Adv. Appl. Prob. **21**, 485–487 (1989)
Printed in N. Ireland
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PERIODIC QUEUES IN HEAVY TRAFFIC

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Abstract

An analytic approach to the diffusion approximation in queueing due to Burman (1979) is applied to the $M(t)/G/1/\infty$ queueing system with periodic Poisson arrivals. We show that under heavy traffic the virtual waiting time process can be approximated by a certain Wiener process with reflecting barrier at 0.

POISSON ARRIVALS; DIFFUSION APPROXIMATION

Introduction

An important property of periodic queues is the absence of a steady state in its usual sense. As a matter of fact, a special 'periodic stationary regime' exists. Consider for example the virtual waiting time process $W(t)$ in the $M(t)/G/1/\infty$ queue with FIFO discipline, Poisson arrival process with periodic intensity λ_t (without loss of generality we can assume that the period equals 1) and general service time distribution function $B(x)$ with finite mean β_1 and variance $\sigma^2 = \beta_2 - \beta_1^2$. For example, λ_t could be $\lambda + \beta \sin 2\pi t$ with $\lambda > \beta \geqq 0$. If $\Lambda_t = \int_0^t \lambda_u \, du$, $\lambda = \Lambda_1$, $\rho = \lambda\beta_1 < 1$, then there exists the family $H_t(x)$ of distribution functions, periodic functions of t (i.e. $H_{t+1}(x) = H_t(x)$), such that $\lim_{t\to\infty} [P(W(t) < x) - H_t(x)] = 0$. This fact is clearly seen from Figure 1 where the dependence of $EW(t)$ on t is given in the case $\lambda_t = 0{\cdot}5(1 + \sin 2\pi t)$, and $B(x)$ is the uniform distribution $[0, 2]$ (results were obtained using an Atari 130 XE). This 'periodic stationarity' imposes essential mathematical difficulties on the analysis of queues. Determination of the functions $H_t(x)$ and even determination of the mean value $\int_0^1 dt \int_0^\infty x \, dH_t(x)$ can be made only with the help of a computer. To simplify the problem in practice, the 'principle of the mean' is usually used: a periodic queue is approximated by the corresponding stationary queue with arrival intensity $\lambda = \lim_{t\to\infty} \frac{1}{t} \int_0^t \lambda_u \, du = \Lambda_1$. Of course we have to find conditions when this approximation is reasonable.

In this note we shall show that this principle holds in heavy traffic in the sense of convergence of a scaled non-stationary process to a stationary process (which is in fact a reflected Brownian motion). To prove this we turn time heterogeneity into space heterogeneity (via a supplementary variable) which allows us to use the methodology of Burman [1].

Main result

Let us consider a sequence of $M(t)/G/1/\infty$ queues (indexed by a parameter n, although we shall usually omit it) with arrival rates $\lambda_t^{(n)}$. As a matter of convenience we suppose that the distribution of service time $B(x)$ does not depend on n.

Received 29 March 1988; revision received 27 January 1989.

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https://doi.org/10.2307/1427175 Published online by Cambridge University Press

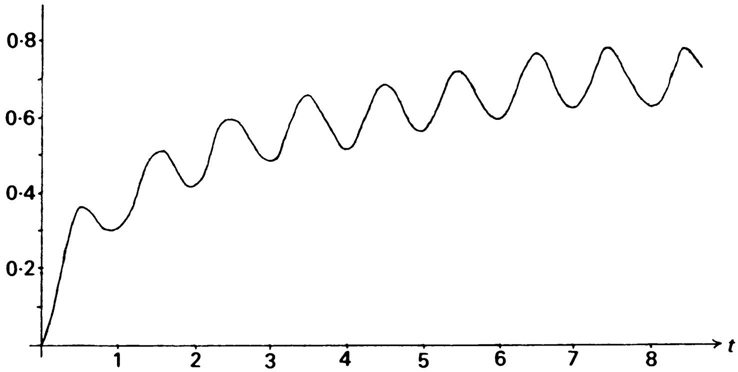

Figure 1

Theorem. If as $n\to\infty$, $\lambda^{(n)}=\int_0^1\lambda_t^{(n)}\,dt$ tends to $1/\beta_1-0$ so that $\sqrt{n}(\rho^{(n)}-1)$, where $\rho^{(n)}=\lambda^{(n)}\beta_1$, tends to $-m$, then the scaled processes $W(nt)/\sqrt{n}$ converge (in the sense of convergence of finite-dimensional distributions) to a Brownian motion with reflecting barrier at the origin with infinitesimal mean m and infinitesimal variance $\beta_2/(2\beta_1)$.

Proof. The main idea of the proof consists in turning time heterogeneity into space heterogeneity. To do this let us denote by $\tau(t)$ the fractional part of t. The process $(W(t),\,\tau(t))$ is a time-homogeneous Markov process with state space $R_+\times[0,1]$, but with transition characteristics which depend on the second coordinate τ of a point $(x,\,\tau)$ of the state space.

Another problem now arises. Under heavy traffic only the first coordinate of the process $(W(t),\,\tau(t))$ will converge to a diffusion process. But this difficulty can be removed by using the method of proving functional limit theorems in queueing theory due to Burman [1].

The infinitesimal generator of the process $(W(t),\,\tau(t))$ is expressed as follows:

$$Af(x,\tau)=\lambda_\tau\int_0^\infty[f(x+u,\tau)-f(x,\tau)]\,dB(u)+f'_\tau(x,\tau)-f'_x(x,\tau),\quad\text{if}\quad x>0,\quad 0<\tau<1,$$

$$Af(0,\tau)=\lambda_\tau\int_0^\infty[f(u,\tau)-f(0,\tau)]\,dB(u)+f'_\tau(0,\tau),\quad\text{if}\quad 0<\tau<1,$$

and operates on functions $f(x,\,\tau)$ satisfying the boundary condition $f(x,1)=f(x,0)$.

For the generator A_n of the scaled process $(W(nt)/\sqrt{n},\,\tau(nt))$ these formulas become:

$$\begin{aligned}A_nf(x,\tau)&=n\lambda_\tau\int_0^\infty\Big[f\Big(x+\frac{1}{\sqrt{n}}u,\tau\Big)-f(x,\tau)\Big]\,d\tilde{B}(u)+nf'_\tau(x,\tau)-\sqrt{n}f'_x(x,\tau)\\&=\sqrt{n}\lambda_\tau f'_x(x,\tau)\beta_1+\tfrac{1}{2}\lambda_\tau\beta_2f''_{xx}(x,\tau)+nf'_\tau(x,\tau)-\sqrt{n}f'_x(x,\tau)+o(1),\end{aligned}$$

if $x>0$, $0<\tau<1$; and

$$\begin{aligned}A_nf(0,\tau)&=n\lambda_\tau\int_0^\infty\Big[f\Big(\frac{u}{\sqrt{n}},\tau\Big)-f(0,\tau)\Big]\,dB(u)+nf'_\tau(0,\tau)\\&=\sqrt{n}\lambda_\tau f'_x(0,\tau)\beta_1+\tfrac{1}{2}\lambda_\tau\beta_2f''_{xx}(0,\tau)+nf'_\tau(0,\tau).\end{aligned}$$

The boundary condition does not change.

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For any twice continuously differentiable function $f(x)$ define

$$f_n(x, \tau) = f(x) + \frac{1}{\sqrt{n}} f'(x) g(\tau) + \frac{1}{n} f''(x) h(\tau),$$

where the functions $g(\tau)$, $h(\tau)$ will be defined below. For such functions we have;

$$A_n f_n(x, \tau) = \sqrt{n} f'(x)[\lambda_\tau \beta_1 - 1 + g'(\tau)] + f''(x)[(\lambda_\tau \beta_1 - 1) g(\tau) + \tfrac{1}{2}\lambda_\tau \beta_2 + h'(\tau)] + o(1),$$

$$A_n f_n(0, \tau) = \sqrt{n} f'(0)[\lambda_\tau \beta_1 + g'(\tau)] + f''(0)[\lambda_\tau \beta_1 g(\tau) + \tfrac{1}{2}\lambda_\tau \beta_2 + h'(\tau)] + o(1).$$

The boundary condition becomes;

$$g(1) = g(0), \quad h(1) = h(0).$$

From the above it is easy to see that $A_n f_n$ can converge to a limit which does not depend on τ only if the functions $g(\tau)$, $h(\tau)$ satisfy the equations:

$$\lambda_\tau \beta_1 - 1 + g'(\tau) = \frac{1}{\sqrt{n}} c_1, \quad (\lambda_\tau \beta_1 - 1) g(\tau) + \frac{\lambda_\tau \beta_2}{2} + h'(\tau) = c_2,$$

where c_1, c_2 are to be determined from the boundary conditions.

The first equation yields

$$g(\tau) = (\tau - \Lambda_\tau \beta_1) + \frac{1}{\sqrt{n}} c_1 \tau + g(0).$$

By the boundary condition $g(1) = g(0)$ we get

$$c_1 = -\sqrt{n}(1 - \rho^{(n)}) = -m + o(1),$$

so $g(\tau) = \rho\tau - \Lambda_\tau \beta_1 + g(0)$. This allows us to obtain

$$h(\tau) = c_2 \tau + \tfrac{1}{2}\Lambda_\tau \beta_2 + \int_0^t (1 - \lambda_u \beta_1) g(u)\, du + h(0).$$

The boundary condition $h(1) = h(0)$ gives

$$C_2 = \beta_2/(2\beta_1) + o(1),$$

Using the functions $g(\tau)$, $h(\tau)$ we get

$$\lim_{n\to\infty} A_n f_n(x, \tau) = -m f'(x) + \frac{\beta_2}{2\beta_1} f''(x) = A_0 f(x).$$

The generator A_0 corresponds to a diffusion process with infinitesimal mean m and infinitesimal variance $\beta_2/(2\beta_1)$.

Similar analysis of behaviour at the boundary $x = 0$ implies that functions $f(x)$ from the domain of the generator A_0 have to satisfy the condition $f(0) = 0$. This means that the generator A_0 corresponds to diffusion with a reflecting barrier at the origin.

To complete the proof it is sufficient to refer to results in [1] and [2] which imply that the above allow us to guarantee the desired convergence.

References

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