Can. J. Math., Vol. XXVI, No. 5, 1974, pp. 1245–1256

UNIFORMLY LIPSCHITZIAN FAMILIES OF TRANSFORMATIONS IN BANACH SPACES

K. GOEBEL, W. A. KIRK, AND R. L. THELE

1. Introduction. The observations of this paper evolved from the concept of 'asymptotic nonexpansiveness' introduced by two of the writers in a previous paper [**10**]. Let X be a Banach space and $K \subseteq X$. A mapping $T : K \to K$ is called *asymptotically nonexpansive* if for each $x, y \in K$

$$||T^i(x) - T^i(y)|| \leqq k_i||x - y||, \qquad i = 1, 2, \ldots,$$

where $\{k_i\}$ is a fixed sequence of real numbers such that $k_i \to 1$ as $i \to \infty$. It is proved in [**10**] that if K is a bounded closed and convex subset of a uniformly convex space X then every asymptotically nonexpansive mapping $T : K \to K$ has a fixed point. This theorem generalizes the fixed point theorem of Browder-Göhde-Kirk [**2**; **12**; **16**] for nonexpansive mappings (mappings T for which $||T(x) - T(y)|| \leqq ||x - y||$, $x, y \in K$) in a uniformly convex space. (A generalization along similar lines also has been obtained by Edelstein [**4**].)

The theorem for asymptotically nonexpansive mappings was subsequently generalized by Goebel and Kirk in [**11**] in which a fixed point theorem is obtained in uniformly convex spaces for continuous mappings $T : K \to K$ (K as above) which have the property that there exists an integer $N \geqq 1$ such that $||T^i(y) - T^i(y)|| \leqq \gamma||x - y||$, $i \geqq N$, where $\gamma > 1$ is a fixed constant sufficiently near 1 (see the example, Section 4). The argument for this theorem provides the general technique for our proof of the main theorem of Section 2. This latter theorem is for a left reversible semigroup $\mathscr{T}$ of transformations of a given convex set K into itself, and it is shown that under appropriate assumptions on the space, weaker than uniform convexity, a point $x_0 \in K$ exists with the property that $x_0 = T(x_0)$ for all T in some right ideal J. The principal assumption on the semigroup is that it possess a right ideal each of whose mappings has Lipschitz constant sufficiently near 1, and this assumption is strong enough to ensure that x_0 is fixed under all continuous mappings in the semigroup.

Section 3 contains a few observations about semigroups of asymptotically nonexpansive mappings, among them a generalization of a theorem of R. DeMarr [**3**].

We now give some definitions. In what follows we consider a semigroup $\mathscr{T}$ of transformations of $U \to U$ where U is a subset of the Banach space X, and we

Received April 19, 1973 and in revised forms, April 8, 1974 and June 25, 1974. This research was supported by NSF Grant GP-18045.

assume that $\mathscr{T}$ is *left reversible*, that is, every two right ideals in $\mathscr{T}$ have non-empty intersection. For $T \in \mathscr{T}$, let $||T||_L$ denote the Lipschitz norm of T relative to U:

$$||T||_L = \sup\{||T(x) - T(y)||/||x - y|| : x, y \in U, x \neq y\}.$$

The *modulus of convexity* of X is the function $\delta : [0, 2] \to [0, 1]$ defined by

$$\delta(\epsilon) = \inf\{1 - ||\tfrac{1}{2}(x + y)|| : x, y \in X, ||x|| \leqq 1, ||y|| \leqq 1, ||x - y|| \geqq \epsilon\}.$$

Let $\epsilon_0 = \epsilon_0(X) = \sup\{\epsilon : \delta(\epsilon) = 0\}$. The number ϵ_0 is called the *characteristic of convexity* of X and it is known (see Goebel [**8**]) that $\epsilon_0 = 0$ if and only if X is uniformly convex, while if $\epsilon_0 < 2$ then X is uniformly non-square [**15**] and isomorphic to a uniformly convex space [**6**] (hence reflexive).

It is also known [**14**; **20**] that the function δ is strictly increasing on $[\epsilon_0, 2]$ and continuous on $[0, 2)$, and moreover (cf. [**22**; **23**]):

(*) $\quad ||x|| \leqq d, \quad ||y|| \leqq d, \quad ||x - y|| \geqq \epsilon \Rightarrow ||\tfrac{1}{2}(x + y)|| \leqq (1 - \delta(\epsilon/d))d.$

We now list some properties derived from the modulus of convexity of X which will be used later.

For $\epsilon \in [0, 2]$ we define

$$f(\epsilon) = \sup\{||x + y|| : ||x|| \leqq 1, \quad ||y|| \leqq 1, \quad ||x - y|| \geqq \epsilon\}.$$

Then $f(\epsilon) = 2(1 - \delta(\epsilon))$ so
(1) f is continuous on $[0, 2)$, and
(2) f is strictly decreasing on $[\epsilon_0, 2]$.

It is easily seen that $f \circ f(\epsilon) = f^2(\epsilon) \geqq \epsilon$. Moreover, for $||x|| \leqq 1$, $||y|| \leqq 1$ we have $f(||x + y||) \geqq ||x - y||$ and so if $\epsilon > \epsilon_0$ it follows that $f^2(\epsilon) \leqq \epsilon$, yielding

(3) $\quad f^2(\epsilon) = \epsilon$ if $\epsilon \in (\epsilon_0, 2]$.

Obviously,

(4) $\quad f^2(\epsilon) = f(2)$ if $\epsilon \in [0, \epsilon_0]$.

Now if $\epsilon_0 < 1$, then continuity of f implies the existence of $r \in (\epsilon_0, 1)$ such that $f(r) = 2r$. Thus if $1 \leqq k < 1/r$, then since $1/k > r > \epsilon_0$ we have by (2)

$$f(1/k) < f(r) = 2r < 2/k.$$

Using (2) and (3), $2/k > \epsilon_0$ implies $f(2/k) < f^2(1/k) = 1/k$, so

(5) $\quad kf(2/k) < 1, \qquad k \in [1, 1/r).$

We also note that since $\lim_{\epsilon \to 2^-} \delta(\epsilon) = 1 - \epsilon_0/2$ it follows that

(6) $\quad \lim_{\epsilon \to 2^-} f(\epsilon) = \epsilon_0.$

Finally, we remark that we use the symbol $S(x; \rho)$, $x \in X$, $\rho > 0$, to denote the closed spherical ball $\{y \in X : ||x - y|| \leqq \rho\}$.

Acknowledgment. We are indebted to Teck-Cheong Lim for a communication which prompted our reasoning in the final paragraph of the proof of Theorem 2.1, and thereby substantially strengthened the conclusion in our original version of that theorem.

2. Asymptotically lipschitzian semigroups. The following results are formulated for Banach spaces X for which $\epsilon_0(X) < 1$. Thus these results hold for a class of spaces which includes all uniformly convex spaces.

Our principal result is Theorem 2.1. Condition (ii) in this theorem also offers slightly greater generality over the usual assumption that the transformations map K into K. This weaker formulation results in no complication.

THEOREM 2.1. *Let K be a nonempty bounded closed convex subset of a Banach space X for which $\epsilon_0 = \epsilon_0(X) < 1$. Then there exists a constant $\gamma > 1$ (depending on X) such that if $\mathscr{T} = \{T_\alpha : \alpha \in A\}$ is a left reversible semigroup of transformations of U into U, $K \subseteq U$, satisfying:*

(i) $||T||_L \leqq k < \gamma$ *for all T in some right ideal $J_1 \subseteq \mathscr{T}$,*

(ii) *for each $\epsilon > 0$, $\text{dist}(T(x), K) \leqq \epsilon$ for all T in some right ideal $J_2 \subseteq \mathscr{T}$, then there exists a point $x_0 \in K$ such that for some right ideal $J \subseteq \mathscr{T}$, $x_0 = T(x_0)$ for all $T \in J$. Further, if all the mappings of $\mathscr{T}$ are continuous, then $T(x_0) = x_0$ for all $T \in \mathscr{T}$.*

We should remark that if $\mathscr{T}$ consists of a single mapping, then as an immediate consequence of Theorem 2.1, Theorem 1 of [**11**] is generalized from the context of uniformly convex spaces to the wider class of spaces considered here. The referee points out that such a generalization is also immediate from a recent result of Per Enflo. Enflo has shown [**6**] that a space X for which $\epsilon_0(X) < 2$ may be renormed so as to be uniformly convex and so that the function f is altered by an arbitrarily small amount. This fact may be used to reduce the problem quickly to the uniformly convex setting. Although such a reduction would also suffice for the proof of Theorem 2.1, we give a direct proof below in the $\epsilon_0(X) < 1$ setting.

As noted in [**11**], if there is a metric s on K such that

$$(**) \qquad \alpha||x - y|| \leqq s(x, y) \leqq \beta||x - y||, \qquad x, y \in K$$

and if $T : K \to K$ is nonexpansive with respect to s then

$$||T^i(x) - T^i(y)|| \leqq (1/\alpha)s(T^i(x), T^i(y)) \leqq (1/\alpha)s(x, y) \leqq (\beta/\alpha)||x - y||.$$

This observation and Theorem 2.1 yield:

COROLLARY 2.2. *Let X be a Banach space for which $\epsilon_0(X) < 1$ and suppose $K \subseteq X$ is nonempty, bounded, closed, and convex. Then there exists a constant $\gamma > 1$ such that if $\mathscr{T}$ is a left reversible semigroup of transformations of K into K and if each mapping $T \in \mathscr{T}$ is nonexpansive with respect to a metric $s(x, y)$ on K satisfying (**) where $\beta/\alpha < \gamma$, then $\mathscr{T}$ has a common fixed point in K.*

Proof of Theorem 2.1. Choose $\gamma = 1/r$ in (5). For $\alpha \in A$, $y \in K$ let

$$\mathscr{T}_\alpha = \{T_\alpha \circ T : T \in \mathscr{T}\}, \qquad \mathscr{T}_\alpha(y) = \{T_\alpha \circ T(y) : T \in \mathscr{T}\}.$$

Fix $y \in K$ and let

$$R_y = \{\rho > 0 : \mathscr{T}_\alpha(y) \subset S(x; \rho) \text{ for some } \alpha \in A,\ x \in K\}.$$

Let $\rho_0 = \rho_0(y) = \inf R_y$. ($R_y \neq \emptyset$ because R_y contains the diameter of K.)

Now for $\epsilon > 0$ let

$$C_\epsilon = \bigcup_{\alpha \in A} \left(\bigcap_{T \in \mathscr{T}_\alpha} S(T(y); \rho_0 + \epsilon) \right).$$

Then C_ϵ is nonempty for each $\epsilon > 0$, and since left reversibility of $\mathscr{T}$ implies that the family

$$\left\{ \bigcap_{T \in \mathscr{T}_\alpha} S(T(y); \rho_0 + \epsilon) : \alpha \in A \right\}$$

is directed by set inclusion, C_ϵ is convex. Since X is reflexive the sets $\bar{C}_\epsilon$ comprise a family of weakly compact sets with the finite intersection property, so there exists a point $z = z(y) \in K$ such that

$$z \in C = \bigcap_{\epsilon > 0} (\bar{C}_\epsilon \cap K).$$

Note that if $\rho_0 = 0$ then the conclusion of the theorem follows, for if $\epsilon > 0$ then $\alpha \in A$ may be chosen so that $\mathscr{T}_\alpha$ is uniformly lipschitzian with Lipschitz constant k, and so that

$$||z - T(y)|| \leqq \epsilon, \qquad T \in \mathscr{T}_\alpha;$$

whence for any $T \in \mathscr{T}_\alpha$,

$$\begin{aligned} ||z - T(z)|| &\leqq ||z - T^2(y)|| + ||T^2(y) - T(z)|| \\ &\leqq \epsilon + k||T(y) - z|| \\ &\leqq \epsilon(1 + k). \end{aligned}$$

Thus we may assume $\rho_0 > 0$, or $d(z) = 0$ where for each point $w \in K$,

$$d(w) = \inf_{\alpha \in A} \{\sup ||w - T(w)|| : T \in \mathscr{T}_\alpha\}.$$

Since the case $d(w) = 0$ is treated in the final paragraph, assume $d(z) > 0$ and let $\epsilon > 0$ satisfy $\epsilon < d(z)$. Choose $\alpha \in A$ so that

(7) $\quad ||z - T_\alpha(z)|| \geqq d(z) - \epsilon, \quad \operatorname{dist}(T_\alpha(z), K) < \epsilon, \quad \text{and} \quad ||T_\alpha||_L \leqq k.$

(Note that the last two choices are true for all T in some right ideal J, and since for each $\alpha \in A$, $\mathscr{T}_\alpha(z) \subsetneqq S(z; d(z) - \epsilon)$ there exists some $T_\alpha \in J$ for which the first choice holds.)

By definition of ρ_0 there exists $\beta \in A$ such that for $T \in \mathscr{T}_\beta$,

$$||z - T(y)|| \leqq \rho_0 + \epsilon.$$

Since $T_\alpha \circ T_\beta = T_\delta$ for some $\delta \in A$, by left reversibility of $\mathscr{T}$ there exists $\mu \in A$ such that $T_\mu \in \mathscr{T}_\delta \cap \mathscr{T}_\beta$. If $T \in \mathscr{T}_\mu$ then there exists $\tilde{T} \in \mathscr{T}$ such that $T = T_\delta \circ \tilde{T}$, and thus

$$(8) \qquad \begin{aligned} ||T_\alpha(z) - T(y)|| &= ||T_\alpha(z) - T_\alpha \circ T_\beta \circ \tilde{T}(y)|| \\ &\leqq k||z - T_\beta \circ \tilde{T}(y)|| \\ &\leqq k(\rho_0 + \epsilon). \end{aligned}$$

Also, since $T \in \mathscr{T}_\mu$ implies $T \in \mathscr{T}_\beta$, we have

$$(9) \qquad ||z - T(y)|| \leqq \rho_0 + \epsilon.$$

Letting $m = (z + T_\alpha(z))/2$ we have by (7), (8), (9), and property (*) of the modulus of convexity,

$$(10) \qquad ||m - T(y)|| \leqq \tfrac{1}{2}k(\rho_0 + \epsilon)f\left(\frac{d(z) - \epsilon}{k(\rho_0 + \epsilon)}\right), \qquad T \in \mathscr{T}_\mu.$$

Moreover, since $z \in K$ and $\operatorname{dist}(T_\alpha(z), K) < \epsilon$, there exists $\bar{m} \in K$ such that $||m - \bar{m}|| < \epsilon$. Therefore

$$(11) \qquad ||\bar{m} - T(y)|| \leqq \tfrac{1}{2}k(\rho_0 + \epsilon)f\left(\frac{d(z) - \epsilon}{k(\rho_0 + \epsilon)}\right) + \epsilon, \qquad T \in \mathscr{T}_\mu.$$

Thus $\mathscr{T}_\mu(y)$ is contained in a spherical ball centered at $\bar{m}$ with radius

$$\tfrac{1}{2}k(\rho_0 + \epsilon)f\left(\frac{d(z) - \epsilon}{k(\rho_0 + \epsilon)}\right) + \epsilon$$

and thus

$$\rho_0 \leqq \tfrac{1}{2}k(\rho_0 + \epsilon)f\left(\frac{d(z) - \epsilon}{k(\rho_0 + \epsilon)}\right) + \epsilon.$$

Letting $\epsilon \to 0$ we have

$$\rho_0 \leqq \frac{k}{2}\rho_0 f\left(\frac{d(z)}{k\rho_0} -\right)$$

where

$$f\left(\frac{d(z)}{k\rho_0} -\right) = \lim_{\epsilon \to 0} f\left(\frac{d(z) - \epsilon}{k(\rho_0 + \epsilon)}\right).$$

This implies that $f(d(z)/k\rho_0 -) \geqq 2/k$. Now $d(z)/k\rho_0 < 2$, for otherwise by (6) $\epsilon_0 = \lim_{\xi \to 2^-} f(\xi) \geqq 2/k$, contradicting $1/k > \epsilon_0$. Thus f is continuous at $d(z)/k\rho_0$ and we have

$$f\left(\frac{d(z)}{k\rho_0}\right) \geqq \frac{2}{k}.$$

There are now two cases to consider:

(a) If $d(z)/k\rho_0 > \epsilon_0$, then, by (3),

$$\frac{d(z)}{k\rho_0} = f^2\left(\frac{d(z)}{k\rho_0}\right) \leqq f\left(\frac{2}{k}\right)$$

and this implies

(12) $d(z) \leqq kf(2/k)\rho_0$.

(b) If $d(z)/k\rho_0 \leqq \epsilon_0$, then

(13) $d(z) \leqq [k\epsilon_0]\rho_0$.

Let $\eta = \max\{k\epsilon_0, kf(2/k)\}$. Then by (5) (and the fact that $1/k > \epsilon_0$) we have $\eta < 1$. Thus (12) and (13) yield for fixed $\eta < 1$,

$$d(z) \leqq \eta d(y).$$

Also by (9) we have $||z - T(y)|| \leqq \rho_0 + \epsilon$, $T \in \mathscr{T}_\beta$, so for $T \in \mathscr{T}_\beta$,

$$||z - y|| \leqq ||z - T(y)|| + ||T(y) - y||,$$

yielding

$$||z - y|| \leqq \rho_0(y) + d(y) \leqq 2d(y).$$

Thus we have established that for each $y \in K$ the corresponding $z = z(y)$ satisfies, for $\eta < 1$,

(14) $d(z) \leqq \eta d(y), \qquad ||z - y|| \leqq 2d(y).$

Now fix $x_0 \in K$ and define the sequence $\{x_n\}$ by $x_{n+1} = z(x_n)$, $n = 0, 1, \ldots$. As previously noted, if $\rho(x_n) = 0$ or $d(x_n) = 0$ for some n, then the theorem will follow. Otherwise, using (14),

$$||x_{n+1} - x_n|| = ||z(x_n) - x_n|| \leqq 2d(x_n) \leqq 2\eta^n d(x_0),$$

and this implies $\{x_n\}$ is a Cauchy sequence. So there exists $w \in K$ such that $x_n \to w$ as $n \to \infty$. Let $\alpha_n \in A$ and $\epsilon_n \to 0$ be chosen so that

$$||x_n - T(x_n)|| \leqq d(x_n) + \epsilon_n, \qquad T \in \mathscr{T}_{\alpha_n},$$
$$||T||_L \leqq k, \qquad T \in \mathscr{T}_{\alpha_n}.$$

Thus for $T \in \mathscr{T}_{\alpha_n}$,

$$||w - T(w)|| \leqq ||w - x_n|| + ||x_n - T(x_n)|| + ||T(x_n) - T(w)||$$
$$\leqq ||w - x_n|| + d(x_n) + \epsilon_n + k||x_n - w||.$$

Since $d(x_n) \to 0$ as $n \to \infty$, the above can be made arbitrarily small, proving $d(w) = 0$.

Thus in any case $d(x_0) = 0$ for some $x_0 \in K$. Because of left reversibility of $\mathscr{T}$, the sets $\{\mathscr{T}_\alpha : \alpha \in A\}$ are directed by set inclusion (and this induces a partial order on A). If $x_\alpha \in \mathscr{T}_\alpha(x_0)$, $\alpha \in A$, then $d(x_0) = 0$ implies that the net $\{x_\alpha : \alpha \in A\}$ converges to x_0. If $T \in \mathscr{T}$ is continuous then $\{Tx_\alpha : \alpha \in A\}$ converges to Tx_0. But $\{T\mathscr{T}_\alpha(x_0) : \alpha \in A\}$ is a subnet of $\{\mathscr{T}_\alpha(x_0) : \alpha \in A\}$ and $Tx_\alpha \in T\mathscr{T}_\alpha(x_0)$, $\alpha \in A$. Thus the net $\{Tx_\alpha : \alpha \in A\}$ also converges to x_0, and $Tx_0 = x_0$. This with (i) proves existence of the ideal J of the theorem, and if all the mappings of $\mathscr{T}$ are continuous then $\mathscr{T}$ has a common fixed point in K.

Remark 2.1. *Let $\mathscr{T}$ be a left reversible semigroup of transformations of $U \to U$, $K \subseteq U \subseteq X$, satisfying* (i) *and* (ii) *of Theorem* 2.1. *Define $\varphi_y : K \to R$ by*

$$\varphi_y(x) = \inf_{\alpha \in A} \left\{ \sup_{T \in \mathscr{T}_\alpha} ||x - T(y)|| \right\}$$

where $\mathscr{T}_\alpha = \{T_\alpha \circ T : T \in \mathscr{T}\}$. Then φ_y is nonexpansive and convex on K, and therefore attains its minimum on K if K is weakly compact.

Proof. Let $y \in K$. With $\mathscr{T}_\alpha$ as defined above there exists $\alpha \in A$ such that $\operatorname{dist}(T(y), K) \leqq 1$ for all $T \in \mathscr{T}_\alpha$. For such T there exists $x_T \in K$ such that $||T(y) - x_T|| = 1$. Therefore

$$\begin{aligned} ||T(y) - x|| &\leqq ||T(y) - x_T|| + ||x_T - x|| \\ &\leqq 1 + \operatorname{diam}(K). \end{aligned}$$

Thus $\sup_{T \in \mathscr{T}_\alpha} ||x - T(y)|| < \infty$ and $\varphi_y(x)$ is defined.

To see that φ_y is *nonexpansive* let $T \in \mathscr{T}_\alpha$, $x, x' \in K$. Then

$$||x - T(y)|| \leqq ||x - x'|| + ||x' - T(y)||,$$

so

$$\sup_{T \in \mathscr{T}_\alpha} ||x - T(y)|| \leqq \sup_{T \in \mathscr{T}_\alpha} ||x' - T(y)|| + ||x - x'||,$$

and thus

$$\inf_{\alpha \in A} \left\{ \sup_{T \in \mathscr{T}_\alpha} ||x - T(y)|| \right\} \leqq \inf_{\alpha \in A} \left\{ \sup_{T \in \mathscr{T}_\alpha} ||x' - T(y)|| \right\} + ||x - x'||$$

yielding $\varphi_y(x) \leqq \varphi_y(x') + ||x - x'||$.

We now show that φ_y is *convex*. Let $x, x' \in K$, $\beta \in (0, 1)$, $T \in \mathscr{T}_\alpha$. Then

$$||\beta x + (1 - \beta)x' - T(y)|| \leqq \beta||x - T(y)|| + (1 - \beta)||x' - T(y)||;$$

$$\sup_{T \in \mathscr{T}_\alpha} ||\beta x + (1 - \beta)x' - T(y)|| \leqq \beta \sup_{T \in \mathscr{T}_\alpha} ||x - T(y)|| + (1 - \beta) \sup_{T \in \mathscr{T}_\alpha} ||x' - T(y)||.$$

Now let $\epsilon > 0$ and choose $\alpha, \alpha' \in A$ so that

$$\sup_{T \in \mathscr{T}_\alpha} ||x - T(y)|| \leqq \varphi_y(x) + \epsilon;$$

$$\sup_{T \in \mathscr{T}_{\alpha'}} ||x' - T(y)|| \leqq \varphi_y(x') + \epsilon.$$

https://doi.org/10.4153/CJM-1974-119-9 Published online by Cambridge University Press

Then if $\alpha'' \in A$ is chosen so that $\mathcal{T}_{\alpha''} \subseteq \mathcal{T}_\alpha \cap \mathcal{T}_{\alpha'}$, we have

$$\begin{aligned}\varphi_y(\beta x + (1-\beta)x') &\leqq \sup_{T \in \mathcal{T}_{\alpha''}} \|\beta x + (1-\beta)x' - T(y)\| \\ &\leqq \beta \sup_{T\in\mathcal{T}_{\alpha''}} \|x - T(y)\| \\ &\qquad + (1-\beta) \sup_{T\in\mathcal{T}_{\alpha''}} \|x' - T(y)\| \\ &\leqq \beta(\varphi_y(x) + \epsilon) + (1-\beta)(\varphi_y(x') + \epsilon).\end{aligned}$$

Therefore

$$\varphi_y(\beta x + (1-\beta)x') \leqq \beta\varphi_y(x) + (1-\beta)\varphi_y(x').$$

It follows from the foregoing that φ_y is lower semi-continuous in the weak topology on K and thus attains its minimum on K.

We note that the above remark yields an alternative approach to the proof of Theorem 2.1. Specifically, with $\rho_0(y) = \inf_{x\in K} \varphi_y(x)$ one immediately obtains existence of a point $z = z(y)$ such that $\varphi_y(z) = \rho_0(y)$, i.e., $z \in C$.

We conclude this section with one final comment. As noted earlier, the above results hold for the class of Banach spaces X for which $\epsilon_0(X) < 1$. This is a class of spaces which lies between the uniformly convex spaces and reflexive spaces which possess 'normal structure'. (See [**8**].) It is in this latter class that the fixed point theorem of Kirk [**16**] is formulated. However, we do not know whether the results of this section (or even those of [**10**; **11**]) hold in this more general class of spaces.

3. Semigroups of asymptotically nonexpansive mappings. It was shown in [**10**] that if K is a closed and convex subset of a uniformly convex space and if $T : K \to K$ is asymptotically nonexpansive then the fixed point set of T in K is closed and convex. If $\mathcal{T}$ is a commutative semigroup of asymptotically nonexpansive mappings with $T_1, T_2 \in \mathcal{T}$ having fixed point sets F_1, F_2 respectively, then $T_2 : F_1 \to F_1$ and thus $F_1 \cap F_2 \neq \emptyset$. As in the nonexpansive case, it follows that the fixed point sets of the mappings of $\mathcal{T}$ have the finite intersection property and since they are each weakly compact they have nonempty intersection, thus yielding a common fixed point for $\mathcal{T}$ (as a special case of Theorem 2.1).

We remark that if two asymptotically nonexpansive mappings T_1, T_2 commute then it is easy to see that their composition $T_1 \circ T_2$ is also asymptotically nonexpansive. Thus there is no loss in generality in assuming that $\mathcal{T}$ in the following theorem is a semigroup as opposed to a commutative family. This theorem generalizes a theorem R. DeMarr [**3**] proved for nonexpansive mappings. (Also see T. Mitchell [**21**].) In this setting the fixed point sets may not be convex.

THEOREM 3.1. *Let K be a nonempty compact convex subset of a Banach space X*

https://doi.org/10.4153/CJM-1974-119-9 Published online by Cambridge University Press

and let $\mathscr{T}$ be a commutative semigroup of asymptotically nonexpansive mappings of K into K. Then there exists a point $x \in K$ such that $T(x) = x$ for each $T \in \mathscr{T}$.

Proof. Use Zorn's lemma to obtain a set $K_1 \subseteq K$ which is minimal with respect to being nonempty, closed, convex, and satisfying:

(*) For each $x \in K_1$ and $T \in \mathscr{T}$, every subsequential limit of the sequence $\{T^n(x)\}$ lies in K_1.

Next let M be a subset of K_1 minimal with respect to being nonempty, closed, and satisfying:

(**) For each $x \in M$ and $T \in \mathscr{T}$, every subsequential limit of the sequence $\{T^n(x)\}$ lies in M.

Note that if $x \in M$ and $\lim_{i\to\infty} T^{n_i}(x) = w$ for some $T \in \mathscr{T}$ then $\lim_{i\to\infty} T^{n_i+1}(x) = T(w) \in M$ by (**). Therefore

$$H_T = M \cap T(M) \neq \emptyset.$$

Now let $x \in H_T$ and $U \in \mathscr{T}$, and suppose $U^{n_j}(x) \to z$ as $j \to \infty$. Then since $x \in M$, (**) implies $z \in M$. Also $x \in T(M)$ implies $x = T(y)$ for some $y \in M$ so by commutativity of $\mathscr{T}$,

$$U^{n_j}(x) = U^{n_j}(T(y)) = T(U^{n_j}(y)) \to z \quad \text{as } j \to \infty.$$

On the other hand because K is compact $\{U^{n_j}(y)\}$ has a convergent subsequence $\{U^{m_j}(y)\}$ which, by (**), converges to some point $v \in M$. Since T is continuous,

$$T(U^{m_j}(y)) \to T(v) \quad \text{as } j \to \infty$$

and it follows that $z = T(v)$, i.e., $z \in T(M)$.

Therefore $z \in H_T$ and by minimality of M, $H_T = M$. Thus $T(M) \supset M$, and since $T \in \mathscr{T}$ was arbitrary,

$$M \subset \bigcap_{T\in\mathscr{T}} T(M).$$

Now assume $\delta = \operatorname{diam}(M) > 0$. As shown by DeMarr [**3**] there exists $r < \delta$ such that for some $x \in K_1$,

$$\sup\{\|x - z\| : z \in M\} \leqq r.$$

Let

$$C = \{x \in K_1 : M \subseteq S(x; r)\}.$$

Then C is a nonempty closed and convex subset of K_1 (see [**1**]) and since $\delta > 0$, C is a proper subset of K_1. Let $z \in C$ and suppose $\lim_{i\to\infty} T^{n_i}(z) = w$ for some $T \in \mathscr{T}$. To see that $w \in C$ let $\epsilon > 0$ and $y \in M$. Choose i so that

$$\|w - T^{n_i}(z)\| \leqq \epsilon$$

and so that T^{n_i} has Lipschitz constant less than $1 + \epsilon$. Choose $t \in M$ so that

$T^{n_i}(t) = y$. Then

$$\begin{aligned}||w - y|| &\leqq ||w - T^{n_i}(z)|| + ||T^{n_i}(z) - T^{n_i}(t)|| \\ &\leqq \epsilon + (1 + \epsilon)||z - t|| \\ &\leqq \epsilon + (1 + \epsilon)r.\end{aligned}$$

Since ϵ is arbitrary we conclude $||w - y|| \leqq r$ which, since $w \in K_1$ by (*), implies $w \in C$. This contradicts the minimality of K_1 so $\delta = 0$ and the proof is complete.

We close this section with one final observation. Freudenthal and Hurewicz [**7**] proved that if $T : M \to M$ is nonexpansive and if $M = T(M)$ then T is necessarily an isometry. One can draw the same conclusion if T is merely assumed to be asymptotically nonexpansive. To see this, assume M is a compact metric space and let T be an asymptotically nonexpansive mapping of M *onto* M. Thus $d(T^i(x), T^i(y)) \leqq k_i d(x, y)$, $x, y \in M$, where $k_i \to 1$ as $i \to \infty$. Fix $x_0, y_0 \in M$ and define $\{x_n\}$, $\{y_n\}$ by $T(x_{n+1}) = x_n$, $T(y_{n+1}) = y_n$, $n = 0, 1, 2, \ldots$. Let $\epsilon > 0$. Because M is compact there exist $m, n, m > n$, such that $d(x_m, x_n) < \epsilon$, $d(y_m, y_n) < \epsilon$, where $m - n = N$ may be assumed to be arbitrarily large. Then

$$\begin{aligned}d(x_m, x_n) &\geqq k_m^{-1} d(T^m(x_m), T^m(x_n)) \\ &= k_m^{-1} d(x_0, T^N(x_0)).\end{aligned}$$

Similarly

$$d(y_m, y_n) \geqq k_m^{-1} d(y_0, T^N(y_0)).$$

Therefore

$$\begin{aligned}d(T(x_0), T(y_0)) &\geqq k_{N-1}^{-1} d(T^N(x_0), T^N(y_0)) \\ &\geqq k_{N-1}^{-1}[d(x_0, y_0) - d(x_0, T^N(x_0)) - d(y_0, T^N(y_0))] \\ &\geqq k_{N-1}^{-1}[d(x_0, y_0) - 2\epsilon].\end{aligned}$$

Letting $N \to \infty$, $d(T(x_0), T(y_0)) \geqq d(x_0, y_0) - 2\epsilon$, and since ϵ is arbitrary this implies T is an expansive mapping of M into itself, hence an isometry by another result of Freudenthal and Hurewicz [**7**].

THEOREM 3.2. *If M is a compact metric space and if T is an asymptotically nonexpansive mapping of M onto itself, then T is an isometry.*

4. An example. The constant γ of Theorem 2.1 is obtained by setting $\gamma = 1/r$ in (5). It is easy to show that this implies $\gamma = (5/4)^{1/2}$ if X is the Hilbert space l^2. (We note that Theorem 2.1 is essentially the same as Theorem 1 of [**11**] when $X = l^2$ and $\mathscr{T}$ consists of a single mapping.) It is reasonable to ask whether this is the largest constant for which the result holds, and we do not know the answer. The following example, however, shows that there is a least

https://doi.org/10.4153/CJM-1974-119-9 Published online by Cambridge University Press

upper bound γ_0 of the numbers γ for which Theorem 2.1 holds in l^2; indeed, $\gamma_0 \in [(5/4)^{1/2}, 2]$.

Let

$$K_1 = \{x \in l^2 : ||x|| \leqq 1\}; K_1^+ = \{x \in K_1 : x_i \geqq 0, i = 1, 2, \ldots\};$$
$$S_1 = \{x \in l^2 : ||x|| = 1\}; \; S_1^+ = S_1 \cap K_1^+.$$

Let $e = (1, 0, 0, \ldots)$ and define $A : K_1^+ \to K_1^+$ by

$$A(x) = (1 - ||x||)e + P(x)$$

where P is the shift operator in l^2. Then

$$1 \geqq ||A(x)||^2 = 1 - 2||x|| + 2||x||^2 \geqq 1/2$$

and

$$||A(x) - A(y)|| = ||(||y|| - ||x||) + [P(x) - P(y)]|| \leqq \sqrt{2}||x - y||,$$
$$x, y \in K_1^+.$$

Now define $R : K_1^+ \to S_1^+$ by

$$R(x) = A(x)/||A(x)||, \qquad x \in K_1^+.$$

It can be shown that $||R(x) - R(y)|| \leqq 2||x - y||$, and that the mapping $T = P \circ R$ maps $K_1^+ \to S_1^+$. Moreover, $T(x) \neq x$ for all $x \in K_1^+$. Also, since $T^n = P^n \circ R$,

$$||T^n(x) - T^n(y)|| \leqq 2||x - y||, \qquad n = 1, 2, \ldots,$$

and it follows that Theorem 2.1 is false if $X = l^2$ and $\gamma \geqq 2$.

References

1. L. P. Belluce and W. A. Kirk, *Fixed point theorems for families of contractive mappings*, Pacific J. Math. *18* (1966), 213–217.
2. F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. USA *54* (1965), 1041–1044.
3. R. De Marr, *Common fixed points for commuting contraction mappings*, Pacific J. Math. *13* (1963), 1139–1141.
4. M. Edelstein, *The construction of an asymptotic center with a fixed point property*, Bull. Amer. Math. Soc. *78* (1972), 206–208.
5. ——— *Fixed point theorems in uniformly convex Banach spaces* (to appear).
6. P. Enflo, *Banach spaces which can be given an equivalent uniformly convex norm*, Israel J. Math. *13* (1972), 281–288.
7. H. Freudenthal and W. Hurewicz, *Dehnungen, Verkürzungen, Isometrien*, Fund. Math. *26* (1930), 120–122.
8. K. Goebel, *Convexity of balls and fixed-point theorems for mappings with nonexpansive square*, Compositio Math. *22* (1970), 269–274.
9. ——— *On the minimal displacement of points under lipschitzian mappings*, Pacific J. Math. *45* (1973), 151–163.
10. K. Goebel and W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. *35* (1972), 171–174.
11. ——— *A fixed point theorem for transformations whose iterates have uniform Lipschitz constant*, Studia Math. *47* (1973), 135–140.

12. D. Göhde, *Zum Prinzip der kontraktiven Abbildung*, Math. Nachr. *30* (1965), 251–258.
13. R. D. Holmes and A. T. Lau, *Nonexpansive actions of topological semigroups and fixed points*, J. London Math. Soc. *5* (1972), 330–336.
14. V. I. Gurarii, *On the differential properties of the modulus of convexity in a Banach space* (in Russian), Mat. Issled. *2* (1967), 141–148.
15. R. C. James, *Uniformly non-square Banach spaces*, Ann. of Math. *80* (1964), 542–550.
16. W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly *72* (1965), 1004–1006.
17. ——— *Fixed point theorems for non-lipschitzian mappings of asymptotically nonexpansive type*, Israel J. Math. (to appear).
18. T.-C. Lim, *A fixed point theorem for families of nonexpansive mappings*, Pacific J. Math. (to appear).
19. ——— *Characterizations of normal structure*, Proc. Amer. Math. Soc. *43* (1974), 313–319.
20. Ju. I. Milman, *Geometric theory of Banach spaces, II. Geometry of the unit ball*, Uspehi Mat. Nauk. *26* (1971), 73–150.
21. T. Mitchell, *Fixed points of reversible semigroups of nonexpansive mappings*, Kōdai Math. Sem. Rep. *22* (1970), 322–323.
22. Z. Opial, *Lecture notes on nonexpansive and monotone mappings in Banach spaces*, Center for Dynamical Systems, Brown University, Providence, R.I., USA, 1967.
23. H. Schaefer, *Über die Methode sukessiver Approximationen*, Jber, Deutsch. Math.-Verein *59* (1957), 131–140.

Maria Curie Sklodowska University,
Lublin, Poland;
The University of Iowa,
Iowa City, Iowa