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σ_4-ACTIONS ON HOMOTOPY SPHERES

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Let σ_4 denote the group of all permutations of $\{a, b, c, d\}$. It has 24 elements, partitioned into five conjugacy classes: (1) the identity 1; (2) 6 transpositions: $(ab), \ldots, (cd)$; (3) 8 elements of order 3: $(abc), \ldots, (bcd)$; (4) 6 elements of order 4: $(abcd), \ldots, (adcb)$; (5) 3 elements of order 2: $x = (ab)(cd)$, $y = (ac)(bd)$, $z = (ad)(bc)$.

In this paper, we study the differentiable actions of σ_4 on odd-dimensional homotopy spheres modelled on the linear actions, with the fixed point set of each transposition a codimension two homotopy sphere.

A simple $(2n - 1)$-knot is a differentiable embedding of a homotopy sphere K^{2n-1} into a homotopy sphere Σ^{2n+1} such that $\pi_j(\Sigma - K) = \pi_j(S^1)$ for $j < n$. For $n \geqq 3$, the isotopy type of a simple $(2n - 1)$-knot is determined by any one of its Seifert matrices, an integral matrix with

$$\det (A + \epsilon A') = \pm 1,$$

where A' denotes the transpose of A and $\epsilon = (-1)^n$ [**5**, p. 186].

For $g \in \sigma_4$, we let $F(g)$ denote the fixed point set of g under the σ_4-action.

In this paper, we will construct infinitely many distinct differentiable σ_4-actions on $(2n + 1)$-homotopy spheres Σ^{2n+1} with $F((ab))$ a simple $(2n - 1)$-knot for $n \geqq 3$. Actually, for $n = 2k$, we will show that the cobordism classes of σ_4-action on $(4k + 1)$-homotopy spheres (see the definition in Section 3 below) contains infinitely many copies of the integers **Z**.

1. In this section, we review the linear representations of σ_4. We know that σ_4 is a semi-direct product of the normal subgroup $H = \{1, x, y, z\}$ and the subgroup σ_3 generated by (ab) and (abc).

There are five inequivalent irreducible real representations for σ_4: the trivial representation I; the sign representation ϵ mapping odd permutations to -1, and even ones to 1; the 2-dimensional representation θ induced from the one on $\sigma_4/H = \sigma_3$ by sending (ab) to a reflection along an axis and (abc) to the rotation by $2\pi/3$; a 3-dimensional representation ψ permuting four vectors $\{e_1, e_2, e_3, e_4\}$ in $\mathbf{R}^3$ with the first three linearly independent and $e_1 + e_2 + e_3 + e_4 = 0$, e.g. with respect to the basis

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$\{e_1, e_2, e_3\}$ we have

$$(ab) \to \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ (abc) \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \text{ and } (ab)(cd) \to \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix},$$

[**11**, p. 77]; and $\epsilon\psi$, the tensor product of ϵ and ψ. We have the following table (compare the character table in [**9**, p. 43]):

	I	ϵ	θ	ψ	$\epsilon\psi$
dim of representation	1	1	2	3	3
codim of $F((ab))$	0	1	1	1	2
codim of $F((abc))$	0	0	2	2	2
codim of $F(x)$	0	0	0	2	2

Therefore, we have the following six types of linear action of σ_4 on $\mathbf{R}^m$ with codim $F((ab)) = 2$: (0) $(m-2)I + \epsilon + \epsilon$; (I) $(m-3)I + \epsilon + \theta$; (II) $(m-4)I + \theta + \theta$; (III) $(m-3)I + \epsilon\psi$; (IV) $(m-4)I + \epsilon + \psi$; (V) $(m-5)I + \theta + \psi$.

We refer to a σ_4-action as one of type (r) (where r is one of the above six types) if the slice representation at a fixed point [**1**, p. 171] is the same as that of the linear action of type (r). Types (0), (I), and (II) have been studied in [**7**], [**8**].

By examining the character table in [**9**, p. 43], we see the two restrictions $\epsilon\psi|\sigma_3 = \epsilon + \theta$ and $\psi|\sigma_3 = I + \theta$.

2. A σ_4-action of type (0) is a semifree involution, and a σ_4-action of type (I) or (II) is a σ_3-action. Thus the following theorem is essentially proved in [**7**] and [**8**].

THEOREM 1. *Let C be a Seifert matrix of the form*

$$C = A(A - \epsilon A')^{-1}A$$

with

$$|\det(A + \epsilon A')| = 1 = |\det(A - \epsilon A')|.$$

Then there exists a differentiable σ_4-action of type (r) $(r = 0, \mathrm{I}, \mathrm{II})$ on a homotopy sphere Σ^{2n+1}, $n \geqq 3$, such that the simple knot $(\Sigma, F((ab)))$ has C as its Seifert matrix.

Recall that σ_4 can be expressed as the semi-direct product of H and σ_3, i.e., $0 \to H \to \sigma_4 \xrightarrow{p} \sigma_3 \to 0$, with $\sigma_3 = \{1, t, t^2, w_1, w_2, w_3\}$, where $P((ab)) = w_1$, $P((abc)) = t$, etc.

THEOREM 2. *Let $r =$ III, IV, or V. Let C be a Seifert matrix of the form* $\begin{pmatrix} B & O \\ O & -B \end{pmatrix}$ *with*

$$B = A(A - \epsilon A')^{-1}A,$$

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and

$$|\det (A + \epsilon A')| = 1 = |\det (A - \epsilon A')|.$$

Then there exists a differentiable σ_4-action of type (r) on a homotopy sphere Σ^{2n+1}, $n \geqq 3$, such that the simple knot $(\Sigma, F((ab)))$ has C as its Seifert matrix.

Proof. Consider a linear action of type (III), (IV) or (V) on a standard sphere. Since $H = Z_2 + Z_2$, and codim $F(x) =$ codim $F(y) =$ codim $F(z) = 2$, the orbit space $S^{2n+1}/H = S^{2n+1}/\{x\}/\{y\}$ is again a sphere M_1. Also, we have an induced σ_3-action on $M_1 = S^{2n+1}/H$. But we have

$$0 \rightarrow Z_3 \rightarrow \sigma_3 \rightarrow Z_2 \rightarrow 0.$$

Thus $t \in \sigma_3$ gives rise to a Z_3-action on M_1. Finally, w_1 generates an involution T on $M_1/\{t\}$. That is, we have the following sequence, where each φ_i is an orbit map.

$$\begin{array}{ccccccc} S^{2n+1} & \xrightarrow{\varphi_1} & S^{2n+1}/H & \xrightarrow{\varphi_2} & S^{2n+1}/H/\{t\} & \xrightarrow{\varphi_3} & S^{2n+1}/\sigma_4 \\ & & \| & & \| & & \| \\ & & M_1 & & M_2 & & M_3 \end{array}$$

Using the explicit expression of the representations mentioned in Section 1, it is easy to see that $\varphi_1(F((ab)))$ is a $(2n - 1)$-disk in M_1, and $\varphi_2\varphi_1(F((ab)))$ is a $(2n - 1)$-disk D_1 in M_2. Also, $\varphi_1(F(x) \cup F(y) \cup F(z))$ is a union of three $(2n - 1)$-disks whose common intersection $\varphi_1(F(H))$ is a $(2n - 2)$-sphere, and $\varphi_2\varphi_1(F(x) \cup F(y) \cup F(z))$ is a $(2n - 1)$-disk D_2 in M_2. Furthermore, the $(2n - 1)$-sphere $D_1 \cup D_2$ is the fixed point set of the involution T on M_2. We write $D_1' = \varphi_3(D_1)$ and $D_2' = \varphi_3(D_2)$ in $M_3 = S^{2n+1}/\sigma_4$.

If the action is of type (III) or (IV), then codim $F((abc)) = 2$. Thus M_2 is a sphere, and so is M_3. But when the action is of type (V), then codim $F((abc)) = 4$, and M_2 is no longer a manifold.

Let W be a small open neighborhood of D_2' in M_3. Then we define $U = (\varphi_3 \circ \varphi_2 \circ \varphi_1)^{-1}(W)$.

From the principal orbit type theorem [**1**, p. 179], we know that we may embed a disk D^{2n+1} in M_3 near D_1' such that

$$D^{2n+1} \cap \varphi_3\varphi_2\varphi_1(F(g)) = \emptyset \quad \text{for all } g \in \sigma_4 \quad \text{and}$$
$$D^{2n+1} \cap W = \emptyset.$$

We construct a simple knot K^{2n-1} in D^{2n+1} with A as its Seifert matrix, [**3**, p. 255-257]. We then take the connected sum of K^{2n-1} and D_1' in M_3 (the connected sum operation # takes place away from $\varphi_3\varphi_2\varphi_1(F(g))$ and W). Now we use $D_2' \cup D_1' \# K^{2n-1}$ as the branched point set to construct a 2-fold branched covering

$$\Sigma_2 \xrightarrow{\beta_3} M_3$$

[**2**]. The branched covering transformation gives us an involution T' on Σ_2.

Since a neighborhood of $\varphi_3\varphi_2\varphi_1(F((abc)))$ is disjoint from K^{2n-1}, we may identify $\varphi_2\varphi_1(F((abc)))$ with

$$\beta_3^{-1}\varphi_3\varphi_2\varphi_1(F((abc))).$$

We construct a 3-fold branched covering

$$\Sigma_1 \xrightarrow{\beta_2} \Sigma_2$$

with $\varphi_2\varphi_1(F((abc)))$ as branched point set. In [**8**], we showed that Σ_1 is a homotopy sphere, and that we may lift the involution T' to Σ_1, which together with the branched covering transformation Z_3 on Σ_1 gives us a σ_3-action Σ_1

As in the preceding paragraph, we may identify $(\varphi_3\varphi_2)^{-1}(W)$ with $(\beta_3\beta_2)^{-1}(W)$. We then construct an H-action on a homotopy sphere Σ with an orbit space Σ_1 as follows: we first use $\varphi_1(F(z) \cup F(y))$ as branched point set to construct a 2-fold branched cover

$$\Sigma' \xrightarrow{\alpha_2} \Sigma_1$$

with associated involution G_2, then use $\alpha_2'\varphi_1(F(x))$ as branched point set to construct a 2-fold branched cover

$$\Sigma \xrightarrow{\alpha_1} \Sigma'$$

with involution G_1. We define $\beta_1 = \alpha_2 \circ \alpha_1$. It is easy to see that the H-action on

$$U' = \beta_1^{-1}(\varphi_3\varphi_2)^{-1}(W)$$

is equivalent to the H-action on

$$U = (\varphi_3\varphi_2\varphi_1)^{-1}(W).$$

The map of the complement $\Sigma - U' \to \Sigma' - \varphi(U')$ is a regular covering. According to [**1**, p. 64-67], we may lift G_2 to $\Sigma - U'$ inducing an H'-action on $\Sigma - U'$, where H' is a semidirect product of Z_2 and Z_2. Since the lifting is unique [**1**, p. 66], and we may take $H|\partial U = H'|\partial U$, then $H' = H$.

Now we have an H-action on Σ, and a σ_3-action on $\Sigma_1 = \Sigma/H$. We then use the argument in the preceding paragraph to lift the σ_3-action via the regular covering

$$\Sigma - U' \xrightarrow{\beta_1} \Sigma_1 - \beta_1(U'),$$

thus inducing a J-action on $\Sigma - U'$, where J is defined by $0 \to H \to J \to \sigma_3 \to 0$. Since

$$U' = (\beta_3\beta_2\beta_1)^{-1}(D_2')$$

is σ_4-equivariantly diffeomorphic to

$$U = (\varphi_3\varphi_2\varphi_1)^{-1}(D_2'),$$

then from the uniqueness of the lifting [**1**, p. 66], we conclude that $J = \sigma_4$ and we thus obtain a σ_4-action on Σ. We have the following sequence, with $\beta_1 = \alpha_2\alpha_1$:

$$\begin{array}{ccccccccc} \Sigma & \xrightarrow{\alpha_1} & \Sigma/\{x\} & \xrightarrow{\alpha_2} & \Sigma/H & \xrightarrow{\beta_2} & \Sigma_1/\{t\} & \xrightarrow{\beta_3} & \Sigma/\sigma_4. \\ & & \| & & \| & & \| & & \| \\ & & \Sigma' & & \Sigma_1 & & \Sigma_2 & & M_3 \end{array}$$

From the construction of the σ_1-action, we see that $\beta_1(F((ab)))$ is a disk, and $(\Sigma_1, \beta_1(F(x)) \cup \beta_1(F((ab))))$ has B as its Seifert matrix [**6**, p. 52]. Since α_1 (respectively α_2) is a branched covering map with a trivial knot $F(x)$ (respectively $\alpha_1(F(y) \cup F(z))$) as its branched point set, and x reverses the orientation of $F((ab))$, we conclude that the knot $(\Sigma, F((ab)))$ has $C = \begin{pmatrix} B & O \\ O & -B \end{pmatrix}$ as its Seifert matrix.

As in [**7**] or [**8**] we have the following corollary.

COROLLARY. *For each* $\gamma = 0, \mathrm{I}, \dots, \mathrm{V}$; *and* $n \geqq 3$, *there exist infinitely many* σ_4-*actions on* $(2n + 1)$-*homotopy spheres* (*having the same orbit space*) *of type* (r).

3. Let $I = [0, 1]$ denote the unit interval. We call two σ_4-actions γ_1, γ_2 on a homotopy sphere Σ^{2n+1} *cobordant* if there exists a σ_4-action γ on $\Sigma \times I$ such that

$$\gamma|\Sigma \times 0 = \gamma_1 \quad \text{and} \quad \gamma|\Sigma \times 1 = \gamma_2.$$

The cobordance is an equivalence relation. Thus we have the notion of cobordism classes.

THEOREM 3. *For* $k \geqq 2$, *the cobordism classes of* σ_4-*actions on* $(4k + 1)$-*homotopy spheres contains infinitely many copies of* $\mathbf{Z}$ *for each type* (r), $r = 0, \mathrm{I}, \dots, \mathrm{V}$.

Proof. A σ_4-action of type (0) is an involution, hence this case of the theorem follows from that in [**7**, (4.3)].

For an action of type (I) or (II), which is just a σ_3-action, we restrict the action to $\mathbf{Z}_2 \subseteq \sigma_3$. The theorem follows from Theorem 1 above and [**7**, (4.3)].

For actions of type (III), (IV), (V), we cannot use the above proof directly since the Seifert matrices $C = \begin{pmatrix} B & O \\ O & -B \end{pmatrix}$ are null-cobordant. Instead, we consider the involution induced by the σ_3-action on Σ^{2n+1}/H.

Let $\mathbf{Z}_{(2)}$ denote the ring of 2-adic integers. It follows from [**1**, pp. 122-124] that both $F((ab))$ and the orbit space $\Sigma/\{x\}$ are $\mathbf{Z}_{(2)}$-spheres, and so therefore is $\Sigma/\{x\}/\{y\} = \Sigma/H$. Let β: $\Sigma \to \Sigma/H$ denote the orbit map. By restricting to $\mathbf{Z}_2 \subseteq \sigma_3$, generated by (ab), we have an involution T on Σ/H with fixed point set

$$F(T) = \beta(F(x)) \cup \beta(F((ab))).$$

With the notion of Seifert manifolds and Seifert matrices for the $\mathbf{Z}_{(2)}$-knot $(\Sigma/H, F(T))$ [**7**, Section 4], we may use the arguments in [**4**], [**10**, (6.6)] and [**7**, (4.2)] to show that the $\mathbf{Z}_{(2)}$-knot cobordism classes can be mapped surjectively to $C_\epsilon(\mathbf{Z}_{(2)})$ (see [**7**, (4.2)]). We note that for the action constructed in Theorem 2 above, the corresponding knot $(\Sigma/H, F(T))$ has B as its Seifert matrix, and the $\mathbf{Z}_{(2)}$-knot $(\Sigma/H/T, F(T))$ has A as its Seifert matrix [**6**]. If two σ_4-actions γ_1, γ_2 are cobordant, then the corresponding $\mathbf{Z}_{(2)}$-knots $(\Sigma/H/T_1, F(T_1))$, $(\Sigma/H/T_2, F(T_2))$ are also cobordant. Levine [**4**, p. 243] constructed an infinite sequence of linearly independent elements of $C_{+1}(\mathbf{Z}_{(2)})$:

$$A_k = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & -k & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} k = 1, 2, \ldots,$$

that is, for each k, the set of direct sums $\bigoplus_n A_k$ (n copies of A_k) generates a copy of $\mathbf{Z}$ in $C_{+1}(\mathbf{Z}_{(2)})$. Since

$$|\det (A_k + \epsilon A_k')| = 1 = |\det (A_k - \epsilon A_k')|,$$

we may use $\bigoplus_n A_k$ as A in Theorem 2 to construct a σ_4-action. The corresponding $\mathbf{Z}_{(2)}$-knot $(\Sigma/H/T, F(T))$ has $\bigoplus_n A_k$ as its Seifert matrix.

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