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CERTAIN EXTENSIONS OF THE MEHLER FORMULA

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1. **Introduction.** For the Hermite polynomials $H_n(z)$ defined by

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(z) = \exp(2zt - t^2) \tag{1}$$

it is easy to see from the Rodrigues formula that

$$H_n(ax) = (-1)^n a^{-n} \exp(a^2x^2) D_x^n \exp(-a^2x^2), \tag{2}$$

where, as usual, $D_z = d/dz$.

In recent papers ([1], [2]) Carlitz has proved the following formulae:

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{t^k}{k!} H_{k+n}(x)H_{k+m}(y) &= (1-4t^2)^{-(m+n+1)/2} \exp\left[\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right] \\ &\times \sum_{r=0}^{\min(m,n)} 2^{2r} r! \binom{m}{r}\binom{n}{r} t^r H_{m-r}\left(\frac{x-2yt}{\sqrt{1-4t^2}}\right) H_{n-r}\left(\frac{y-2xt}{\sqrt{1-4t^2}}\right),\end{aligned} \tag{3}$$

$$\begin{aligned}\sum_{n_1,\ldots,n_k=0}^{\infty} H_{n_1+\cdots+n_k}(x)H_{n_1}(y_1)\ldots H_{n_k}(y_k) \frac{u_1^{n_1}\ldots u_k^{n_k}}{n!\ldots n_k!} \\ = (1-4\sum u_i^2)^{-1/2} \exp\left[x^2 - \frac{(x-2\sum y_i u_i)^2}{1-4\sum u_i^2}\right],\end{aligned} \tag{4}$$

where, on the right-hand side of (4), the range of each summation is from $i=1$ to $i=k$ $(k=1, 2, \ldots)$.

Both (3) and (4) provide elegant generalizations of the bilinear generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)H_n(y) = (1-4t^2)^{-1/2} \exp\left[\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right], \tag{5}$$

which is well known as Mehler formula [3, p. 198]. The object of the present note is to show how effectively certain operational techniques may be applied to give easy and direct proofs of (3) and (4). We first derive here the operational formula

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$$\exp(tD_xD_y)\{\exp(-a^2x^2-b^2y^2)\} = (1-4a^2b^2t^2)^{-1/2}\exp\left[-a^2x^2-\frac{(by-2a^2bxt)^2}{1-4a^2b^2t^2}\right], \tag{6}$$

which is believed to be new, and then apply it to prove (3). Formula (4) is shown to be an immediate consequence of Glaisher's operational formula

$$\exp(tD_x^2)\{\exp(-x^2)\} = (1+4t)^{-1/2}\exp\left[\frac{-x^2}{1+4t}\right]. \tag{7}$$

In what follows we shall also use the known results

$$D_x^rH_n(x) = 2^r r!\binom{n}{r}H_{n-r}(x) \tag{8}$$

$$\exp(tD_x)f(x) = f(x+t). \tag{9}$$

2. To prove (6), we replace x, y in the Mehler formula (5) by ax and by respectively. Making use of the operational formula (2) we are led at once to (6).

In order to prove (3), we note that

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{t^k}{k!}H_{k+n}(x)H_{k+m}(y)\\
&= (-1)^{m+n}\exp(x^2+y^2)D_x^nD_y^m\exp(tD_xD_y)\{\exp(-x^2-y^2)\}\\
&= (1-4t^2)^{-1/2}\exp(x^2+y^2)(-D_x)^n(-D_y)^m\exp\left[-x^2-\frac{(y-2xt)^2}{1-4t^2}\right]\\
&= (1-4t^2)^{-(m+1)/2}\exp(x^2+y^2)\\
&\quad\times(-1)^nD_x^n\left[H_m\left(\frac{y-2xt}{\sqrt{1-4t^2}}\right)\exp\left\{-\frac{(x-2yt)^2}{1-4t^2}-y^2\right\}\right]\\
&= (1-4t^2)^{-(m+1)/2}\exp(x^2).\\
&\quad\times\sum_{r=0}^{n}(-1)^n\binom{n}{r}D_x^{n-r}\exp\left[-\frac{(x-2yt)^2}{1-4t^2}\right]D_x^rH_m\left(\frac{y-2xt}{\sqrt{1-4t^2}}\right)\\
&= (1-4t^2)^{-(m+n+1)/2}\exp\left[\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]\\
&\quad\times\sum_{r=0}^{\min(m,n)}2^{2r}r!\binom{m}{r}\binom{n}{r}t^rH_{n-r}\left(\frac{x-2yt}{\sqrt{1-4t^2}}\right)H_{m-r}\left(\frac{y-2xt}{\sqrt{1-4t^2}}\right)
\end{aligned}$$

which evidently proves (3).

Further, by using (2) and (1), the left-hand side of (4) can be transformed into

$$\begin{aligned}
&\exp(x^2)\exp\left[-\left(\sum u_i^2\right)D_x^2-2\left(\sum u_iy_i\right)D_x\right]\exp(-x^2)\\
&\qquad= \exp(x^2)\exp\left[-\left(\sum u_i^2\right)D_x^2\right]\exp\left[-\left(x-\sum 2u_iy_i\right)^2\right],
\end{aligned}$$

which, in view of (7), leads us to the desired result (4).

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