\sigma}}^{(1)})$ and $X_{\boldsymbol{\tau}}^{(n)} = \bigcup_{k\in\mathbb{Z}} T^k\phi_n(X_{\boldsymbol{\sigma}}^{(n)})$ for every $n \geq 1$.*

Proof. We start by proving that $X_{\boldsymbol{\tau}}^{(n)} \subseteq \bigcup_{k\in\mathbb{Z}} T^k\phi_n(X_{\boldsymbol{\sigma}}^{(n)})$. Let $y \in X_{\boldsymbol{\tau}}^{(n)}$ and $\ell \in \mathbb{N}$. There exist $N > n$ and $b \in \mathcal{B}_n$ such that $y_{[-\ell,\ell]}$ occurs in $\tau_{[n,N)}(b)$. In addition, since ϕ_N is letter-onto, there exists $a \in \mathcal{A}_N$ for which b occurs in $\phi_N(a)$. Then, $y_{[-\ell,\ell]}$ occurs in $\tau_{[n,N)}\phi_N(b)$ and, consequently, also in $\phi_n\sigma_{[n,N)}(b)$ as $\tau_{[n,N)}\phi_N = \phi_n\sigma_{[n,N)}$. Hence, by taking the limit $\ell \to \infty$, we can find $(k', x) \in \mathbb{Z} \times X_{\boldsymbol{\sigma}}^{(n)}$ such that $y = T^{k'}\phi_n(x)$. Therefore, $y \in \bigcup_{k\in\mathbb{Z}} T^k\phi_n(X_{\boldsymbol{\sigma}}^{(n)})$. To prove the other inclusion, we use Lemma 2.6 to compute:

$$\begin{aligned}\phi_n(X_{\boldsymbol{\sigma}}^{(n)}) &= \bigcap_{N>n} \bigcup_{k\in\mathbb{Z}} T^k\phi_n\sigma_{[n,N)}(\mathcal{A}_N^{\mathbb{Z}}) = \bigcap_{N>n} \bigcup_{k\in\mathbb{Z}} T^k\tau_{[n,N)}\phi_N(\mathcal{A}_N^{\mathbb{Z}}) \\ &\subseteq \bigcap_{N>n} \bigcup_{k\in\mathbb{Z}} T^k\tau_{[n,N)}(\mathcal{B}_N^{\mathbb{Z}}) = X_{\boldsymbol{\tau}}^{(n)}.\end{aligned}$$

□

As we mentioned before, the following proposition is a consequence of the main result in [**AEG15**]. We provide a combinatorial proof in the Appendix.

PROPOSITION 4.4. *Let $\boldsymbol{\sigma}$ be a letter-onto, everywhere growing, and proper directive sequence. Suppose that $X_{\boldsymbol{\sigma}}$ is aperiodic. Then, there exist a contraction $\boldsymbol{\sigma}' = (\sigma_n')_{n\in\mathbb{N}}$,*

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a letter-onto, everywhere growing, proper and recognizable $\boldsymbol{\tau} = (\tau_n)_{n\in\mathbb{N}}$ *generating* $X_{\boldsymbol{\sigma}}$*, and a letter-onto factor* $\boldsymbol{\phi}\colon \boldsymbol{\sigma}' \to \boldsymbol{\tau}$, $\boldsymbol{\phi} = (\phi_n)_{n\in\mathbb{N}}$*, such that* $\phi_0 = \sigma'_0$.

The next proposition is the main technical result of this section. To state it, it is convenient to introduce the following concept. The directive sequences $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are *equivalent* if $\boldsymbol{\sigma} = \boldsymbol{\nu}'$, $\boldsymbol{\tau} = \boldsymbol{\nu}''$ for some contractions $\boldsymbol{\nu}'$, $\boldsymbol{\nu}''$ of a directive sequence $\boldsymbol{\nu}$. Observe that equivalent directive sequences generate the same $\mathcal{S}$-adic subshift.

PROPOSITION 4.5. *Let* $\boldsymbol{\phi}\colon \boldsymbol{\sigma} \to \boldsymbol{\tau}$ *be a letter-onto factor between the everywhere growing and proper directive sequences. Then, there exist a letter-onto and proper factor* $\boldsymbol{\psi}\colon \boldsymbol{\sigma}' \to \boldsymbol{\nu}$*, where:*

(1) $\boldsymbol{\sigma}'$ *is a contraction of* $\boldsymbol{\sigma}$*;*
(2) $\boldsymbol{\nu}$ *is letter-onto, everywhere growing, proper, equivalent to* $\boldsymbol{\tau}$, $\mathrm{AR}(\boldsymbol{\nu}) \le \mathrm{AR}(\boldsymbol{\sigma})$*, and the first coordinate of* $\boldsymbol{\psi}$ *and* $\boldsymbol{\phi}$ *coincide;*
(3) *if* $\boldsymbol{\tau}$ *is recognizable, then* $\boldsymbol{\nu}$ *is recognizable.*

Proof. Let us write $\boldsymbol{\sigma} = (\mathcal{A}_{n+1}^{+} \to \mathcal{A}_n^{+})_{n\in\mathbb{N}}$ and $\boldsymbol{\tau} = (\mathcal{B}_{n+1}^{+} \to \mathcal{B}_n^{+})_{n\in\mathbb{N}}$. Up to contractions, we can suppose that for every $n \ge 1$, $\#\mathcal{A}_n = \mathrm{AR}(\boldsymbol{\sigma})$ and that τ_n is $|\phi_n|_1^4$-proper (for the last property, we used that $\boldsymbol{\tau}$ is everywhere growing and proper).

Using that ϕ_{n+1} is letter-onto, we can compute:

$$\tau_n(\mathcal{B}_{n+1}^{++}) \supseteq \tau_n(\phi_{n+1}(\mathcal{A}_{n+1}^{++})) = \phi_n(\sigma_n(\mathcal{A}_{n+1}^{++})) \subseteq \phi_n(\mathcal{A}_n^{+}),$$

where in the middle step, we used the commutativity property of $\boldsymbol{\phi}$. We deduce that

$$\tau_n(\mathcal{B}_{n+1}^{++}) \cap \phi_n(\mathcal{A}_n^{+}) \neq \emptyset \quad \text{for every } n \in \mathbb{N}.$$

This and the fact that τ_n is a $|\phi_n|_1^4$-proper morphism allow us to use Lemma 3.6 to find morphisms $\mathcal{B}_{n+1}^{+} \xrightarrow{q_{n+1}} \mathcal{D}_{n+1}^{+} \xrightarrow{p_n} \mathcal{B}_n^{+}$ such that

$$\text{(i) } \#\mathcal{D}_{n+1} \le \#\mathcal{A}_n, \quad \text{(ii) } \tau_n = p_n q_{n+1}, \quad \text{(iii) } q_{n+1} \text{ is letter-onto and proper.}$$

We define $\nu_0 := p_0$, the morphisms $\nu_n := q_n p_n\colon \mathcal{D}_{n+1}^{+} \to \mathcal{D}_n^{+}$ and $\psi_n := q_n \phi_n\colon \mathcal{A}_n^{+} \to \mathcal{D}_n^{+}$, $n \ge 1$, and the sequences $\boldsymbol{\nu} = (\nu_n)_{n\in\mathbb{N}}$ and $\boldsymbol{\psi} = (\psi_n)_{n\in\mathbb{N}}$, where $\psi_0 := \phi_0$. We are going to show that these objects satisfy the conclusion of the proposition.

We start by observing that it follows from the definitions that the diagram below commutes for all $n \ge 1$:

$$\begin{array}{ccccc}
\mathcal{A}_n^{+} & \xrightarrow{\phi_n} & \mathcal{B}_n^{+} & \xrightarrow{q_n} & \mathcal{D}_n^{+} \\
{\scriptstyle\sigma_n}\uparrow & & {\scriptstyle\tau_n}\uparrow & \nwarrow^{p_n} & \uparrow{\scriptstyle\nu_n} \\
\mathcal{A}_{n+1}^{+} & \xrightarrow[\phi_{n+1}]{} & \mathcal{B}_{n+1}^{+} & \xrightarrow[q_{n+1}]{} & \mathcal{D}_{n+1}^{+}
\end{array}$$

In particular, $\nu_n \nu_{n+1} = q_n \tau_n p_{n+1}$, so $\langle \nu_{[n,n+1]} \rangle \ge \langle \tau_n \rangle$. Being $\boldsymbol{\tau}$ everywhere growing, this implies that $\boldsymbol{\nu}$ has the same property. We also observe that condition (iii) implies that $\nu_n = q_n p_n$ is letter-onto and proper. Altogether, these arguments prove that, up to contracting the first levels, $\boldsymbol{\nu}$ is everywhere growing and proper.

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Next, we note that $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$ are equivalent as both are contractions of $(p_0, q_1, p_1, q_2, \ldots)$. This implies, by Lemma 2.3, that $\boldsymbol{\nu}$ is recognizable if $\boldsymbol{\tau}$ is recognizable. Further, by condition (i), $\boldsymbol{\nu}$ has alphabet rank at most $\mathrm{AR}(\boldsymbol{\sigma})$.

It is only left to prove that $\boldsymbol{\psi}$ is a letter-onto and proper factor. By unraveling the definitions we can compute:

$$\psi_0 = \phi_0 = \tau_0\phi_1 = p_0q_1\phi_1 = \nu_0\psi_1,$$

and from the diagram, we have $\sigma_n\psi_n = \psi_{n+1}\tau_n$ for all $n \geq 1$. Therefore, $\boldsymbol{\psi}$ is a factor. Finally, since q_n is letter-onto and proper by condition (iii) and $\boldsymbol{\phi}$ was assumed to be letter-onto, $\psi_n = q_n\phi_n$ is letter-onto and proper. □

4.2. *Rank of factors of $\mathcal{S}$-adic subshifts.* In this section, we will prove Theorem 1.3 and its consequences. We start with a technical lemma.

The next lemma will allow us to assume without loss of generality that our directive sequences are letter-onto.

LEMMA 4.6. *Let $\boldsymbol{\tau} = (\tau_n \colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\in\mathbb{N}}$ be an everywhere growing and proper directive sequence. If $\tilde{\mathcal{A}}_n = \mathcal{A}_n \cap \mathcal{L}(X_{\boldsymbol{\sigma}}^{(n)})$, $\tilde{\tau}_n$ is the restriction of τ_n to $\tilde{\mathcal{A}}_{n+1}$ and $\tilde{\boldsymbol{\tau}} = (\tilde{\tau}_0, \tilde{\tau}_1, \ldots)$, then $\tilde{\boldsymbol{\tau}}$ is letter-onto and $X_{\tilde{\boldsymbol{\tau}}}^{(n)} = X_{\boldsymbol{\tau}}^{(n)}$ for every $n \in \mathbb{N}$. Conversely, if $\boldsymbol{\tau}$ is letter-onto, then $\mathcal{A}_n \subseteq \mathcal{L}(X_{\boldsymbol{\tau}}^{(n)})$ for every $n \in \mathbb{N}$.*

Proof. By Lemma 2.5, $\tilde{\tau}_n$ is letter-onto mapping $\tilde{\mathcal{A}}_{n+1}^+$ into $\tilde{\mathcal{A}}_n$. Moreover, that lemma also gives that for every $x \in X_{\boldsymbol{\tau}}^{(n)}$ and $N > n$, there exists a $\tau_{[n,N)}$-factorization (k', x') of x in $X_{\boldsymbol{\tau}}^{(N)}$. This together with the inclusion $X_{\boldsymbol{\tau}}^{(N)} \subseteq \tilde{\mathcal{A}}_N^{\mathbb{Z}}$ imply that

$$Z := \bigcap_{N>n} \bigcup_{k\in\mathbb{Z}} T^k \tau_{[n,N)}(\tilde{\mathcal{A}}_N^{\mathbb{Z}}) \supseteq X_{\boldsymbol{\tau}}^{(n)}.$$

Now, $\tilde{\boldsymbol{\tau}}$ is everywhere growing and proper, so we can apply Lemma 2.6 to obtain that $X_{\tilde{\boldsymbol{\tau}}}^{(n)} = Z \supseteq X_{\boldsymbol{\tau}}^{(n)}$. Since it is clear that $X_{\tilde{\boldsymbol{\tau}}}^{(n)} \subseteq X_{\boldsymbol{\tau}}^{(n)}$ as $\tilde{\mathcal{A}}_N \subseteq \mathcal{A}_N$ for every N, we conclude that $X_{\tilde{\boldsymbol{\tau}}}^{(n)} = X_{\boldsymbol{\tau}}^{(n)}$.

If $\boldsymbol{\tau}$ is letter-onto, then $\mathcal{A}_n \subseteq \mathcal{L}(\bigcup_{k\in\mathbb{Z}} T^k\tau_{[n,N)}(\mathcal{A}_N^{\mathbb{Z}}))$ for every $N > n$, and hence, by the formula in Lemma 2.6, $\mathcal{A}_n \subseteq \mathcal{L}(X_{\boldsymbol{\tau}}^{(n)})$. □

Now we are ready to prove Theorem 1.3. We re-state it in a more precise way.

THEOREM 1.3. *Let $\pi \colon (X, T) \to (Y, T)$ be a factor map between aperiodic subshifts. Suppose that X is generated by the everywhere growing and proper directive sequence $\boldsymbol{\sigma} = (\sigma_n \colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\in\mathbb{N}}$ of alphabet rank K. Then, Y is generated by a letter-onto, everywhere growing, proper, and recognizable directive sequence $\boldsymbol{\tau}$ of alphabet rank at most K.*

Moreover, if $\boldsymbol{\sigma}$ is letter-onto, then, up to contracting the sequences, there exists a proper factor $\boldsymbol{\phi} \colon \boldsymbol{\sigma} \to \boldsymbol{\tau}$ such that $\pi(\sigma_0(x)) = \phi_0(x)$ for all $x \in X_{\boldsymbol{\sigma}}^{(1)}$ and $|\sigma_0(a)| = |\phi_0(a)|$ for all $a \in \mathcal{A}_1$.

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Proof. Thanks to Lemma 4.6, we can assume without loss of generality that $\boldsymbol{\sigma}$ is letter-onto. Moreover, in this case,

$$\mathcal{A}_n \subseteq \mathcal{L}(X_{\boldsymbol{\sigma}}^{(n)}) \quad \text{for every } n \in \mathbb{N}. \tag{22}$$

Let us write $\boldsymbol{\sigma} = (\sigma_n \colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\in\mathbb{N}}$. By contracting $\boldsymbol{\sigma}$, we can further assume that σ_0 is r-proper and π has radius r. Then, Lemma 2.4 gives us a proper morphism $\tau \colon \mathcal{A}_1^+ \to \mathcal{B}^+$, where $\mathcal{B}$ is the alphabet of Y, such that

$$\pi(\sigma_0(x)) = \tau(x) \text{ for all } x \in X_{\boldsymbol{\sigma}}^{(1)} \quad \text{and} \quad |\sigma_0(a)| = |\tau(a)| \text{ for every } a \in \mathcal{A}_1. \tag{23}$$

In particular, $\pi(\sigma_{[0,n)}(x)) = \tau\sigma_{[1,n)}(x)$ and $|\sigma_{[0,n)}(a)| = |\tau\sigma_{[1,n)}(a)|$ for all $n \in \mathbb{N}$, $x \in X_{\boldsymbol{\sigma}}^{(n)}$, and $a \in \mathcal{A}_n$, so (23) holds for any contraction of $\boldsymbol{\sigma}$.

We define $\tilde{\boldsymbol{\sigma}} = (\tau, \sigma_1, \sigma_2, \ldots)$ and observe this is a letter-onto, everywhere growing and proper sequence generating Y. This and that Y is aperiodic allow us to use Proposition 4.4 and obtain, after a contraction, a letter-onto factor $\tilde{\boldsymbol{\phi}} \colon \tilde{\boldsymbol{\sigma}} \to \tilde{\boldsymbol{\tau}}$, where $\tilde{\phi}_0 = \tilde{\sigma}_0 = \tau$ and $\tilde{\boldsymbol{\tau}}$ is a letter-onto, everywhere growing, proper, and recognizable directive sequence generating Y. The sequence $\tilde{\boldsymbol{\tau}}$ has all the properties required by the theorem but having alphabet rank bounded by K. To overcome this, we use Proposition 4.5 with $\tilde{\boldsymbol{\phi}}$ and do more contractions to obtain a letter-onto and proper factor $\boldsymbol{\phi} \colon \tilde{\boldsymbol{\sigma}} \to \boldsymbol{\tau}$ such that $\phi_0 = \tilde{\phi}_0 = \tau$ and $\boldsymbol{\tau}$ is a letter-onto, everywhere growing, proper, and recognizable directive sequence generating Y and satisfying $\mathrm{AR}(\boldsymbol{\tau}) \le \mathrm{AR}(\tilde{\boldsymbol{\sigma}}) = \mathrm{AR}(\boldsymbol{\sigma})$.

It is left to prove the last part of the theorem. Observe that since $\tilde{\boldsymbol{\sigma}}$ and $\boldsymbol{\sigma}$ differ only at their first coordinate, $\boldsymbol{\phi}$ is also a factor from $\boldsymbol{\sigma}$ to $\boldsymbol{\tau}$. Further, by (23) and the fact that $\phi_0 = \tau$, we have $\pi(\sigma_0(x)) = \tau(x) = \phi_0(x)$ and $|\sigma_0(a)| = |\phi_0(a)|$ for every $x \in X_{\boldsymbol{\sigma}}^{(1)}$ and $a \in \mathcal{A}_1$. □

COROLLARY 4.7. *Let (X, T) be an aperiodic minimal subshift generated by an everywhere growing and proper directive sequence of alphabet rank K. Then, the topological rank of X is at most K.*

Proof. We can use Theorem 1.3 to obtain an everywhere growing, proper, and recognizable directive sequence $\boldsymbol{\tau} = (\tau_n \colon \mathcal{B}_{n+1}^+ \to \mathcal{B}_n^+)_{n\in\mathbb{N}}$ generating X and having alphabet rank at most K. Due to Lemma 4.6, we can assume that $\boldsymbol{\tau}$ is letter-onto. In particular, $\mathcal{B}_n \subseteq \mathcal{L}(X_{\boldsymbol{\tau}}^{(n)})$ for every $n \in \mathbb{N}$.

We claim that $X_{\boldsymbol{\tau}}^{(n)}$ is minimal. Indeed, if $Y \subseteq X_{\boldsymbol{\tau}}^{(n)}$ is a subshift, then $\tau_{[0,n)}(Y)$ is closed (as $\tau_{[0,n)} \colon X_{\boldsymbol{\tau}}^{(n)} \to X_{\boldsymbol{\tau}}$ is continuous), so $\bigcup_{k\in\mathbb{Z}} T^k\tau_{[0,n)}(Y) = \bigcup_{|k|\le|\tau_{[0,n)}|} T^k\tau_{[0,n)}(Y)$ is a subshift in $X_{\boldsymbol{\tau}}$ which, by minimality, is equal to it. Thus, any point $x \in X_{\boldsymbol{\tau}}^{(n)}$ has a $\tau_{[0,n)}$-factorization (k, z) with $z \in Y$. The recognizability property of $(X_{\boldsymbol{\tau}}^{(n)}, \tau_{[0,n)})$ then implies that $Y = X_{\boldsymbol{\tau}}^{(n)}$.

Now, we prove that for any $n \in \mathbb{N}$, there exists $N > n$ such that $\tau_{[n,N)}$ is positive. This would imply that the topological rank of X is at most K and hence would complete the proof. Let $n \in \mathbb{N}$ and R be a constant of recognizability for $(X_{\boldsymbol{\tau}}^{(n)}, \tau_{[0,n)})$. Since $X_{\boldsymbol{\tau}}^{(n)}$ is minimal, there exists a constant $L \ge 1$ such that two consecutive occurrences of a word $w \in \mathcal{L}(X_{\boldsymbol{\tau}}^{(n)}) \cap \mathcal{B}_n^{2R+1}$ in a point $x \in X_{\boldsymbol{\tau}}^{(n)}$ are separated by at most L. Let $N > n$ be big enough so that $\langle\tau_{[0,N)}\rangle \ge L + 2R$. Then, for all $a \in \mathcal{B}_N \subseteq \mathcal{L}(X_{\boldsymbol{\tau}}^{(N)})$ and $w \in \mathcal{L}(X_{\boldsymbol{\tau}}^{(n)}) \cap$

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$\mathcal{B}_n^{2R+1}$, w occurs at a position $i \in \{R, R+1, \ldots, |\tau_{[0,N)}(a)| - R\}$ of $\tau_{[0,N)}(a)$. Since R is a recognizability constant for $(X_{\boldsymbol{\tau}}^{(n)}, \tau_{[0,n)})$, we deduce that for all $a \in \mathcal{B}_N$ and $b \in \mathcal{B}_n$, b occurs in $\tau_{[n,N)}(a)$. Thus, $\tau_{[n,N)}$ is positive. □

We can now prove Corollary 1.4.

COROLLARY 1.4. *Let (X, T) be an aperiodic minimal subshift generated by an everywhere growing directive sequence of finite alphabet rank. Then, the topological rank of (X, T) is finite.*

Proof. We are going to prove that X is generated by an everywhere growing and proper directive sequence $\boldsymbol{\tau}$ of finite alphabet rank. This would imply, by Corollary 4.7, that the topological rank of X is finite. Let $\boldsymbol{\sigma} = (\sigma_n : \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\in\mathbb{N}}$ be an everywhere growing directive sequence of finite alphabet rank generating X. We contract $\boldsymbol{\tau}$ in a way such that $\#\mathcal{A}_n \le K$ for every $n \ge 1$.

We are going to inductively define subshifts $X_n, n \in \mathbb{N}$. We start with $X_0 := X$. We now assume that X_n is defined for some $n \in \mathbb{N}$. Then the set $X'_{n+1} = \{x \in X_{\boldsymbol{\sigma}}^{(n+1)} : \sigma_n(x) \in X_n\}$ is a subshift. We define X_{n+1} as any minimal subshift contained in X'_{n+1}. It follows from the definition of X_{n+1} that $\bigcup_{k\in\mathbb{Z}} T^k\sigma_n(X_{n+1}) \subseteq X_n$. Being X_n minimal,

$$\bigcup_{k\in\mathbb{Z}} T^k\sigma_n(X_{n+1}) = X_n. \tag{24}$$

Let $\tilde{\mathcal{A}}_n = \mathcal{A}_n \cap \mathcal{L}(X_n)$. Equation (24) and the fact that $\boldsymbol{\sigma}$ is everywhere growing allow us to assume without loss of generality that, after a contraction of $\boldsymbol{\sigma}$, the following holds for every $n \in \mathbb{N}$:

$$\text{if } a \in \tilde{\mathcal{A}}_{n+1} \text{ and } w \in \mathcal{L}(X_n) \text{ has length 3, then } w \text{ occurs twice in } \sigma_n(a). \tag{25}$$

Let us fix a word $w_n = a_n b_n c_b \in \mathcal{L}(X_n)$ of length 3. Then, by (25), we can decompose $\sigma_n(a) = u_n(a)v_n(a)$ in a way such that

$$u_n(a) \text{ ends with } a_n,\ v_n(a) \text{ starts with } b_n c_n, \text{ and } |v_n(a)| \ge 2. \tag{26}$$

To define $\boldsymbol{\tau}$, we need to introduce additional notation first. Let $\mathcal{B}_n$ be the alphabet consisting of tuples $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ such that $ab \in \mathcal{L}(X_n)$. Also, if $w = w_1 \ldots w_{|w|} \in \mathcal{L}(X_n)$ has length $|w| \ge 2$, then $\chi_n(w) := \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]\left[\begin{smallmatrix} w_2 \\ w_3 \end{smallmatrix}\right] \ldots \left[\begin{smallmatrix} w_{|w|-1} \\ w_{|w|} \end{smallmatrix}\right] \in \mathcal{B}_n^+$, and if $w' = \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] \ldots \left[\begin{smallmatrix} w_{|w|-1} \\ w_{|w|} \end{smallmatrix}\right] \in \mathcal{B}_0^+$, then $\eta(w') := w_1 \ldots w_{|w|-1} \in \mathcal{A}_0^+$. Observe that $\eta : \mathcal{B}_0^+ \to \mathcal{A}_0^+$ is a morphism.

We now define $\boldsymbol{\tau}$. Let $\tau_n : \mathcal{B}_{n+1}^+ \to \mathcal{B}_n^+$ be the unique morphism such that $\tau_n(\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]) = \chi_n(v_n(a)u_n(a)b_n)$ for every $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \mathcal{B}_{n+1}$. Observe that since $v_n(a)u_n(a)b_n \in \mathcal{L}(X_n)$, it is indeed the case that $\tau_n(\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]) \in \mathcal{B}_n^+$. We set $\boldsymbol{\tau} = (\eta\tau_0, \tau_1, \tau_2, \ldots)$.

It follows from (26) that for every $n \in \mathbb{N}$ and $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \mathcal{B}_{n+1}$, $\tau_n(\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right])$ starts with $\left[\begin{smallmatrix} b_n \\ c_n \end{smallmatrix}\right]$ and ends with $\left[\begin{smallmatrix} a_n \\ b_n \end{smallmatrix}\right]$. Thus, $\boldsymbol{\tau}$ is proper. Moreover, since $|v_n(a)| \ge 2$, we have $|v_n(a)u_n(a)b_n| \ge 3$ and thus $|\tau_n(\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right])| \ge 2$. Therefore, $\langle\tau_n\rangle \ge 2$ and $\boldsymbol{\tau}$ is everywhere growing. Also, $\#\mathcal{B}_n \le \#\mathcal{A}_n^2 \le K^2$ for every $n \in \mathbb{N}$, so the alphabet rank of $\boldsymbol{\tau}$ is finite.

It remains to prove that $X = X_{\boldsymbol{\tau}}$. By minimality, it is enough to prove that $X \supseteq X_{\boldsymbol{\tau}}$. Observe that since $\tau_n\chi_{n+1}(ab) = \chi_n(v_n(a)u_n(b)b_n)$, the word $\tau_n\chi_{n+1}(ab)$ occurs

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in $\chi_n\sigma_n(ab)$. Moreover, for every $w = w_1 \dots w_{|w|} \in \mathcal{L}(X_{\boldsymbol{\sigma}}^{(n)})$, $\tau_n\chi_{n+1}(w)$ occurs in $\chi_n\sigma_n(w)$. Then, by using the symbol $\sqsubseteq$ to denote the 'subword' relation, we can write for every $n \in \mathbb{N}$ and $ab \in \mathcal{L}(X_{\boldsymbol{\sigma}}^{(n)})$:

$$\begin{aligned}\tau_{[0,n)}\chi_n(ab) &\sqsubseteq \tau_{[0,n-1)}\chi_{n-1}\sigma_{n-1}(ab)\\ &\sqsubseteq \tau_{[0,n-2)}\chi_{n-2}\sigma_{[n-2,n)}(ab) \sqsubseteq \cdots \sqsubseteq \chi_0\sigma_{[0,n)}(ab).\end{aligned}$$

Hence, $\eta\tau_{[0,n)}\left(\begin{bmatrix}a\\b\end{bmatrix}\right) \sqsubseteq \eta\chi_0\sigma_{[0,n)}(ab) \sqsubseteq \sigma_{[0,n)}(ab)$. We conclude that $X_{\boldsymbol{\tau}} \subseteq X_{\boldsymbol{\sigma}} = X$. □

COROLLARY 4.8. *Let (X, T) be a minimal subshift of topological rank K and $\pi\colon (X, T) \to (Y, T)$ a factor map, where Y is an aperiodic subshift. Then, the topological rank of Y is at most K.*

Proof. By Theorem 1.1, (X, T) is generated by a proper and primitive directive sequence $\boldsymbol{\sigma}$ of alphabet rank equal to K. In particular, $\boldsymbol{\sigma}$ is everywhere growing and proper, so we can use Theorem 1.3 to obtain an everywhere growing, proper, and recognizable directive sequence $\boldsymbol{\tau} = (\tau_n\colon \mathcal{B}_{n+1}^+ \to \mathcal{B}_n^+)_{n\geq 0}$ generating (Y, T) and having alphabet rank at most K. Then, the hypothesis of Corollary 4.7 holds for (Y, T), and thus the topological rank of (Y, T) is at most K. □

The following notion will be used in the proof of the theorem below: $\boldsymbol{\sigma} = (\sigma_n\colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n)_{n\geq 0}$ has *exact alphabet rank at most K* if $\#\mathcal{A}_n \leq K$ for all $n \geq 1$.

COROLLARY 1.5. *Let (X, T) be an $\mathcal{S}$-adic subshift generated by an everywhere growing and proper sequence of alphabet rank K, and $\pi_j\colon (X_{j+1}, T) \to (X_j, T)$, $j = 0, \dots, L$ be a chain of aperiodic symbolic factors, with $X_L = X$. Suppose that $L > \log_2(K)$. Then π_j is a conjugacy for some j.*

Proof. We start by using Theorem 1.3 with the identity function $\mathrm{id}\colon (X, T) \to (X, T)$ to obtain a letter-onto, everywhere growing, proper, and recognizable directive sequence $\boldsymbol{\sigma_L}$ of alphabet rank at most K generating X. By doing a contraction, we can assume that $\boldsymbol{\sigma_L}$ has exact alphabet rank at most K.

By Theorem 1.3 applied to π_{L-1} and $\boldsymbol{\sigma_L}$, there exists, after a contraction of $\boldsymbol{\sigma_L}$, a letter-onto factor $\boldsymbol{\phi_{L-1}}\colon \boldsymbol{\sigma_L} \to \boldsymbol{\sigma_{L-1}}$, where $\boldsymbol{\sigma_{L-1}}$ is letter-onto, everywhere growing, proper, recognizable, has alphabet rank at most K, generates X_{L-1}, and, if $\phi_{L-1,0}$ and $\sigma_{L,0}$ are the first coordinates of $\boldsymbol{\phi_{L-1}}$ and $\boldsymbol{\sigma_L}$, respectively, then $\pi_{L-1}(\sigma_{L,0}(x)) = \phi_{L-1,0}(x)$ for every $x \in X_{\boldsymbol{\sigma_L}}^{(1)}$ and $|\sigma_{L,0}(a)| = |\phi_{L-1,0}(a)|$ for every letter a in the domain of $\sigma_{L,0}$. By contracting these sequences, we can also suppose that $\boldsymbol{\sigma_{L-1}}$ has exact alphabet rank at most K. The same procedure applies to π_{L-2} and $\boldsymbol{\sigma_{L-1}}$. Thus, by continuing in this way, we obtain for every $j = 0, \dots, L-1$ a letter-onto factor $\boldsymbol{\phi_j}\colon \boldsymbol{\sigma_{j+1}} \to \boldsymbol{\sigma_j}$ such that the following holds.

- $\boldsymbol{\sigma_j}$ is letter-onto, everywhere growing, proper, recognizable, has exact alphabet rank at most K, generates X_j, $\pi_j(\sigma_{j+1,0}(x)) = \phi_{j,0}(x)$ for every $x \in X_{\boldsymbol{\sigma_{j+1}}}^{(1)}$, and $|\sigma_{j+1,0}(a)| = |\phi_{j,0}(a)|$ for every $a \in \mathcal{A}_{j+1,1}$.

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Here, we are using the notation $\boldsymbol{\sigma_j} = (\sigma_{j,n}\colon \mathcal{A}^+_{j,n+1} \to \mathcal{A}^+_{j,n})_{n\in\mathbb{N}}$, $\boldsymbol{\phi_j} = (\phi_{j,n}\colon \mathcal{A}^+_{j+1,n} \to \mathcal{A}^+_{j,n})_{n\in\mathbb{N}}$ and $X^{(n)}_j = X^{(n)}_{\boldsymbol{\sigma_j}}$. We note that:

($\triangle_1$) for every $x \in X^{(1)}_{j+1}$, $\pi_j(\sigma_{j+1,0}(x)) = \phi_{j,0}(x) = \sigma_{j,0}\phi_{j,1}(x)$ since $\phi_{j,0} = \sigma_{j,0}\phi_{j,1}$;

($\triangle_2$) $X^{(1)}_j = \bigcup_{k\in\mathbb{Z}} T^k\phi_{j,1}(X^{(1)}_{j+1})$ by Lemma 4.3.

Hence, the following diagram commutes:

$$\begin{array}{ccccccccc}
X^{(1)}_0 & \xleftarrow{\phi_{0,1}} & \cdots & X^{(1)}_j & \xleftarrow{\phi_{j,1}} & X^{(1)}_{j+1} & \cdots & \xleftarrow{\phi_{L-1,1}} & X^{(1)}_L \\
\downarrow{\scriptstyle \sigma_{0,0}} & & & \downarrow{\scriptstyle \sigma_{j,0}} & & \downarrow{\scriptstyle \sigma_{j+1,0}} & & & \downarrow{\scriptstyle \sigma_{L,0}} \\
X^{(0)}_0 & \xleftarrow[\pi_0]{} & \cdots & X^{(0)}_j & \xleftarrow[\pi_j]{} & X^{(0)}_{j+1} & \cdots & \xleftarrow[\pi_{L-1}]{} & X^{(0)}_L
\end{array}$$

CLAIM 4.8.1. *If $(X^{(1)}_{j+1}, \phi_{j,1})$ is recognizable, then π_j is a conjugacy.*

Proof. Let us assume that $(X^{(1)}_{j+1}, \phi_{j,1})$ is recognizable, and let, for $i = 0, 1$, $x^i \in X^{(1)}_{j+1}$ such that $y = \pi_j(x^0) = \pi_j(x^1)$. We have to show that $x^0 = x^1$. First, we use Lemma 2.5 to find a centered $\sigma_{j+1,0}$-factorization (k^i, z^i) of x^i in $X^{(1)}_{j+1}$. Then, equation $\triangle_1$ allows us to compute:

$$T^{k^0}\sigma_{j,0}\phi_{j,1}(z^0) = T^{k^0}\pi_j(\sigma_{j+1,0}(z^0)) = \pi_j(x^0) = \pi_j(x^1) = T^{k^1}\sigma_{j,0}\phi_{j,1}(z^1).$$

This implies that (k^i, z^i) is a $\sigma_{j,0}\phi_{j,1}$-factorization of y in $X^{(1)}_{j+1}$ for $i = 0, 1$. Moreover, these are centered factorizations as, by $\bullet$, $|\sigma_{j,0}\phi_{j,1}(a)| = |\sigma_{j+1,0}(a)|$ for all $a \in \mathcal{A}_{j+1,1}$. Now, being $(X^{(1)}_j, \sigma_{0,j})$ and $(X^{(1)}_{j+1}, \phi_{j,1})$ recognizable, Lemma 2.3 gives that $(X^{(1)}_{j+1}, \sigma_{j,1}\phi_{j,1})$ is recognizable, and thus we have that $(k^0, z^0) = (k^1, z^1)$. Therefore, $x^0 = x^1$ and π is a conjugacy. □

Now we can finish the proof. We assume, by contradiction, that π_j is not a conjugacy for all j. Then, by the claim,

$$(X^{(1)}_j, \phi_{1,j}) \text{ is not recognizable for every } j \in \{0, \ldots, L-1\}. \tag{27}$$

Let

$$\boldsymbol{\nu} = (\phi_{0,1}, \phi_{1,1}, \phi_{2,1}, \ldots, \phi_{L-1,1}, \sigma_{L,1}, \sigma_{L,2}, \sigma_{L,3}, \ldots).$$

The idea is to use Theorem 2.7 with $\boldsymbol{\nu}$ to obtain a contradiction. To do so, we first note that, since $\boldsymbol{\nu}$ and $\boldsymbol{\sigma}^{(L)}$ have the same 'tail', $X^{(m+L)}_{\boldsymbol{\nu}} = X^{(m+1)}_L$ for all $m \in \mathbb{N}$. Moreover, $\triangle_2$ and the previous relation imply that

$$\begin{aligned}
X^{(j)}_{\boldsymbol{\nu}} &= \bigcup_{k\in\mathbb{Z}} T^k\phi_{j,1}(X^{(j+1)}_{\boldsymbol{\nu}}) = \cdots = \bigcup_{k\in\mathbb{Z}} T^k\phi_{j,1}\ldots\phi_{L-1,1}(X^{(L)}_{\boldsymbol{\nu}}) \\
&= \bigcup_{k\in\mathbb{Z}} T^k\phi_{j,1}\ldots\phi_{L-1,1}(X^{(1)}_L) = \bigcup_{k\in\mathbb{Z}} T^k\phi_{j,1}\ldots\phi_{L-2,1}(X^{(1)}_{L-1}) = \cdots = X^{(1)}_j.
\end{aligned}$$

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This and (27) imply that for every $j \in \{1, \dots, L-1\}$, the level $(X_{\boldsymbol{\nu}}^{(j)}, \phi_{j,1})$ of $\boldsymbol{\nu}$ is not recognizable. Being $\boldsymbol{\nu}$ everywhere growing as $\boldsymbol{\sigma_L}$ has this property, we conclude that Theorem 2.7 can be applied and, therefore, that $X_0^{(1)} = X_{\boldsymbol{\nu}}$ is periodic. However, then $X_0 = \bigcup_{k\in\mathbb{Z}} T^k \sigma_{0,0}(X_0^{(1)})$ is periodic, contrary to our assumptions.

A system (X, T) is *coalescent* if every endomorphism $\pi : (X, T) \to (X, T)$ is an automorphism. This notion has been relevant in the context of topological dynamics; see for example [**Dow97**].

COROLLARY 4.9. *Let (X, T) be an $\mathcal{S}$-adic subshift generated by an everywhere growing and proper directive sequence of finite alphabet rank. Then, (X, T) is coalescent.*

Remark 4.10. A linearly recurrent subshift of constant C is generated by a primitive and proper directive sequence of alphabet rank at most $C(C+1)^2$ [**Dur00**, Proposition 6]. In [**DHS99**], the authors proved the following.

THEOREM 4.11. [**DHS99**, Theorem 3] *For a linearly recurrent subshift X of constant C, in any chain of factors $\pi_j : (X_j, T) \to (X_{j+1}, T)$, $j = 0, \dots, L$, with $X_0 = X$ and $L \geq (2C(2C+1)^2)^{4C^3(2C+1)^2}$, there is at least one π_j which is a conjugacy.*

Thus, Theorem 1.5 is not only a generalization of this result to a much larger class of systems, but also improves the previous super-exponential constant to a logarithmic one.

In Proposition 28 of [**DHS99**], the authors proved that Cantor factors of linearly recurrent systems are either subshifts or odometers. Their proof only uses that this kind of system satisfies the strong coalescence property that we proved in Corollary 4.9 for finite topological rank systems. Therefore, by the same proof, we have the following.

COROLLARY 4.12. *Let $\pi : (X, T) \to (Y, T)$ be a factor map between minimal systems. Assume that (X, T) has finite topological rank and that (Y, T) is a Cantor system. Then, (Y, T) is either a subshift or a odometer.*

Proof. We sketch the proof from [**DHS99**] that we mentioned above.

Let $(\mathcal{P}_n)_{n\in\mathbb{N}}$ be a sequence of clopen partitions of Y such that $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$ and their union generates the topology of Y. Also, let Y_n be the subshift obtained by codifying the orbits of (Y, T) by using the atoms of $\mathcal{P}_n$. Then, the fact that $\mathcal{P}_n$ is a clopen partition induces a factor map $\pi_n : (Y, T) \to (Y_n, T)$. Moreover, since $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$, there exists a factor map $\xi_n : (Y_{n+1}, T) \to (Y_n, T)$ such that $\xi_n \pi_{n+1} = \pi_n$. Hence, we have the following chain of factors:

$$(X, T) \xrightarrow{\pi} (Y, T) \xrightarrow{\pi_n} (Y_n, T) \xrightarrow{\xi_{n-1}} (Y_{n-1}, T) \xrightarrow{\xi_{n-2}} \cdots \xrightarrow{\xi_1} (Y_0, T).$$

We conclude, by also using the fact that the partitions $\mathcal{P}_n$ generate the topology of Y, that (Y, T) is conjugate to the inverse limit $\varprojlim_{n\to\infty}(Y_n; \xi_n)$.

Now we consider two cases. If Y_n is periodic for every $n \in \mathbb{N}$, then Y is the inverse limit of periodic system, and hence an odometer. In the other case, we have, by Corollary 1.5,

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that ξ_n is a conjugacy for all big enough $n \in \mathbb{N}$, and thus that (Y, T) is conjugate to one of the subshifts Y_n. □

5. *Fibers of symbolic factors*

The objective of this section is to prove Theorem 1.6, which states that factor maps $\pi : (X, T) \to (Y, T)$ between $\mathcal{S}$-adic subshifts of finite topological rank are always *almost* k-to-1 for some k bounded by the topological rank of X. We start with some lemmas from topological dynamics.

LEMMA 5.1. [**Aus88**] *Let $\pi : X \to Y$ be a continuous map between compact metric spaces. Then $\pi^{-1} : Y \to 2^X$ is continuous at every point of a residual subset of Y.*

The next lemma gives a sufficient condition for a factor map π to be almost k-to-1. Recall that $E(X, T)$ stands for the Ellis semigroup of (X, T).

LEMMA 5.2. *Let $\pi : (X, T) \to (Y, T)$ be a factor map between topological dynamical systems, with (Y, T) minimal and $K \geq 1$ an integer. Suppose that for every $y \in Y$, there exists $u \in E(2^X, T)$ such that $\#u \circ \pi^{-1}(y) \leq K$. Then, π is almost k-to-1 for some $k \leq K$.*

Proof. First, we observe that by the description of $u \circ A$ in terms of nets at the end of §2.1, we have

$$\#u \circ A \leq \#A, \quad \text{for all } u \in E(2^X, T),\ A \in 2^X. \tag{28}$$

Now, by the previous lemma, there exists a residual set $\tilde{Y} \subseteq Y$ of continuity points for π^{-1}. Let $y, y' \in \tilde{Y}$ be arbitrary. Since Y is minimal, there exists a sequence $(n_\ell)_\ell$ such that $\lim_\ell T^{n_\ell} y = y'$. If $w \in E(2^X, T)$ is the limit of a convergent subnet of $(T^{n_\ell})_\ell$, then $wy = y'$. By the continuity of π^{-1} at y' and (28), we have

$$\#\pi^{-1}(y') = \#\pi^{-1}(wy) = \#w \circ \pi^{-1}(y) \leq \#\pi^{-1}(y).$$

We deduce, by symmetry, that $\#\pi^{-1}(y') = \#\pi^{-1}(y)$. Hence, $k := \pi^{-1}(y)$ does not depend on the chosen $y \in \tilde{Y}$. To end the proof, we have to show that $k \leq K$. We fix $y \in \tilde{Y}$ and take, using the hypothesis, $u \in E(2^X, T)$ such that $\#u \circ \pi^{-1}(y) \leq K$. As above, by minimality, there exists $v \in E(2^X, T)$ such that $vuy = y$. Then, by the continuity of π^{-1} at y,

$$\pi^{-1}(y) = \pi^{-1}(vuy) = (vu) \circ \pi^{-1}(y) = v \circ (u \circ \pi^{-1}(y)).$$

This and (28) imply that $k = \#\pi^{-1}(y) \leq \#u \circ \pi^{-1}(y) \leq K$. □

Let $\sigma : \mathcal{A}^+ \to \mathcal{B}^+$ be a morphism, (k, x) a centered σ-factorization of $y \in \mathcal{B}^{\mathbb{Z}}$ in $\mathcal{A}^{\mathbb{Z}}$, and $\ell \in \mathbb{Z}$. Note that there exists a unique $j \in \mathbb{Z}$ such that $\ell \in [c_{\sigma,j}(k, x), c_{\sigma,j+1}(k, x))$ (recall the notion of *cut* from Definition 2.1). In this context, we say that $(c_{\sigma,j}(k, x), x_j)$ is *the symbol of (k, x) covering position ℓ of y*.

THEOREM 1.6. *Let $\pi : (X, T) \to (Y, T)$ be a factor between subshifts, with (Y, T) minimal and aperiodic. Suppose that X is generated by a proper and everywhere growing directive sequence $\boldsymbol{\sigma}$ of alphabet rank K. Then, π is almost k-to-1 for some $k \leq K$.*

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Proof. Let $\boldsymbol{\sigma} = (\sigma_n \colon \mathcal{A}_{n+1} \to \mathcal{A}_n)_{n\geq 0}$ be a proper and everywhere growing directive sequence of alphabet rank at most K generating X. Due the possibility of contracting $\boldsymbol{\sigma}$, we can assume without loss of generality that $\#\mathcal{A}_n \leq K$ for every $n \geq 1$ and that σ_0 is r-proper, where r is the radius of π. Then, by Lemma 2.4, Y is generated by an everywhere growing directive sequence of the form $\boldsymbol{\tau} = (\tau, \sigma_1, \sigma_2, \ldots)$, where $\tau \colon \mathcal{A}_1^+ \to \mathcal{B}^+$ is such that $\tau(x) = \pi(\sigma_0(x))$ for every $x \in X_{\boldsymbol{\tau}}^{(1)} = X_{\boldsymbol{\sigma}}^{(1)}$. We will use the notation $\tau_{[0,n)} = \tau\sigma_{[1,n)}$. Further, for $y \in Y$ and $n \geq 1$, we write $F_n(y)$ to denote the set of $\tau_{[0,n)}$-factorizations of y in $Y_{\boldsymbol{\tau}}^{(n)}$.

Before continuing, we prove the following claim.

CLAIM 5.2.1. *There exist $\ell_n \in \mathbb{Z}$ and $G_n \subseteq \mathbb{Z} \times \mathcal{B}_{n+1}$ with at most K elements such that if $(k, x) \in F_n(y)$, then the symbol of (k, x) covering position ℓ_n of y is in G_n.*

Proof. First, since Y is aperiodic, there exists $L \in \mathbb{N}$ such that

$$\text{all words } w \in \mathcal{L}(Y) \text{ of length } \geq L \text{ have at least period greater than } |\tau_{[0,n)}|. \tag{29}$$

We assume, by contradiction, that the claim does not hold. In particular, for every $\ell \in [0, L)$, there exist $K + 1\tau_{[0,n)}$-factorizations (x, k) of y in $Y_{\boldsymbol{\tau}}^{(n)}$ such that their symbols covering position ℓ of y are all different. Now, since $\#\tau_{[0,n)}(\mathcal{A}_{n+1}) \leq K$, we can use the pigeonhole principle to find two of such factorizations, say (k, x) and (k', x'), such that if (c, a) and (c', a') are their symbols covering position ℓ of y, then $a = a'$ and $c < c'$. Then,

$$y_{(c,c+|\tau_{[0,n)}(a)|]} = \tau_{[0,n)}(a) = y_{(c',c'+|\tau_{[0,n)}(a)|]}$$

and, thus, $y_{(c,c'+|\tau_{[0,n)}(a)|]}$ is $(c' - c)$-periodic. Being $\ell \in (c', c + |\tau_{[0,n)}(a)|)$, we deduce that the local period of $y_{[0,L)}$ at ℓ is at most $c' - c \leq |\tau_{[0,n)}|$. Since this is true for every $\ell \in [0, L)$ and since, by Theorem 3.7, $\operatorname{per}(y_{[0,L)}) = \operatorname{per}(y_{[0,L)}, y_{[0,\ell)})$ for some $\ell \in [0, L)$, we conclude that $\operatorname{per}(y_{[0,L)}) \leq |\tau_{[0,n)}|$. This contradicts (29) and proves thereby the claim. □

Now we prove the theorem. It is enough to show that the hypothesis of Lemma 5.2 holds. Let $y \in Y$ and $\tilde{F}_n(y) \subseteq F_n(y)$ be such that $\#\tilde{F}_n(y) = \#G_n$ and the set consisting of all the symbols of factorizations $(k, x) \in \tilde{F}_n(y)$ covering position ℓ_n of y is equal to G_n. Let $z \in \pi^{-1}(y)$ and (k, x) be a $\sigma_{[0,n)}$-factorization of z in $X_{\boldsymbol{\sigma}}^{(n)}$. Then, $T^k\tau_{[0,n)}(x) = T^k\pi(\sigma_{[0,n)}(x)) = \pi(z) = y$ and (k, x) is a $\tau_{[0,n)}$-factorization of y in $Y_{\boldsymbol{\tau}}^{(n)}$. Thus, we can find $(k', x') \in \tilde{F}_n(y)$ such that the symbols of (k, x) and (k', x') covering position ℓ_n of y are the same; let (m, a) be this common symbol. Since $\boldsymbol{\sigma}$ is proper, we have

$$z_{[m-\langle\sigma_{[0,n-1)}\rangle, m+|\sigma_{[0,n)}(a)|+\langle\sigma_{[0,n-1)}\rangle]} = z'_{[m-\langle\sigma_{[0,n-1)}\rangle, m+|\sigma_{[0,n)}(a)|+\langle\sigma_{[0,n-1)}\rangle]},$$

where $z' = T^{k'}\sigma_{[0,n)}(x') \in X$ is the point that (k', x') factorizes in $(X_{\boldsymbol{\sigma}}^{(n)}, \sigma_{[0,n)})$. Then, as $\ell_n \in (m, m + |\sigma_{[0,n)}(a)|]$,

$$z_{(\ell_n-\langle\sigma_{[0,n-1)}\rangle, \ell_n+\langle\sigma_{[0,n-1)}\rangle]} = z'_{(\ell_n-\langle\sigma_{[0,n-1)}\rangle, \ell_n+\langle\sigma_{[0,n-1)}\rangle]}.$$

Thus, $\operatorname{dist}(T^{\ell_n}z, T^{\ell_n}P_n(y)) \leq \exp(-\langle\sigma_{[0,n-1)}\rangle)$, where $P_n(y) \subseteq \pi^{-1}(y)$ is the set of all points $T^{k''}\sigma_{[0,n)}(x'') \in X$ such that $(k'', x'') \in \tilde{F}_n(y)$. Since this holds for every $n \geq 1$, we obtain that $d_{\mathrm{H}}(T^{\ell_n}\pi^{-1}(y), T^{\ell_n}P_n(y))$ converges to zero as n goes to infinity (where,

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we recall, d_{H} is the Hausdorff distance). By taking an appropriate convergent subnet $u \in E(2^X, T)$ of $(T^{\ell_n})_{n\in\mathbb{N}}$, we obtain $\#u \circ \pi^{-1}(y) \leq \sup_{n\in\mathbb{N}} \#P_n = \sup_{n\in\mathbb{N}} \#G_n \leq K$. This proves that the hypothesis of Lemma 5.2 holds. Therefore, π is almost k-to-1 for some $k \leq K$.

6. *Number of symbolic factors*

In this section, we prove Theorem 1.7. To do this, we split the proof into three subsections. First, in Lemma 6.3 of §6.1, we deal with the case of Theorem 1.7 in which the factor maps are distal. Next, we show in Lemma 6.7 from §6.2 that in certain technical situation—which will arise when we consider non-distal factor maps—it is possible to reduce the problem to a similar one, but where the alphabet is smaller. Then, we prove Theorem 1.7 in §6.3 by a repeated application of the previous lemmas.

6.1. *Distal factor maps.*

We start with some definitions. If (X, T) is a system, then we always give X^k the diagonal action $T^{[k]} := T \times \cdots \times T$. If $\pi\colon (X, T) \to (Y, T)$ is a factor map and $k \geq 1$, then we define $R_\pi^k = \{(x^1, \ldots, x^k) \in X^k : \pi(x^1) = \cdots = \pi(x^k)\}$. Observe that R_π^k is a closed $T^{[k]}$-invariant subset of X^k.

The next lemma follows from classical ideas from topological dynamics. See, for example, theorem 6 in ch. 10 of [**Aus88**].

LEMMA 6.1. *Let $\pi\colon (X, T) \to (Y, T)$ be a distal almost k-to-1 factor between minimal systems, $z = (z^1, \ldots, z^k) \in R_\pi^k$ and $Z = \overline{\mathrm{orb}}_{T^{[k]}}(z)$. Then, π is k-to-1 and Z is minimal.*

We will also need the following lemma.

LEMMA 6.2. [**Dur00**, Lemma 21] *Let $\pi_i\colon (X, T) \to (Y_i, T)$, $i = 0, 1$, be two factors between aperiodic minimal systems. Suppose that π_0 is finite-to-one. If $x, y \in X$ are such that $\pi_0(x) = \pi_0(y)$ and $\pi_1(x) = T^p\pi_1(y)$, then $p = 0$.*

LEMMA 6.3. *Let (X, T) be an infinite minimal subshift of topological rank K and J an index set of cardinality $\#J > K(144K^7)^K$. Suppose that for every $j \in J$ there exists a distal symbolic factor $\pi_j\colon (X, T) \to (Y_j, T)$. Then, there are $i \neq j \in J$ such that (Y_i, T) is conjugate to (Y_j, T).*

Proof. We start by introducing the necessary objects for the proof and doing some general observations about them. First, thanks to Theorem 1.6, we know that π_j is almost k_j-to-1 for some $k_j \leq K$, so, by the pigeonhole principle, there exist $J_1 \subseteq J$ and $k \leq K$ such that $\#J_1 \geq \#J/K > (144K^7)^K$ and $k_j = k$ for every $j \in J_1$. For $j \in J_1$, we fix $z^j = (z_1^j, \ldots, z_k^j) \in R_{\pi_j}^k$ with $z_n^j \neq z_m^j$ for all $n \neq m$. Let $Z_j = \overline{\mathrm{orb}}_{T^{[k]}}(z^j)$ and $\rho\colon X^k \to X$ be the factor map that projects onto the first coordinate. By Lemma 6.1, π_j is k-to-1 and Z_j minimal. This implies that if $x = (x_1, \ldots, x_k) \in Z_j$, then

$$\{x_1, \ldots, x_k\} = \pi_j^{-1}(\pi_j(x_n)) \quad \text{for all } n \in \{1, \ldots, k\}, \tag{30}$$

$$x_n \neq x_m \quad \text{for all } n, m \in \{1, \ldots, k\}. \tag{31}$$

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Indeed, since Z_j is minimal, $(T^{[k]})^{n_\ell} z \to x$ for some sequence $(n_\ell)_\ell$, so,

$$\inf_{n\neq m} \operatorname{dist}(x_n, x_m) \geq \inf_{n\neq m, l\in\mathbb{Z}} \operatorname{dist}(T^l z_n, T^l z_m) > 0,$$

where in the last step is due to the fact that π_j is distal. This gives (31). For (30), we first note that $\{x_1, \ldots, x_k\} \subseteq \pi_j^{-1}(\pi_j(x_n))$ as $x \in R_{\pi_j}$, and then that the equality must hold since $\#\pi_j^{-1}(\pi_j(x_n)) = k = \#\{x_1, \ldots, x_k\}$ by (31).

The next step is to prove that asymptotic pairs in Z_j are well-behaved.

CLAIM 6.3.1. *Let $j \in J_1$ and $(x^j = (x_1^j, \ldots, x_k^j), \tilde{x}^j = (\tilde{x}_1^j, \ldots, \tilde{x}_k^j))$ be a right asymptotic pair in Z_j, this is,*

$$\lim_{n\to-\infty} \operatorname{dist}((T^{[k]})^n x^j, T^{[k]}\tilde{x}^j) = 0 \quad \text{and} \quad x^j \neq \tilde{x}^j. \tag{32}$$

Then, $(x_n^j, \tilde{x}_n^j)$ is right asymptotic for every $n \in \{1, \ldots, k\}$.

Proof. Suppose, with the aim to obtain a contradiction, that $(x_n^j, \tilde{x}_n^j)$ is not right asymptotic for some $n \in \{1, \ldots, k\}$. Observe that (32) implies that

$$\text{for every } m \in \{1, \ldots, k\}, \text{ either } (x_m^j, \tilde{x}_m^j) \text{ is right asymptotic or } x_n^j = \tilde{x}_n^j. \tag{33}$$

Therefore, $x_n^j = \tilde{x}_n^j$. Using this and that $x^j, \tilde{x}^j \in R_{\pi_j}^k$, we can compute:

$$\pi_j(x_m^j) = \pi_j(x_n^j) = \pi_j(\tilde{x}_n^j) = \pi_j(\tilde{x}_l^j) \quad \text{for all } m, l \in \{1, \ldots, k\},$$

and thus, by (30),

$$\{x_1^j, \ldots, x_k^j\} = \pi_j^{-1}(\pi_j(x_n^j)) = \pi_j^{-1}(\pi_j(\tilde{x}_n^j)) = \{\tilde{x}_1^j, \ldots, \tilde{x}_k^j\}.$$

The last equation, (31), and that $x^j \neq \tilde{x}^j$ imply that there exist $m \neq l \in \{1, \ldots, k\}$ such that $\tilde{x}_l^j = x_m^j$. This last equality and (33) tell us that x_m^j and x_l^j are either asymptotic or equal. However, in both cases a contradiction occurs: in the first one with the distality of π and in the second one with (31). □

Let $j \in J_1$. Since Y_j is infinite, Z_j is a infinite subshift. It is a well-known fact from symbolic dynamics that this implies that there exists a right asymptotic pair $(x^j = (x_1^j, \ldots, x_k^j), \tilde{x}^j = (\tilde{x}_1^j, \ldots, \tilde{x}_k^j))$ in Z_j. We are now going to use Theorem 2.8 to prove the following.

CLAIM 6.3.2. *There exists $i, j \in J_1$, $i \neq j$, such that $Z_i = Z_j$.*

Proof. On one hand, by the previous claim, $(x_n^j, \tilde{x}_n^j) \in X^2$ is right asymptotic for every $n \in \{1, \ldots, k\}$ and $j \in J_1$. Let $p_n^j \in \mathbb{Z}$ be such that $(T^{p_n^j} x_n^j, T^{p_n^j} \tilde{x}_n^j)$ is centered right asymptotic. On the other hand, Theorem 2.8 asserts that the set

$$\{x_{(0,\infty)} : (x, \tilde{x}) \text{ is centered right asymptotic in } X\}$$

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has at most $144K^7$ elements. Since $\#J_1 > (144K^7)^K$, we conclude, by the pigeonhole principle, that there exist $i, j \in J_1$, $i \neq j$, such that

$$T^{p_n^i} x_n^i \text{ and } T^{p_n^j} x_n^j \text{ agree on } (0, \infty) \text{ for every } n \in \{1, \dots, k\}. \tag{34}$$

We are going to show that $Z_i = Z_j$.

Using (34), we can find $u \in E(X, T)$ such that $uT^{p_n^i} x_n^i = uT^{p_n^j} x_n^j$ for every n. Then, by putting $y_n^i = ux_n^i$, $y_n^j = ux_n^j$ and $q_n = p_n^j - p_n^i$, we have

$$y^i := (y_1^i, \dots, y_k^i) \in Z_i,\ y^j := (y_1^j, \dots, y_k^j) \in Z_j \text{ and } y_n^i = T^{q_n} y_n^j.$$

Hence, $\pi(y_n^i) = T^{q_n}\pi(y_n^j)$ and Lemma 6.2 can be applied to deduce that $q := q_n$ has the same value for every n. We conclude that $y^i = T^q y^j \in T^q Z_j = Z_j$, that $Z_i \cap Z_j$ is not empty, and, therefore, that $Z_i = Z_j$ as these are minimal systems. □

We can now finish the proof. Let $i \neq j \in J_1$ be the elements given by the previous claim, so that $Z := Z_i = Z_j$. Let $y \in Y_i$ and $x = (x_1, \dots, x_k) \in \rho^{-1}\pi_i^{-1}(y) \cap Z$. Then, by (30), $\pi_i^{-1}(y) = \{x_1, \dots, x_k\} = \pi_j^{-1}(\pi_j(x_1))$, and so $\pi_j\pi_i^{-1}(y)$ contains exactly one element, which is $\pi_j(x_1)$. We define $\psi : Y_i \to Y_j$ by $\psi(y) = \pi_j(x_1)$.

Observe that $\pi_i^{-1} : Y_i \to 2^X$ is continuous (as π_i is distal and hence open) and commutes with T. Being π_j a factor map, ψ is continuous and commutes with T. Therefore, $\psi : (Y_i, T) \to (Y_j, T)$ is a factor map. A similar construction gives a factor map $\phi : Y_j \to Y_i$ which is the inverse function of ψ. We conclude that ψ is a conjugacy and, thus, that Y_i and Y_j are conjugate. □

6.2. *Non-distal factor maps and asymptotic pairs lying in fibers.* To deal with non-factor maps, we study asymptotic pairs belonging to fibers of this kind of factor. The starting point is the following lemma.

LEMMA 6.4. *Let $\pi : (X, T) \to (Y, T)$ be a factor between minimal subshifts. Then, either π is distal or there exists a fiber $\pi^{-1}(y)$ containing a pair of right or left asymptotic points.*

Proof. Assume that π is not distal. Then, we can find a fiber $\pi^{-1}(y)$ and proximal points $x, x' \in \pi^{-1}(y)$, with $x \neq x'$. This implies that for every $k \in \mathbb{N}$, there exist a (may be infinite) interval $I_k = (a_k, b_k) \subseteq \mathbb{Z}$, with $b_k - a_k \geq k$, for which x and x' coincide on I, and I_k is maximal (with respect to the inclusion) with this property. Since $x \neq x'$, then $a_k > -\infty$ or $b_k < \infty$. Hence, there exists an infinite set $E \subseteq \mathbb{N}$ such that $a_k > -\infty$ for every $k \in E$ or $b_k < \infty$ for every $k \in E$. In the first case, we have that $(T^{b_k}(x, x'))_{k\in E}$ has a left asymptotic pair (z, z') as an accumulation point, while in the second case, it is a right asymptotic pair (z, z') which is an accumulation point of $(T^{a_k}(x, x'))_{k\in E}$. In both cases we have that $(z, z') \in R_\pi^2$ since $(T^{b_k}(x, x'))_{k\in E}$ and $(T^{a_k}(x, x'))_{k\in E}$ are contained in R_π^2 and R_π^2 is closed. Therefore, the fiber $\pi^{-1}(\pi(z))$ contains a pair z, z' of asymptotic points. □

The next lemma allows us to pass from morphisms $\sigma : X \to Y$ to factors $\pi : X' \to Y$ in such a way that X' is defined on the same alphabet as X and has the 'same' asymptotic pairs. We remark that its proof is simple, but tedious.

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LEMMA 6.5. *Let $X \subseteq \mathcal{A}^+$ be an aperiodic subshift, $\sigma\colon \mathcal{A}^+ \to \mathcal{B}^+$ be a morphism, and $Y = \bigcup_{k\in\mathbb{Z}} T^k\sigma(X)$. Define the morphism $i_\sigma\colon \mathcal{A}^+ \to \mathcal{A}^+$ by $i_\sigma(a) = a^{|\sigma(a)|}$, $a \in \mathcal{A}$, and $X' = \bigcup_{k\in\mathbb{Z}} T^k i_\sigma(X)$. Then, centered asymptotic pairs in X' are of the form $(i_\sigma(x), i_\sigma(\tilde{x}))$, where $(x, \tilde{x})$ is a centered asymptotic pair in X, and there exists a factor map $\pi\colon (X', T) \to (Y, T)$ such that $\pi(i_\sigma(x)) = \tau(x)$ for all $x \in X$.*

Proof. Our first objective is to prove that (X, i_σ) is recognizable. We start by observing that

$$\text{if } (k, x), (\tilde{k}, \tilde{x}) \text{ are centered } i_\sigma\text{-factorizations of } y \in X', \quad \text{then } x_0 = \tilde{x}_0. \tag{35}$$

Indeed, since the factorizations are centered, we have $x_0 = i_\sigma(x_0)_k = y_0 = i_\sigma(\tilde{x}_0)_{\tilde{k}} = \tilde{x}_0$.

Let Λ be the set of tuples $(k, x, \tilde{k}, \tilde{x})$ such that $(k, x), (\tilde{k}, \tilde{x})$ are centered i_σ-factorizations of the same point. Moreover, for $\mathcal{R} \in \{=, >\}$, let $\Lambda_{\mathcal{R}}$ be the set of those $(k, x, \tilde{k}, \tilde{x}) \in \Lambda$ satisfying $k \,\mathcal{R}\, \tilde{k}$.

CLAIM 6.5.1. *If $(k, x, \tilde{k}, \tilde{x}) \in \Lambda_=$, then $(0, Tx, 0, T\tilde{x}) \in \Lambda_=$, and if $(k, x, \tilde{k}, \tilde{x}) \in \Lambda_>$, then $(|i_\sigma(x_0)| - k + \tilde{k}, \tilde{x}, 0, Tx) \in \Lambda_>$.*

Proof. If $(k, x, \tilde{k}, \tilde{x}) \in \Lambda_=$, then, since $x_0 = \tilde{x}_0$ by (35), we can write $i_\sigma(Tx) = T^k i_\sigma(x) = T^{\tilde{k}} i_\sigma(\tilde{x}) = i_\sigma(T\tilde{x})$. Thus, $(0, Tx, 0, T\tilde{x}) \in \Lambda_=$. Let now $(k, x, \tilde{k}, \tilde{x}) \in \Lambda_>$ and $y := T^k i_\sigma(x) = T^{\tilde{k}} i_\sigma(\tilde{x})$. We note that

$$T^{|i_\sigma(x_0)|-k+\tilde{k}} i_\sigma(\tilde{x}) = T^{|i_\sigma(x_0)|-k} y = T^{|i_\sigma(x_0)|} i_\sigma(x) = i_\sigma(Tx),$$

so $(|i_\sigma(x_0)| - k + \tilde{k}, \tilde{x})$ and $(0, Tx)$ are i_σ-factorizations of the same point. Now, since $x_0 = \tilde{x}_0$ (by (35)) and $(k, x), (\tilde{k}, \tilde{x})$ are centered, we have $k, \tilde{k} \in [0, |i_\sigma(x_0)|)$. This and the fact that $k > \tilde{k}$ imply that $k - \tilde{k} \in (0, |i_\sigma(x_0)|)$. Therefore, $|i_\sigma(x_0)| - k + \tilde{k} \in (0, |i_\sigma(x_0)|)$ and, consequently, $(|i_\sigma(x_0)| - k + \tilde{k}, \tilde{x}, 0, Tx) \in \Lambda_>$. □

We prove now that (X, i_σ) is recognizable. Let $(k, x, \tilde{k}, \tilde{x}) \in \Lambda$. We have to show that $(k, x) = (\tilde{k}, \tilde{x})$. First, we consider the case in which $k = \tilde{k}$. In this situation, the previous claim implies that $(0, Tx, 0, T\tilde{x}) \in \Lambda_=$. We use again the claim, but with $(0, Tx, 0, T\tilde{x})$, to obtain that $(0, T^2x, 0, T^2\tilde{x}) \in \Lambda_=$. By continuing in this way, we get $(0, T^n x, 0, T^n\tilde{x}) \in \Lambda_=$ for any $n \geq 0$. Then, (35) implies that $x_n = \tilde{x}_n$ for all $n \geq 0$. A similar argument shows that $x_n = \tilde{x}_n$ for any $n \leq 0$, and so $(k, x) = (\tilde{k}, \tilde{x})$. We now do the case $k > \tilde{k}$. Another application of the claim gives us $(p_1, \tilde{x}, 0, Tx) \in \Lambda_>$ for some $p_1 \in \mathbb{Z}$. As before, we iterate this procedure to obtain that $(p_2, Tx, 0, T\tilde{x}) \in \Lambda_>$, $(p_3, T\tilde{x}, 0, T^2x) \in \Lambda_>$, and so on. From these relations and (35), we deduce that $x_0 = \tilde{x}_0$, $\tilde{x}_0 = (Tx)_0 = x_1$, $x_1 = (Tx)_0 = (T\tilde{x})_0 = \tilde{x}_1$, $\tilde{x}_1 = (T\tilde{x})_0 = (T^2x)_0 = x_2$, etc. We conclude that $x_n = \tilde{x}_n = x_0$ for any $n \geq 0$. Then, by compacity, the periodic point $\cdots x_0.x_0x_0\cdots$ belongs to X, contrary to our aperiodicity hypothesis on X. Thus, the case $k > \tilde{k}$ does not occur. This proves that (X, i_σ) is recognizable.

Using the property we just proved, we can define the factor map $\pi\colon X' \to Y$ as follows: if $x' \in X'$, then we set $\pi(x') = T^k\tau(x) \in Y$, where (k, x) is the unique centered i_σ-factorization of x' in X. To show that π is indeed a factor map, we first observe that

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since

$$|\tau(a)| = |i_\sigma(a)| \quad \text{for all } a \in \mathcal{A}, \tag{36}$$

π commutes with T. Moreover, thanks to condition (iii) in Remark 2.2, π is continuous. Finally, if $y \in Y$, then by the definition of Y, there exist a centered $(k, x)\tau$-factorization of y in X. Thus, by (36), (k, x) is a centered i_σ factorization of $x' := T^k i_\sigma(x)$. Therefore, $\pi(x') = y$ and π is onto. Altogether, these arguments show that π is a factor map. That $\pi(i_\sigma(x)) = \tau(x)$ for every $x \in X$ follows directly from the definition of π.

It is left to prove the property about the asymptotic pairs. We only prove it for left asymptotic pairs since the other case is similar. We will use the following notation: if Z is a subshift, then $A(Z)$ denotes the set of centered left asymptotic pairs. To start, we observe that $(i_\sigma(x), i_\sigma(x')) \in A(X')$ for every $(x, \tilde{x}) \in A(X)$. Let now $(z, \tilde{z}) \in A(X')$, and (k, x) and $(\tilde{k}, \tilde{x})$ be the unique centered i_σ-factorizations of z and $\tilde{z}$ in X, respectively. We have to show that $k = \tilde{k} = 0$ and that $(x, \tilde{x}) \in A(X)$. Due to condition (iii) in Remark 2.2, (X, i_σ) has a recognizability constant. This and the fact that $(z, \tilde{z})$ is centered left asymptotic imply that (k, x) and $(\tilde{k}, \tilde{x})$ have a common cut in $(-\infty, 0]$, this is, that there exist $p, q \leq 0$ such that

$$m := -k - |i_\sigma(x_{[p,0)})| = -\tilde{k} - |i_\sigma(\tilde{x}_{[q,0)})| \in (-\infty, 0].$$

We take m as big as possible with this property. Then, $x_p \neq \tilde{x}_q$. Moreover, being $z_m = x_p$ and $\tilde{z}_m = \tilde{x}_p$ by the definition of i_σ, we have that $z_m \neq \tilde{z}_m$ and consequently, by also using that $(z, \tilde{z})$ is centered left asymptotic, that $m \geq 0$. We conclude that $m = 0$, i.e. that $k + |i_\sigma(x_{[p,0)})| = \tilde{k} + |i_\sigma(\tilde{x}_{[q,0)})| = 0$. Hence, $k = \tilde{k} = p = q = 0$. Now, it is clear that $x_{(-\infty,p]} = \tilde{x}_{(-\infty,q]}$, so from the last equations, we obtain that $(x, \tilde{x}) \in A(X)$. This completes the proof. □

We will also need the following lemma to slightly strengthen Proposition 2.8.

LEMMA 6.6. *Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be an aperiodic subshift with L asymptotic tails. Then, (X, T) has at most $2L^2 \cdot \#\mathcal{A}^2$ centered asymptotic pairs.*

Proof. Let $\mathcal{P}_r$ be the set of centered right asymptotic pairs in X and $\mathcal{T}_r = \{x_{(0,\infty)} : (x, \tilde{x}) \in \Lambda\} \subseteq \mathcal{A}^{\mathbb{N}_{\geq 1}}$ be the set of right asymptotic tails, where $\mathbb{N}_{\geq 1} = \{1, 2, \ldots\}$. We are going to prove that

$$\#\mathcal{P}_r \leq \#\mathcal{T}_r^2 \cdot \#\mathcal{A}^2. \tag{37}$$

Once this is done, we will have by symmetry the same relation for the centered left asymptotic pairs $\mathcal{P}_l$, and thus we are going to be able to conclude that the number of centered asymptotic pairs in X is at most $(\#\mathcal{T}_r^2 + \#\mathcal{T}_l^2) \cdot \#\mathcal{A}^2 \leq 2L^2 \cdot \#\mathcal{A}^2$, completing the proof.

Let $(x, \tilde{x}) \in \mathcal{P}_r$ and $\mathcal{R}_x = \{k \leq 0 : x_{(k,\infty)} \in \mathcal{T}_r\}$. We claim that $\#\mathcal{R}_x \leq \#\mathcal{T}_r$. Indeed, if this is not the case, then, by the pigeonhole principle, we can find $k' < k$ and $w \in \mathcal{T}_r$ such that $w = x_{(k,\infty)} = x_{(k',\infty)}$. However, this implies that w has period $k - k'$, and so X contains a point of period $k - k'$, contrary to the aperiodicity hypothesis. Thus, $\mathcal{R}_x$ is

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finite and, since $\mathcal{R}_x$ is non-empty as it contains $x_{(0,\infty)}$, $k_x := \min \mathcal{R}_x$ is a well-defined non-positive integer.

Let now $\phi\colon \mathcal{P}_r \to \mathcal{T}_r^2 \times \mathcal{A}^2$ be the function defined by

$$\phi(x, \tilde{x}) = (x_{(k_x,\infty)}, \tilde{x}_{(k_{\tilde{x}},\infty)}, x_{k_x}, \tilde{x}_{k_{\tilde{x}}}).$$

If ϕ is injective, then (37) follows. Let us then prove that ϕ is injective.

We argue by contradiction and assume that there exist $(x, \tilde{x}) \neq (y, \tilde{y})$ such that $\phi(x, \tilde{x}) = \phi(y, \tilde{y}) = (z, \tilde{z}, a, \tilde{a})$. Without loss of generality, we may assume that $x \neq y$. Then, $x_{(k_x,\infty)} = z = y_{(k_y,\infty)}$ and $x_{k_x} = a = y_{k_y}$. Being $x \neq y$, this implies that (x, y) is asymptotic. Furthermore, it implies that there exist $p < k$ and $q < \ell$ such that $(T^p x, T^q y)$ is centered right asymptotic. In particular, $x_{(p,\infty)} \in \mathcal{T}_r$ and $p < k_x$, contrary to the definition of k_x. We conclude that ϕ is injective and thereby complete the proof of the lemma. □

LEMMA 6.7. *Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a subshift of topological rank K, J be an index set, and, for $j \in J$, let $\tau_j\colon \mathcal{A}^+ \to \mathcal{B}_j^+$ be a morphism. Suppose that for every $j \in J$:*

(I) *$Y_j = \bigcup_{k\in\mathbb{Z}} T^k \tau_j(X)$ is aperiodic;*

(II) *for every fixed $a \in \mathcal{A}$, $|\tau_j(a)|$ is equal to a constant ℓ_a independent of $j \in J$.*

Then, one of the following situations occur.

(1) *There exist $i, j \in J$, $i \neq j$, such that (Y_i, T) is conjugate to (Y_j, T).*

(2) *There exist $\phi\colon \mathcal{A}^+ \to \mathcal{A}_1^+$ with $\#\mathcal{A}_1 < \#\mathcal{A}$, a set $J_1 \subseteq J$ having at least $\#J/2\#\mathcal{A}^2(144K^7)^2 - K(144K^7)^K$ elements, and morphisms $\tau_j'\colon \mathcal{C}_1^+ \to \mathcal{B}_j$, $j \in J_1$, such that $\tau_j = \tau_j'\phi$. In particular, the hypothesis of this lemma holds for $X_1 := \bigcup_{k\in\mathbb{Z}} T^k\phi(X)$ and τ_j', $j \in J_1$.*

Proof. Let $\mathfrak{i}\colon \mathcal{A}^+ \to \mathcal{A}^+$ be the morphism defined by $\mathfrak{i}(a) = a^{\ell_a}$, $a \in \mathcal{A}$, and $X' = \bigcup_{k\in\mathbb{Z}} T^k\mathfrak{i}(X)$. We use Lemma 6.5 with X and τ_j to obtain a factor map $\pi_j\colon (X', T) \to (Y_j, T)$ such that

$$\pi(\mathfrak{i}(x)) = \tau_j(x) \quad \text{for every } x \in X. \tag{38}$$

If π_j is distal for $K(144K^7)^K + 1$ different values of $j \in J$, then by Lemma 6.3, we can find i, j such that (Y_i, T) is conjugate to (Y_j, T). Therefore, we can suppose that there exists $J' \subseteq J$ such that

$$\#J' \geq \#J - K(144K^7)^K \quad \text{and} \quad \pi_j \text{ is not distal for every } j \in J'. \tag{39}$$

From this and Lemma 6.4, we obtain, for every $j \in J'$, a centered asymptotic pair $(x^{(j)}, \tilde{x}^{(j)})$ in X' such that $\pi_j(x^{(j)}) = \pi_j(\tilde{x}^{(j)})$. This and (38) imply that

$$\tau_j(x^{(j)}) = \pi_j(x^{(j)}) = \pi_j(\tilde{x}^{(j)}) = \tau_j(\tilde{x}^{(j)}). \tag{40}$$

Now, by Lemma 6.6, X has at most $2\#\mathcal{A}^2(144K^7)^2$ centered asymptotic pairs and thus, thanks to Lemma 6.5, the same bound holds for X'. Therefore, by the pigeonhole principle, there exist $J_1 \subseteq J$ satisfying $\#J_1 \geq \#J'/2\#\mathcal{A}^2(144K^7)^2 \geq \#J/2\#\mathcal{A}^2(144K^7)^2 - K(144K^7)^K$ and a centered asymptotic pair $(x, \tilde{x})$ in X' such that $(x, \tilde{x}) = (x^{(j)}, \tilde{x}^{(j)})$ for every $j \in J_1$.

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We assume that $(x, \tilde{x})$ is right asymptotic as the other case is similar. Then, (40) implies that if $\ell = \sum_{a\in\mathcal{A}} \ell_a$, then, for every $j \in J_1$,

$$\text{one of the words in } \{\tau_j(x_{[0,\ell)}), \tau_j(\tilde{x}_{[0,\ell)})\} \text{ is a prefix of the other.} \tag{41}$$

This, hypothesis (II), and the fact that, since $(x, \tilde{x})$ is a centered asymptotic pair, $x_0 \neq \tilde{x}_0$ allows us to use Lemma 3.2 with $u := x_{[0,\ell)}$, $v := \tilde{x}_{[0,\ell)}$, $J := J_1$, and $w^j := \tau_j(x_{[0,\infty)})_{[0,\ell)}$, and obtain morphisms $\phi\colon \mathcal{A}^+ \to \mathcal{A}_1^+$ and $\tau'_j\colon \mathcal{A}_1^+ \to \mathcal{B}_j^+$, $j \in J_1$, such that $\#\mathcal{A}_1 < \#\mathcal{A}$, $\tau_j = \tau'_j\phi$, and

$$\text{for every } a \in \mathcal{A}_1,\ \ell'_a := |\tau'_j(c)| \text{ does not depend on the chosen } j \in J. \tag{42}$$

Finally, we observe that X_1 and τ'_j, $j \in J_1$, satisfy the hypothesis of the lemma: condition (I) holds since, by the relation $\tau_j = \tau'_j\phi$, the subshift $X_1 := \bigcup_{k\in\mathbb{Z}} T^k\phi(X)$ satisfies that $\bigcup_{k\in\mathbb{Z}} T^k\tau'_j(X_1) = Y_j$ is aperiodic; condition (II) is given by (42). □

6.3. *Proof of main result.* We now prove Theorem 1.7. We restate it for convenience.

THEOREM 1.7. *Let (X, T) be an minimal subshift of topological rank K. Then, (X, T) has at most $(3K)^{32K}$ aperiodic symbolic factors up to conjugacy.*

Proof. We set $R = (3K)^{32K}$. We prove the theorem by contradiction: assume that there exist $X \subseteq \mathcal{A}^{\mathbb{Z}}$ of topological rank K and, for $j \in \{0, \dots, R\}$, factor maps $\pi_j\colon (X, T) \to (Y_j, T)$ such that (Y_i, T) is not conjugate to (Y_j, T) for every $i \neq j \in \{0, \dots, R\}$. We remark that X must be infinite as, otherwise, it would not have any aperiodic factor.

To start, we build $\mathcal{S}$-representations for the subshifts X and Y_j. Let $\boldsymbol{\sigma} = (\sigma_n\colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+)_{n\in\mathbb{N}}$ be the primitive and proper directive sequence of alphabet rank K generating X given by Theorem 1.1. Let $r \in \mathbb{N}$ be such that every π_j has a radius r and let $\mathcal{B}_j$ be the alphabet of Y_j. By contracting $\boldsymbol{\sigma}$, we can assume that σ_0 is r-proper and $\#\mathcal{A}_n = K$ for all $n \geq 1$. Then, we can use Lemma 2.4 to find morphisms $\tau_j\colon \mathcal{A}_1^+ \to \mathcal{B}_j^+$ such that

$$\pi_j(\sigma_1(x)) = \tau_j(x) \quad \text{for all } x \in X_{\boldsymbol{\sigma}}^{(1)} \quad \text{and} \quad |\tau_j(a)| = |\sigma_0(a)| \quad \text{for all } a \in \mathcal{A}_1. \tag{43}$$

Next, we inductively define subshifts $X_n \subseteq C_n^{\mathbb{Z}}$ and morphisms $\{\tau_{n,j}\colon C_n^+ \to \mathcal{B}_j : j \in J_n\}$ such that:

(i) X_n has topological rank at most K;
(ii) $Y_j = \bigcup_{k\in\mathbb{Z}} \tau_{n,j}(X_n)$;
(iii) for every $c \in C_n$, $\ell_{n,a} := |\tau_{n,j}(c)|$ does not depend on the chosen $j \in J_n$.

First, we set $X_0 = X_{\boldsymbol{\sigma}}^{(1)}$, $C_0 = \mathcal{A}_1$, $J_0 = J$, and, for $j \in J_0$, $\tau_{0,j} = \tau_j$, and note that by the hypothesis and (43), they satisfy conditions (i), (ii), and (iii). Let now $n \geq 0$ and suppose that $X_n \subseteq C_n^{\mathbb{Z}}$ and $\tau_{n,j}$, $j \in J_n$, has been defined in a way such that conditions (i), (ii), and (iii) hold. If $\#J_n/2\#\mathcal{A}^2(144K^7)^2 - K(144K^7)^K \leq 1$, then the procedure stops. Otherwise, we define step $n + 1$ as follows. Thanks to conditions (i), (ii), and (iii), we can use Lemma 6.7, and since there are no two conjugate (Y_i, T), this lemma gives us a morphism $\phi\colon C_n^+ \to C_{n+1}^+$, a set $J_{n+1} \subseteq J_n$, and morphisms $\{\tau_{n+1,j}\colon C_{n+1}^+ \to \mathcal{B}_j^+ : j \in J_{n+1}\}$ such

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that

$$\#C_{n+1} < \#C_n, \#J_{n+1} \geq \#J_n/2\#C_n^2(144K^7)^2 - K(144K^7)^K \quad \text{and} \quad \tau_{n,j} = \tau_{n+1,j}\phi_n.$$

Furthermore, $X_{n+1} := \bigcup_{k\in\mathbb{Z}} T^k\phi_n(X_n)$ and $\tau_{n+1,j}$ satisfy the hypothesis of that lemma, that is, conditions (ii) and (iii) above. Since $(\phi_n \ldots \phi_0\sigma_1, \sigma_2, \sigma_3, \ldots)$ is a primitive and proper sequence of alphabet rank K generating X_{n+1}, Theorem 1.3 implies that condition (i) is met as well.

Since $\#C_0 > \#C_1 > \ldots$, there is a last C_N defined. Our next objective is to prove that $N \geq K$. Observe that $\#C_n \leq K$, so

$$\#J_{n+1} \geq \#J_n/2K^2(144K^7)^2 - K(144K^7)^K \quad \text{for any } n \in \{0, \ldots, N-1\}.$$

Using this recurrence and the inequalities $\#J_0 > (3K)^{32K}$ and $K \geq 2$, it is routine to verify that the following bound holds for every $n \in \{0, \ldots, K-1\}$ such that the nth step is defined:

$$\#J_n/2\#C_n^2(144K^7)^2 - K(144K^7)^K > 1.$$

Therefore, $N \geq K$. We conclude that $\#C_N \leq \#C_0 - K = 0$, which is a contradiction. □

Remark 6.8. In theorem 1 of [**Dur00**], the author proved that linearly recurrent subshifts have finitely many aperiodic symbolic factors up to conjugacy. Since this kind of systems have finite topological rank (see Remark 4.10), Theorem 1.7 generalizes the theorem of [**Dur00**] to the much larger class of minimal finite topological rank subshifts.

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A. *Appendix*

To prove Proposition 4.4, we start with some lemmas concerning how to construct recognizable pairs (Z, τ) for a fixed subshift $Y = \bigcup_{k\in\mathbb{Z}} T^k\tau(Z)$.

A.1. *Codings of subshifts.* If $Y \subseteq \mathcal{B}^{\mathbb{Z}}$ is a subshift, $U \subseteq Y$ and $y \in Y$, we denote by $\mathcal{R}_U(y)$ the set of *return times* of y to U, this is, $\mathcal{R}_U(y) = \{k \in \mathbb{Z} : T^k y \in U\}$. We recall that the set $C_\tau(k, z)$ in the lemma below corresponds to the cuts of (k, z) (see Definition 2.1 for further details).

LEMMA A.1. *Let $Y \subseteq \mathcal{B}^{\mathbb{Z}}$ be an aperiodic subshift, with $\mathcal{B} \subseteq \mathcal{L}(Y)$. Suppose that $U \subseteq Y$ is:*

(I) *d-syndetic, for every $y \in Y$ there exists $k \in [0, d-1]$ with $T^k y \in U$;*
(II) *of radius r, $U \subseteq \bigcup_{u\in\mathcal{A}^r, v\in\mathcal{A}^{r+1}} [u.v]$;*
(III) *ℓ-proper, $U \subseteq [u.v]$ for some $u, v \in \mathcal{A}^\ell$;*
(IV) *ρ-separated, $U, TU, \ldots, T^{\rho-1}U$ are disjoint.*

Then, there exist a letter-onto morphism $\tau : C^+ \to \mathcal{B}^+$ and a subshift $Z \subseteq C^{\mathbb{Z}}$ such that:

(1) *$Y = \bigcup_{n\in\mathbb{Z}} T^n\tau(Z)$ and $C \subseteq \mathcal{L}(Y)$;*

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(2) *(Z, τ) is recognizable with constant $r + d$;*
(3) *$|\tau| \le d$, $\langle\tau\rangle \ge \rho$ and τ is $\min(\rho, \ell)$-proper;*
(4) *$C_\tau(k, z) = \mathcal{R}_U(y)$ for all $y \in Y$ and τ-factorization (k, z) of y in Z.*

Remark A.2. If $U \subseteq Y$ satisfies condition (III), then U is $\rho := \min \mathrm{per}(\mathcal{L}_\ell(Y))$-separated. Indeed, if $U \cap T^k U \neq \emptyset$ for some $k > 0$, then $[v] \cap T^k[v] \neq \emptyset$, where $v \in \mathcal{A}^\ell$ is such that $U \subseteq [v]$. Hence, v is k periodic and $k \ge \rho$.

Proof. Let $y \in Y$. By condition (I), the sets $\mathcal{R}_U(y) \cap [0, \infty)$, $\mathcal{R}_U(y) \cap (-\infty, 0]$ are infinite. Thus, we can write $\mathcal{R}_U(y) = \{\cdots k_{-1}(y) < k_0(y) < k_1(y) \cdots\}$, with $\min\{i \in \mathbb{Z} : k_i(y) > 0\} = 1$. Let $\mathcal{W} = \{y_{[k_i(y), k_{i+1}(y))} : y \in Y,\ i \in \mathbb{Z}\} \subseteq \mathcal{B}^+$. By condition (I), $\mathcal{W}$ is finite, so we can write $C := \{1, \ldots, \#\mathcal{W}\}$ and choose a bijection $\phi : C \to \mathcal{W}$. Then, ϕ extends to a morphism $\tau : C^+ \to \mathcal{B}^+$. As $\mathcal{B} \subseteq \mathcal{L}(Y)$, ϕ is letter-onto. We define $\psi : Y \to C^{\mathbb{Z}}$ by $\psi(y) = (\phi^{-1}(y_{[k_i(y), k_{i+1}(y))}))_{i \in \mathbb{Z}}$ and set $Z = \psi(Y)$. We are going to prove that τ and Z satisfy items (1–4).

Claim A.2.1.

(i) *If $y_{[-d-r, d+r]} = y'_{[-d-r, d+r]}$, then $\psi(y)_0 = \psi(y')_0$.*
(ii) *$\tau(\psi(y)) = T^{k_0(y)} y$.*
(iii) *$T^j \psi(y) = \psi(T^k y)$ for $j \in \mathbb{Z}$ and $k \in [k_j(y), k_{j+1}(y))$.*

Proof. Let $y, y' \in Y$ such that $y_{[-d-r, d+r]} = y'_{[-d-r, d+r]}$. By condition (I), we have $k_{i+1}(y) - k_i(y) \le d$ for all $i \in \mathbb{Z}$ and, thus, $|k_0(y)|, |k_1(y)| \le d$. Since U has radius r and $y_{[-d-r, d+r]} = y'_{[-d-r, d+r]}$, we deduce that $k_0(y) = k_0(y')$ and $k_1(y) = k_0(y')$. Hence, $\psi(y)_0 = \phi^{-1}(y_{[k_0(y), k_1(y))}) = \phi^{-1}(y'_{[k_0(y'), k_1(y'))}) = \psi(y')_0$. To prove claim (ii) we compute:

$$\begin{aligned}\tau(\psi(y)) &= \tau(\cdots \phi^{-1}(y_{[k_{-1}(y), k_0(y))}).\phi^{-1}(y_{[k_0(y), k_1(y))}) \cdots) \\ &= \cdots y_{[k_{-1}(y), k_0(y))}.y_{[k_0(y), k_1(y))} \cdots = T^{k_0} y.\end{aligned}$$

Finally, for claim (iii) we write, for $k \in [k_j(y), k_{j+1}(y))$,

$$T^j \psi(y) = \cdots \phi^{-1}(y_{[k_{j-1}(y), k_j(y))}).\phi^{-1}(y_{[k_j(y), k_{j+1}(y))}) \cdots = \psi(T^k y).$$

Now we prove the desired properties of τ and Z.

(1) From claim (i), we see that ψ is continuous and, therefore, Z is closed. By claim (iii), Z is also shift-invariant and, then, a subshift. By claim (ii), $Y = \bigcup_{n \in \mathbb{Z}} T^n \tau(Z)$. The condition $C \subseteq \mathcal{L}(Y)$ follows from the definition of $\mathcal{W}$ and τ.

(2) We claim that the only centered τ-interpretation in Z of a point $y \in Y$ is $(-k_0(y), \psi(y))$. Indeed, this pair is a τ-interpretation in Z by claim (ii), and it is centered because $k_0(y) \le 0 < k_1(y)$ implies $-k_0(y) \in [0, k_1(y) - k_0(y)) = [0, |\psi(y)_0|)$. Let (n, z) be another centered τ-interpretation of y in Z. By the definition of Z, there exists $y' \in Y$ with $z = \psi(y')$. Then, by claim (ii),

$$T^{n + k_0(y')} y' = T^n \tau(\psi(y')) = T^n \tau(z) = y. \tag{44}$$

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Now, on one hand, we have $|\tau(z_0)| = |\tau(\psi(y')_0)| = k_1(y') - k_0(y')$. On the other hand, that $(n, \psi(y'))$ is centered gives that $n \in [0, |\tau(z_0)|)$. Therefore, $n + k_0(y') \in (k_0(y'), k_1(y')]$. We conclude from this, claim (iii), and (44) that $\psi(y') = \psi(y)$. Hence, $y = T^n \tau\psi(y') = T^n \tau\psi(y) = T^{n+k_0(y)} y$, which implies that $n = -k_0(y)$ as Y is aperiodic. This proves that $(-k_0(y), \psi(y))$ is the only τ-interpretation of y in Z. From this and claim (i), we deduce property (2).

(3) Since U is d-syndetic, $|\tau(\psi(y)_i)| = |y_{[k_i(y), k_{i+1}(y))}| = k_{i+1}(y) - k_i(y) \leq d$ for $y \in Y$ and $i \in \mathbb{Z}$, so $|\tau| \leq d$. Similarly, we can obtain $\langle\tau\rangle \geq \rho$ using that U is ρ-separated. Let $u, v \in \mathcal{B}^\ell$ satisfying $U \subseteq [u.v]$. Since $k_i, k_{i+1} \in \mathcal{R}_U(y)$, we have that $u = y_{[k_i(y), k_i(y)+|u|)}$, $v = y_{[k_{i+1}(y)-|v|, k_{i+1}(y))}$ and, thus, that τ is $\min(\ell, \langle\tau\rangle)$-proper. In particular, it is $\min(\ell, \rho)$-proper.

(4) This follows directly from the definition of τ and $\mathcal{R}_U(y)$. □

LEMMA A.3. *For $j \in \{0, 1\}$, let $\sigma_j \colon \mathcal{A}_j^+ \to \mathcal{B}^+$ be a morphism and $X_j \subseteq \mathcal{A}_j^{\mathbb{Z}}$ be a subshift such that $Y := \bigcup_{n\in\mathbb{Z}} T^n \sigma_j(X_j)$ and $\mathcal{A}_j \subseteq \mathcal{L}(X_j)$ for every $j \in \{0, 1\}$. Suppose that:*

(1) *(X_0, σ_0) is recognizable with constant ℓ;*
(2) *σ_1 is ℓ-proper;*
(3) *$C_{\sigma_0}(k^0, x^0)(y) \supseteq C_{\sigma_1}(k^1, x^1)(y)$ for all $y \in Y$ and σ_j-factorizations (k^j, x^j) of y in X_j, $j = 0, 1$.*

Then, there exist a letter-onto and proper morphism $\nu \colon \mathcal{A}_1^+ \to \mathcal{A}_0^+$ such that $\sigma_1 = \sigma_0\nu$ and $X_0 = \bigcup_{k\in\mathbb{Z}} T^k \nu(X_1)$.

Proof. Since σ_1 is ℓ-proper, we can find $u, v \in \mathcal{B}^\ell$ such that $\sigma_1(a)$ starts with u and ends with v for every $a \in \mathcal{A}_1$. We define ν as follows. Let $a \in \mathcal{A}_1$ and $x \in X_1$ such that $a = x_0$. Since σ_1 is ℓ-proper, the word $v.\sigma_1(a)u$ occurs in $\sigma_1(x) \in Y$ at position 0. By item (3), we can find $w \in \mathcal{L}(X_0)$ with $\sigma_1(x_0) = \sigma_0(w)$. We set $\nu(a) = w$. Since (X_0, σ_0) is recognizable with constant ℓ and u, v have length ℓ, w uniquely determined by $v.\sigma_1(a)u$ and, therefore, ν is well defined. Moreover, the recognizability implies that the first letter of $\nu(a)$ depends only on $v.u$, so ν is left-proper. A symmetric argument shows that ν is right-proper and, in conclusion, that it is proper. We also note that ν is letter-onto as $\mathcal{A}_0 \subseteq \mathcal{L}(X_0)$. It follows from the definition of ν that $\sigma_1 = \sigma_0\nu$. Now, let $x \in X_1$ and (k, x') be a centered σ_0-factorization of $\sigma_1(x)$ in X_0. By item (3), $k = 0$ and $\sigma_1(x_j) = \sigma_0(x'_{[k_j, k_{j+1})})$ for some sequence $\cdots < k_{-1} < k_0 < \cdots$ Hence, by the definition of ν, $\nu(x) = x' \in X_0$. This argument shows that $X_0' := \bigcup_{n\in\mathbb{Z}} T^n \nu(X_1) \subseteq X_0$. Then, $\bigcup_{n\in\mathbb{Z}} T^n \sigma_0(X_0') = \bigcup_{n\in\mathbb{Z}} T^n \sigma_0\nu(X_1) = Y$, where in the last step, we used that $\sigma_0\nu = \sigma_1$. Since the points in Y have exactly one σ_0-factorization, we must have $X_0' = X_0$. This ends the proof. □

A.2. *Factors of $\mathcal{S}$-adic sequences.* Now we are ready to prove Proposition 4.4. For convenience, we repeat its statement.

PROPOSITION A.4. *Let $\boldsymbol{\sigma} = (\sigma_n \colon \mathcal{A}_n \to \mathcal{A}_{n-1})_{n\geq 0}$ be a letter-onto, everywhere growing, and proper directive sequence. Suppose that $X_{\boldsymbol{\sigma}}$ is aperiodic. Then, there exists a*

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contraction $\boldsymbol{\sigma}' = (\sigma_{n_k})_{k\in\mathbb{N}}$ and a letter-onto and proper factor $\boldsymbol{\phi}\colon \boldsymbol{\sigma}' \to \boldsymbol{\tau}$, where $\boldsymbol{\tau}$ is letter-onto, everywhere growing, proper, recognizable, and generates $X_{\boldsymbol{\sigma}}$.

Proof. We start by observing that from Lemma 4.6, we can get that

$$\mathcal{A}_n \subseteq \mathcal{L}(X_{\boldsymbol{\sigma}}^{(n)}) \quad \text{for every } n \in \mathbb{N}. \tag{45}$$

Let $p_n = \min\{\mathrm{per}(\sigma_{[0,n)}(a)) : a \in \mathcal{A}_n\}$. Since $\boldsymbol{\sigma}$ is everywhere growing and $X_{\boldsymbol{\sigma}}$ is aperiodic, $\lim_{n\to\infty} p_n = \infty$. Hence, we can contract $\boldsymbol{\sigma}$ in a way such that, for every $n \geq 2$,

$$(\mathrm{I}_n)\ p_n \geq 3|\sigma_{[0,n-1)}|, \quad (\mathrm{II}_n)\ \sigma_{[0,n)} \text{ is } 3|\sigma_{[0,n-1)}|\text{-proper.}$$

For $n \geq 2$, let $U_n = \bigcup_{u,v\in\mathcal{A}_n^2}[\sigma_{[0,n)}(u.v)]$. Observe that U_n is $|\sigma_{[0,n)}|$-syndetic, has radius $2|\sigma_{[0,n)}|$, is $3|\sigma_{[0,n-1)}|$-proper and, by Remark A.2, is p_n-separated. Thus, by (I_n), U is $3|\sigma_{[0,n-1)}|$-separated. We can then use Lemma A.1 with $(X_{\boldsymbol{\sigma}}^{(n)}, \sigma_{[0,n)})$ to obtain a letter-onto morphism $\nu_n\colon \mathcal{B}_n^+ \to \mathcal{A}_0^+$ and a subshift $Y_n \subseteq \mathcal{B}_n^{\mathbb{Z}}$ such that:

(P_n^1) $X_{\boldsymbol{\sigma}} = \bigcup_{k\in\mathbb{Z}} T^k \nu_n(Y_n)$ and $\mathcal{B}_n \subseteq \mathcal{L}(Y_n)$;
(P_n^2) (Y_n, ν_n) is recognizable with constant $3|\sigma_{[0,n)}|$;
(P_n^3) $|\nu_n| \leq |\sigma_{[0,n)}|$, $\langle\nu_n\rangle \geq 3|\sigma_{[0,n-1)}|$, and ν_n is $3|\sigma_{[0,n-1)}|$-proper;
(P_n^4) $C_{\nu_n}(k, y) = \mathcal{R}_{U_n}(x)$ for all $x \in X_{\boldsymbol{\sigma}}$ and ν_n-factorization (k, y) of x in Y_n.

We write $C_{\nu_n}(x) := C_{\nu_n}(k, y)$ if $x \in X_{\boldsymbol{\sigma}}$ and (k, y) is the unique ν_n-factorization of x in Y_n. Observe that $U_{n+1} \subseteq U_n$ for $n \geq 2$. Thus, $C_{\nu_{n+1}}(x) = \mathcal{R}_{U_{n+1}}(x) \subseteq \mathcal{R}_{U_n}(x) = C_{\nu_n}(x)$ for all $x \in X_{\boldsymbol{\sigma}}$. This, (P_n^2) and (P_{n+1}^3) allow us to use Lemma A.3 with (Y_{n+1}, ν_{n+1}) and (Y_n, ν_n) and find a letter-onto and proper morphism $\tau_n\colon \mathcal{B}_{n+1}^+ \to \mathcal{B}_n^+$ such that $\nu_n\tau_n = \nu_{n+1}$ and $Y_n = \bigcup_{k\in\mathbb{Z}} T^k\tau_n(Y_{n+1})$.

Next, we claim that $C_{\nu_n}(x) \supseteq C_{\sigma_{[0,n+1)}}(k, z)$ for all $x \in X_{\boldsymbol{\sigma}}$ and $\sigma_{[0,n+1)}$-factorization (k, z) of x in $X_{\boldsymbol{\sigma}}^{(n+1)}$. Indeed, if $j \in \mathbb{Z}$, then $T^{c_{\sigma_{[0,n+1)},j}(k,z)}x \in [\sigma_{[0,n+1)}(z_{j-1}.z_jz_{j+1})] \subseteq [\sigma_{[0,n)}(a.bc)] \subseteq U_n$, where a is the last letter of $\sigma_n(z_{j-1})$ and bc the first two letters of $\sigma_n(z_jz_{j+1})$, so $c_{\sigma_{[0,n+1)},j}(k, z) \in \mathcal{R}_{U_n}(x) = C_{\nu_n}(x)$, as desired.

Thanks to the claim, (P_n^2), (I_{n+1}), and (45), we can use Lemma A.3 with (Y_n, ν_n) and $(X_{\boldsymbol{\sigma}}^{(n+1)}, \sigma_{[0,n+1)})$ to obtain a proper morphism $\phi_n\colon \mathcal{A}_{n+1}^+ \to \mathcal{B}_n^+$ such that $\sigma_{[0,n+1)} = \nu_n\phi_n$ and $Y_n = \bigcup_{k\in\mathbb{Z}} T^k\phi_n(X_{\boldsymbol{\sigma}}^{(n+1)})$.

Now we can define the morphisms $\tau_1 := \nu_2$ and $\phi_1 := \nu_2\phi_2$ and the sequence:

$$\boldsymbol{\phi} = (\phi_n)_{n\geq 1},\ \boldsymbol{\tau} = (\tau_n)_{n\geq 1} \quad \text{and} \quad \boldsymbol{\sigma}' = (\sigma_{[0,2)}, \sigma_2, \sigma_3, \ldots)_{n\geq 2}.$$

We are going to prove that $\boldsymbol{\phi}$, $\boldsymbol{\sigma}'$, and $\boldsymbol{\tau}$ are the objects that satisfy the conclusion of the proposition.

These sequences are letter-onto as each ν_n and each ϕ_n is letter-onto. Next, we show that $\boldsymbol{\phi}$ is a factor. The relation $\phi_1 = \tau_1\phi_2$ follows from the definitions. To prove the other relations, we observe that from the commutative relations for τ_n and ϕ_n, we have that

$$\nu_n\phi_n\sigma_{n+1} = \sigma_{[0,n+1)}\sigma_{n+1} = \sigma_{[0,n+2)} = \nu_{n+1}\phi_{n+1} = \nu_n\tau_n\phi_{n+1}. \tag{46}$$

In particular, $\nu_n\phi_n\sigma_{n+1}(x) = \nu_n\tau_n\phi_{n+1}(x)$ for any $x \in X_{\boldsymbol{\sigma}}^{(n+2)}$. Since $\phi_n\sigma_{n+1}(x)$ and $\tau_n\phi_{n+1}(x)$ are both elements of Y_n and (Y_n, ν_n) is recognizable, we deduce that $\phi_n\sigma_{n+1}(x) = \tau_n\phi_{n+1}(x)$ for any $x \in X_{\boldsymbol{\sigma}}^{(n+2)}$. Thus, one of the words in $\{\phi_n\sigma_{n+1}(x_0),$

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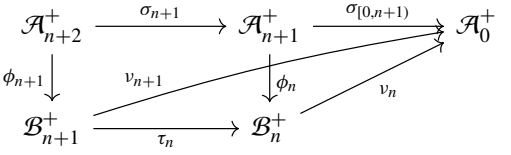
$\tau_n\phi_{n+1}(x_0)\}$ is a prefix of the other. Since $\mathcal{A}_{n+2} \subseteq \mathcal{L}(X_{\boldsymbol{\sigma}}^{(n+2)})$, we deduce that, for any $a \in \mathcal{A}_{n+2}$, one of the words in $\{\tau_n\phi_{n+1}(a), \nu_n\phi_n\sigma_{n+1}(a)\}$ is a prefix of the other. However, by (46), the words $\nu_n\tau_n\phi_{n+1}(a)$ and $\nu_n\phi_n\sigma_{n+1}(a)$ have the same length, so $\phi_n\sigma_{n+1}(a)$ must be equal to $\tau_n\phi_{n+1}(a)$ for every $n \geq 2$. This proves that $\phi_n\sigma_{n+1} = \tau_n\phi_{n+1}$ for every $n \geq 2$ and that $\boldsymbol{\phi}\colon \boldsymbol{\sigma}' \to \boldsymbol{\tau}$ is a factor.

The following commutative diagram, valid for all $n \geq 2$, summarizes the construction so far:



As shown in the diagram, we have that $\nu_n\tau_n = \nu_{n+1}$ for $n \geq 2$. Thus, $\tau_1\tau_2 \ldots \tau_n = \nu_{n+1}$, and hence $\langle\tau_1\tau_2 \ldots \tau_n\rangle \geq \langle\nu_{n+1}\rangle \geq p_n \to_{n\to\infty} \infty$. Therefore, $\boldsymbol{\tau}$ is everywhere growing. Also, by using Lemma 2.3 with $(Y_n, \nu_n) = (Y_n, \tau_1\tau_2 \ldots \tau_{n-1})$, we deduce that (Y_n, τ_{n-1}) is recognizable for every $n \geq 2$, which implies that $\boldsymbol{\tau}$ is recognizable. Finally, as each τ_n is proper, $\boldsymbol{\tau}$ is proper. □

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