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Nagoya Math. J.
Vol. 167 (2002), 35–54

TWO MEROMORPHIC FUNCTIONS SHARING FIVE SMALL FUNCTIONS IN THE SENSE OF $\overline{E}_{k)}(\beta, f) = \overline{E}_{k)}(\beta, g)$ *

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Abstract. In this paper, we proved a result that if two meromorphic functions $f(z)$ and $g(z)$ share five small functions $a_j(z)$ $(j = 1, \ldots, 5)$ in the sense of $\overline{E}_{k)}(a_j, f) = \overline{E}_{k)}(a_j, g)$, $(j = 1, \ldots, 5)$ $(k \geq 22)$, then we have $f(z) \equiv g(z)$.

1. Introduction and main result

In this paper the term "meromorphic function" will mean a meromorphic function in C. We will use the standard notations of Nevanlinna theory and we assume that the reader is familiar with the basic results in Nevanlinna theory as found in [1,5,6]. Now we explain briefly the meaning of the notations used in the paper. First of all, we introduce positive logarithmic function. For $x \geq 0$, define

$$\log^+ x = \max(\log x, 0) = \begin{cases} \log x, & x \geq 1 \\ 0, & 0 \leq x < 1. \end{cases}$$

It is obvious that

$$\log x = \log^+ x - \log^+ \frac{1}{x},$$

holds for all positive number.

Let $f(x)$ be a function which is meromorphic on the disc $|z| \leq R$ $(0 < R < \infty)$. For $0 < r < R$, R. Nevanlinna [1] defined the following functions.

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta.$$

Sometimes we write $m(r, f)$ as $m(r, \infty, f)$ or $m(r, \infty)$, which is the average of the positive logarithm of $|f(z)|$ on the circle $|z| = r$.

Received May 15, 2000.
1991 Mathematics Subject Classification: 30D35.
*Project supported by the NSFC and the RFDP.

https://doi.org/10.1017/S0027763000025423 Published online by Cambridge University Press

$$N(r,f)=\int_0^r \frac{n(t,f)-n(0,f)}{t}\,dt+n(0,f)\log r,$$

where $n(r,f)$ denotes the number of poles of $f(z)$ on the disc $|z|\le t$, multiple poles are counted according to their multiplicities. $n(0,f)$ denotes the multiplicity of pole of $f(z)$ at the origin (if $f(0)\ne\infty$, then $n(0,f)=0$).

$N(r,f)$ is called the counting function of poles of $f(z)$, which can be written as $N(r,\infty,f)$ or $N(r,\infty)$.

$$T(r,f)=m(r,f)+N(r,f).$$

$T(r,f)$ is said to be the characteristic function of $f(z)$ which is obviously a non-negative function.

Let a be a complex number. Obviously, $\frac{1}{f(z)-a}$ is meromorphic on the disc $|z|\le R$. Similar to above definitions, R. Nevanlinna [1] defined the following functions.

$$m(r,\frac{1}{f-a})=\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|f(re^{i\theta})-a|}\,d\theta.$$

$m(r,\frac{1}{f-a})$ can also be written as $m(r,a,f)$ or $m(r,a)$.

$$N(r,\frac{1}{f-a})=\int_0^r \frac{n(t,\frac{1}{f-a})-n(0,\frac{1}{f-a})}{t}\,dt+n(0,\frac{1}{f-a})\log r,$$

where $n(t,\frac{1}{f-a})$ denotes the number of zeros of $f(z)-a$ on the disc $|z|\le t$ counting multiplicities and $n(0,\frac{1}{f-a})$ the multiplicity of zero of $f(z)-a$ at origin.

Sometimes we express $n(t,\frac{1}{f-a})$ as $n(t,a,f)$ or $n(t,a)$. The notation $n(0,\frac{1}{f-a})$ is also denoted by $n(0,a,f)$ or $n(0,a)$. $N(r,\frac{1}{f-a})$, sometimes expressed as $N(r,a,f)$ or $N(r,a)$, is said to be the counting function of $f(z)$ at value a.

Let $f(z)$ be a meromorphic function in $|z|<R$ $(\le\infty)$ and a be any complex number. For $0<r<R$, we denote $\overline{n}(r,\frac{1}{f-a})$ sometimes $\overline{n}(r,a,f)$ or $\overline{n}(r,a)$ the number of distinct zeros of $f(z)-a$ in $|z|\le r$, any of it be counted only once. Let

$$\overline{n}(0,\frac{1}{f-a})=\begin{cases}0, & \text{if } f(0)\ne a,\\ 1, & \text{if } f(0)=a\end{cases}$$

https://doi.org/10.1017/S0027763000025423 Published online by Cambridge University Press

and

$$\overline{N}(r,\frac{1}{f-a})=\int_0^r \frac{\overline{n}(t,\frac{1}{f-a})-\overline{n}(0,\frac{1}{f-a})}{t}\,dt+\overline{n}(0,\frac{1}{f-a})\log r,$$

which is called the reduced counting function of $f(z)-a$ and be sometimes denoted by $\overline{N}(r,a,f)$, or $\overline{N}(r,a)$. Similarly, we have the notations $\overline{n}(r,f)$ (or $\overline{n}(r,\infty,f)$, $\overline{n}(r,\infty)$) and $\overline{N}(r,f)$ (or $\overline{N}(r,\infty,f)$, $\overline{N}(r,\infty)$).

$$T(r,\frac{1}{f-a})=m(r,\frac{1}{f-a})+N(r,\frac{1}{f-a}).$$

By $S(r,f)$ we denote any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to+\infty$, possibly outside a set of r of finite linear measure. We denote the set as E. It is not necessarily the same when it appears. If two meromorphic functions f and g have same a-point with same multiplicities (ignoring multiplicities), then we say f and g share the value a CM (IM).

Let $f(z)$ be a non-constant meromorphic function in the complex plane and let $S(f)$ be the set of meromorphic functions $\beta(z)$ in the complex plane which satisfy

$$T(r,\beta)=S(r,f),$$

Such a meromorphic function $\beta(z)$ is said to be a small function of $f(z)$. Note that $S(f)$ is a field.

For a non-constant meromorphic function f, a small function $\beta\in S(f)\cup\{\infty\}$ and a positive integer k (or $+\infty$), we write $E_{k)}(\beta,f)$ for the set of zeros of $f(z)-\beta$ with multiplicity $\le k$ (counting multiplicity); we write $\overline{E}_{k)}(\beta,f)$ for the set of zeros of $f(z)-\beta$ with multiplicity $\le k$ (each zero counted only once).

If two non-constant meromorphic functions f and g and a small function $\beta\in(S(f)\cap S(g))\cup\{\infty\}$ satisfy

$$E_{+\infty)}(\beta,f)=E_{+\infty)}(\beta,g),$$

then we say that f and g share β CM. If f and g satisfy

$$\overline{E}_{+\infty)}(\beta,f)=\overline{E}_{+\infty)}(\beta,g)$$

then we say that f and g share β IM.

Let $h_1(z)$ and $h_2(z)$ be two non-constant meromorphic functions, and let $a(z)$ (or ∞) be the common small function of $h_1(z)$ and $h_2(z)$.

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We denote by $\overline{N}(r, h_1(z) = a(z) = h_2(z))$ (resp. $\overline{N}_E(r, h_1(z) = a(z) = h_2(z))$) the counting function of those common $a(z)$-points of $h_1(z)$ and $h_2(z)$, regardless of multiplicity (resp. with the same multiplicity). Each point counted only once.

(i) If $\overline{N}(r, \frac{1}{h_j(z)-a(z)}) - \overline{N}_E(r, h_1(z) = a(z) = h_2(z)) = S(r, h_j)$ $(j = 1, 2)$, then we say that $h_1(z)$ and $h_2(z)$ share small function $a(z)$ CM$''$;

(ii) If $\overline{N}(r, \frac{1}{h_j(z)-a(z)}) - \overline{N}(r, h_1(z) = a(z) = h_2(z)) = S(r, h_j)$ $(j = 1, 2)$, then we say that $h_1(z)$ and $h_2(z)$ share small function $a(z)$ IM$''$.

In 1929, Nevanlinna proved the following well-known result which is the so-called Nevanlinna five-values theorem.

Theorem A. ([1]) *Let f and g be two non-constant meromorphic functions. If f and g share five distinct values CM, then $f \equiv g$.*

In 1999, Li Yuhua and Qiao Jianyong get the following result, which extend Theorem A to small functions.

Theorem B. ([2]) *Let f and g be two non-constant meromorphic functions. If f and g share five distinct small functions IM, then $f \equiv g$.*

In this paper, we obtain the following result which is an improvement of Theorem B.

Theorem 1. *Let f and g be two non-constant meromorphic functions, and let a_j $(j = 1, \cdots, 5)$ be distinct small functions of f and g. If f and g satisfy $\overline{E}_{k)}(a_j, f) = \overline{E}_{k)}(a_j, g)$, $(j = 1, \cdots, 5)$, $k(\geq 22)$ is a positive integer, then we have $f(z) \equiv g(z)$.*

Note. Without loss of generality, we suppose that $a_1(z) = b(z),\ a_2(z) = a(z),\ a_3(z) = 1,\ a_4(z) = 0,\ a_5(z) = \infty,\ (a(z), b(z) \not\equiv \infty, 0, 1)$, otherwise, a quasi-Möbius transformation will do.

2. Some lemmas and notations

In the rest of this section, we assume that f and g are distinct non-constant meromorphic functions sharing $a_1(z) = b(z)$, $a_2(z) = a(z)$, $a_3(z) = 1$, $a_4(z) = 0$ and $a_5(z) = \infty$ in the sense of $\overline{E}_{k)}(a_j, f) = \overline{E}_{k)}(a_j, g)$, where $k(\geq 22)$ is a positive integer.

LEMMA 1. ([3]) *Suppose that f, $a(z)$ and $b(z)$ are all meromorphic functions ($\not\equiv \infty$), $a(z)$ and $b(z)$ are distinct small functions of f. Set*

$$L(f,a,b) := \begin{vmatrix} f & f' & 1 \\ a & a' & 1 \\ b & b' & 1 \end{vmatrix}.$$

Then

$$L(f,a,b) \not\equiv 0,$$

and

$$m(r, \frac{L(f,a,b)f^k}{(f-a)(f-b)}) = S(r,f) \quad (k=0,1).$$

LEMMA 2. *Let f and g be two non-constant meromorphic functions sharing four distinct small functions $b_1(z)$, $b_2(z)$, $b_3(z)$ and ∞ IM$''$. Set*

$$H := \frac{L(f,b_1,b_2)(f-g)L(g,b_2,b_3)}{(f-b_1)(f-b_2)(g-b_2)(g-b_3)} - \frac{L(g,b_1,b_2)(f-g)L(f,b_2,b_3)}{(g-b_1)(g-b_2)(f-b_2)(f-b_3)},$$

Then we have $T(r,H) = S(r,f)$.

Proof. From Lemma 1 we get $m(r,H) = S(r,f)$. The zeros of $f - b_1$ and $f - b_3$ contribute to $N(r,H)$ only $S(r,f)$. Let z_∞ be a pole of f with multiplicity p, a pole of g with multiplicity q, and $b_j(z_\infty)(b_j(z_\infty)-1) \neq 0, \infty$ $(j = 1,2,3)$. Without loss of generality, we suppose that $p \geq q$, then when $z \to z_\infty$, we have

$$\frac{L(f,b_1,b_2)(f-g)L(g,b_2,b_3)}{(f-b_1)(f-b_2)(g-b_2)(g-b_3)} \sim (b_2-b_1)(b_3-b_2)(1-\frac{g}{f})\frac{f'g'}{fg^2},$$

$$\frac{L(g,b_1,b_2)(f-g)L(f,b_2,b_3)}{(g-b_1)(g-b_2)(f-b_2)(f-b_3)} \sim (b_2-b_1)(b_3-b_2)(1-\frac{g}{f})\frac{f'g'}{fg^2}.$$

So we know H is analytic at z_∞. Hence the poles of f contribute to $N(r,H)$ also $S(r,f)$. Notice that H can be written as :

$$H \equiv \frac{-(f-g)}{(f-b_1)(g-b_3)}\{[(b_1-b_2)\frac{f'-b_2'}{f-b_2}-(b_1'-b_2')][(b_3-b_2)\frac{g'-b_2'}{g-b_2}-(b_3'-b_2')]$$

$$-\frac{f(g-1)}{g(f-1)}[(b_1-b_2)\frac{g'-b_2'}{g-b_2}-(b_1'-b_2')][(b_3-b_2)\frac{f'-b_2'}{f-b_2}-(b_3'-b_2')]\}.$$

From the formula above we know that the zeros of $f - b_2$ contribute to $N(r,H)$ also $S(r,f)$.

So we have $N(r,H)=S(r,f)$. Hence $T(r,H)=S(r,f)$.

Lemma 3. *Let f and g be two non-constant meromorphic functions, $b_1(z)$, $b_2(z)$, $b_3(z)$ and ∞ be four distinct small functions of f and g. Set*

$$H := \frac{L(f,b_1,b_2)(f-g)L(g,b_2,b_3)}{(f-b_1)(f-b_2)(g-b_2)(g-b_3)} - \frac{L(g,b_1,b_2)(f-g)L(f,b_2,b_3)}{(g-b_1)(g-b_2)(f-b_2)(f-b_3)},$$

$$H^* := \frac{L(f,b_2,b_1)(f-g)L(g,b_1,b_3)}{(f-b_2)(f-b_1)(g-b_1)(g-b_3)} - \frac{L(g,b_2,b_1)(f-g)L(f,b_1,b_3)}{(g-b_2)(g-b_1)(f-b_1)(f-b_3)},$$

$$H^{**} := \frac{L(f,b_1,b_3)(f-g)L(g,b_3,b_2)}{(f-b_1)(f-b_3)(g-b_3)(g-b_2)} - \frac{L(g,b_1,b_3)(f-g)L(f,b_3,b_2)}{(g-b_1)(g-b_3)(f-b_3)(f-b_2)}.$$

Then we have $H\equiv -H^*\equiv -H^{**}$.

Proof. We only prove $H\equiv -H^*$. The proof of $H\equiv -H^{**}$ is similar.

$$\begin{aligned} H \equiv{}& (f-g)\Big\{\frac{1}{b_2-b_1}\Big(\frac{L(f,b_1,b_2)}{f-b_2}-\frac{L(f,b_1,b_2)}{f-b_1}\Big) \\ &\qquad\times\frac{1}{b_3-b_2}\Big(\frac{L(g,b_2,b_3)}{g-b_3}-\frac{L(g,b_2,b_3)}{g-b_2}\Big) \\ &-\frac{1}{b_2-b_1}\Big(\frac{L(g,b_1,b_2)}{g-b_2}-\frac{L(g,b_1,b_2)}{g-b_1}\Big) \\ &\qquad\times\frac{1}{b_3-b_2}\Big(\frac{L(f,b_2,b_3)}{f-b_3}-\frac{L(f,b_2,b_3)}{f-b_2}\Big)\Big\} \\ \equiv{}& (f-g)\Big\{\Big(\frac{f'-b_2'}{f-b_2}-\frac{f'-b_1'}{f-b_1}\Big)\Big(\frac{g'-b_3'}{g-b_3}-\frac{g'-b_2'}{g-b_2}\Big) \\ &-\Big(\frac{g'-b_2'}{g-b_2}-\frac{g'-b_1'}{g-b_1}\Big)\Big(\frac{f'-b_3'}{f-b_3}-\frac{f'-b_2'}{f-b_2}\Big)\Big\} \\ \equiv{}& (f-g)\Big\{\frac{f'-b_2'}{f-b_2}\frac{g'-b_3'}{g-b_3}-\frac{g'-b_2'}{g-b_2}\frac{f'-b_3'}{f-b_3}-\frac{f'-b_1'}{f-b_1}\frac{g'-b_3'}{g-b_3} \\ &+\frac{g'-b_1'}{g-b_1}\frac{f'-b_3'}{f-b_3}+\frac{f'-b_1'}{f-b_1}\frac{g'-b_2'}{g-b_2}-\frac{g'-b_1'}{g-b_1}\frac{f'-b_2'}{f-b_2}\Big\}, \end{aligned}$$

By making an exchange of the positions of b_1, b_2, we have $H\equiv -H^*$. Then we completes the proof of Lemma 3.

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LEMMA 4. *Let f and g be two non-constant meromorphic functions, $b_1(z)$, $b_2(z)$, $b_3(z)$ and ∞ be four distinct small functions of f and g. Set*

$$H := \frac{L(f,b_1,b_2)(f-g)L(g,b_2,b_3)}{(f-b_1)(f-b_2)(g-b_2)(g-b_3)} - \frac{L(g,b_1,b_2)(f-g)L(f,b_2,b_3)}{(g-b_1)(g-b_2)(f-b_2)(f-b_3)},$$

then

$$H \equiv 0 \Longleftrightarrow \tilde{H} \equiv 0,$$

where

$$\tilde{H} := \frac{L(F,\tilde{b_1},\tilde{b_2})(F-G)L(G,\tilde{b_2},\tilde{b_3})}{(F-\tilde{b_1})(F-\tilde{b_2})(G-\tilde{b_2})(G-\tilde{b_3})} - \frac{L(G,\tilde{b_1},\tilde{b_2})(F-G)L(F,\tilde{b_2},\tilde{b_3})}{(G-\tilde{b_1})(G-\tilde{b_2})(F-\tilde{b_2})(F-\tilde{b_3})},$$

$$F = \frac{1}{f-b_1} + b_1, \quad G = \frac{1}{g-b_1} + b_1,$$

$$\tilde{b_1} = b_1, \quad \tilde{b_2} = \frac{1}{b_2-b_1} + b_1, \quad \tilde{b_3} = \frac{1}{b_3-b_1} + b_1.$$

Proof. Without loss of generality, we suppose that $b_1(z) = 0$, $b_2(z) = 1$, $b_3(z) = b(z)$, $(b(z) \not\equiv \infty, 0, 1)$. From Lemma 3 we know that

$$-H \equiv H^* = \frac{L(f,1,0)(f-g)L(g,0,b)}{(f-1)fg(g-b)} - \frac{L(g,1,0)(f-g)L(f,0,b)}{(g-1)gf(f-b)}.$$

So $F = \frac{1}{f}$, $G = \frac{1}{g}$, $\tilde{b_1} = 0$, $\tilde{b_2} = 1$, $\tilde{b_3} = b^{-1}$.

$$\begin{aligned}-\tilde{H} \equiv \tilde{H}^* &= \frac{L(F,1,0)(F-G)L(G,0,b^{-1})}{(F-1)FG(G-b^{-1})} - \frac{L(G,1,0)(F-G)L(F,0,b^{-1})}{(G-1)GF(F-b^{-1})} \\ &= \frac{L(f^{-1},1,0)(f^{-1}-g^{-1})L(g^{-1},0,b^{-1})}{(f^{-1}-1)f^{-1}g^{-1}(g^{-1}-b^{-1})} - \frac{L(g^{-1},1,0)(f^{-1}-g^{-1})L(f^{-1},0,b^{-1})}{(g^{-1}-1)g^{-1}f^{-1}(f^{-1}-b^{-1})} \\ &= \frac{\frac{f'}{f^2}\frac{g-f}{fg}\begin{vmatrix} \frac{1}{g} & \frac{g'}{g^2} \\ \frac{1}{b} & \frac{b'}{b^2}\end{vmatrix}}{\frac{1-f}{f}\frac{1}{f}\frac{1}{g}\frac{b-g}{bg}} - \frac{\frac{g'}{g^2}\frac{g-f}{fg}\begin{vmatrix} \frac{1}{f} & \frac{f'}{f^2} \\ \frac{1}{b} & \frac{b'}{b^2}\end{vmatrix}}{\frac{1-g}{g}\frac{1}{g}\frac{1}{f}\frac{b-f}{bf}}\end{aligned}$$

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$$= \frac{1}{b}\left\{\frac{-f'(f-g)\begin{vmatrix} g & g' \\ b & b' \end{vmatrix}}{(f-1)fg(g-b)} - \frac{-g'(f-g)\begin{vmatrix} f & f' \\ b & b' \end{vmatrix}}{(g-1)gf(f-b)}\right\}$$

$$= \frac{1}{b}H^* = -\frac{1}{b}H.$$

Noting that $b(z) \equiv 0$, so we have

$$H \equiv 0 \Longleftrightarrow \tilde{H} \equiv 0.$$

LEMMA 5. ([8]) *Let f and g be two non-constant meromorphic functions, and let a_j $(j = 1, \cdots, 5)$ be distinct small functions of f and g. Then we have*

$$2T(r,h) < \sum_{j=1}^{5} \overline{N}(r, \frac{1}{h-a_j}) + S(r,f) + S(r,g),$$

where h is f or g.

LEMMA 6. *Let f and g be two non-constant meromorphic functions, and let a_j $(j = 1, \cdots, 5)$ be distinct small functions of f and g. If f and g satisfy*

$$\overline{E}_{k)}(a_j, f) = \overline{E}_{k)}(a_j, g), \quad (j = 1, \cdots, 5) \tag{0}$$

where $k(\geq 22)$ is a positive integer, then we have
(i)

$$S(r) = S(r,f) = S(r,g). \tag{1}$$

(ii)

$$(2 - \frac{3}{k})\{T(r,f) + T(r,g)\} \leq \sum_{j=1}^{5}\{\overline{N}_{k)}(r, a_j, f) + \overline{N}_{k)}(r, a_j, g)\} + S(r). \tag{2}$$

$$(1 - \frac{3}{2k})\{T(r,f) + T(r,g)\} \leq \sum_{j=1}^{5}\overline{N}_{k)}(r, a_j, f) + S(r). \tag{3}$$

Proof. (i) From Lemma 4 we know that

$$2T(r,f) \leq \sum_{j=1}^{5} \overline{N}(r, \frac{1}{f-a_j}) + S(r,f)$$

$$\leq \frac{k}{k+1} \sum_{j=1}^{5} \overline{N}_{k)}(r, \frac{1}{f-a_j}) + \frac{5}{k+1} T(r,f) + S(r,f)$$

$$\leq \frac{k}{k+1} N(r, \frac{1}{f-g}) + \frac{5}{k+1} T(r,f) + S(r,f).$$

$$\leq \frac{k+5}{k+1} T(r,f) + \frac{k}{k+1} T(r,g) + S(r,f).$$

So we have

$$T(r,f) \leq \frac{k}{k-3} T(r,g) + S(r,f). \tag{4}$$

Similarly we have

$$T(r,g) \leq \frac{k}{k-3} T(r,f) + S(r,g). \tag{5}$$

From (4) and (5), we can get

$$S(r) = S(r,f) = S(r,g).$$

(ii) From Lemma 4 we know that

$$2T(r,f) + k \sum_{j=1}^{5} \overline{N}_{(k+1}(r, a_j, f)$$

$$\leq \sum_{j=1}^{5} \overline{N}(r, a_j, f) + k \sum_{j=1}^{5} \overline{N}_{(k+1}(r, a_j, f) + S(r).$$

$$\leq 5T(r,f) + S(r).$$

So we have

$$\sum_{j=1}^{5} \overline{N}_{(k+1}(r, a_j, f) \leq \frac{3}{k} T(r,f) + S(r). \tag{6}$$

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Similarly we have

$$\sum_{j=1}^{5} \overline{N}_{(k+1}(r, a_j, g) \leq \frac{3}{k} T(r, g) + S(r). \tag{7}$$

From Lemma 4 and (6), we know

$$2T(r, f) \leq \sum_{j=1}^{5} \overline{N}_{k)}(r, a_j, f) + \sum_{j=1}^{5} \overline{N}_{(k+1}(r, a_j, f) + S(r)$$

$$\leq \sum_{j=1}^{5} \overline{N}_{k)}(r, a_j, f) + \frac{3}{k} T(r, f) + S(r).$$

Thus

$$(2 - \frac{3}{k}) T(r, f) \leq \sum_{j=1}^{5} \overline{N}_{k)}(r, a_j, f) + S(r).$$

Similarly we have

$$(2 - \frac{3}{k}) T(r, g) \leq \sum_{j=1}^{5} \overline{N}_{k)}(r, a_j, g) + S(r).$$

So we get

$$(2 - \frac{3}{k}) \{T(r, f) + T(r, g)\} \leq \sum_{j=1}^{5} \{\overline{N}_{k)}(r, a_j, f) + \overline{N}_{k)}(r, a_j, g)\} + S(r).$$

Noting (0), we know that

$$\overline{N}_{k)}(r, a_j, f) = \overline{N}_{k)}(r, a_j, g), \quad (j = 1, \cdots, 5)$$

So we get (3).

3. Proof of Theorem 1

In order to prove Theorem 1, we need a slight generalization of Theorem B as follows:

THEOREM B′. *Theorem* B *remains valid if IM is replaced by IM″.*

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In what follows we assume that f and g are distinct meromorphic functions, satisfying the assumptions of Theorem 1, and the set S_0 is as the following:

$$S_0 = \{z|a(z) = 0, 1, \infty, \text{ or } b(z) = 0, 1, \infty, \text{ or } a(z) - b(z) = 0\}$$

Set

$$H_1 := \frac{L(f,0,1)(f-g)L(g,1,a)}{f(f-1)(g-1)(g-a)} - \frac{L(g,0,1)(f-g)L(f,1,a)}{g(g-1)(f-1)(f-a)}.$$

If $H_1 \not\equiv 0$, from Lemma 2 and (3) we have

$$m(r, H_1) = S(r, f),$$

and

$$\begin{aligned} N(r, H_1) \le \sum_{j=2}^{5} \overline{N}_{(k+1}(r, \frac{1}{f-a_j}) \\ + \sum_{j=2}^{5} \overline{N}_{(k+1}(r, \frac{1}{g-a_j}) + S(r,f) + S(r,g) \\ \le \frac{4}{k+1}\{T(r,f) + T(r,g)\} - \frac{2}{k+1}\sum_{j=1}^{5} \overline{N}_{k)}(r, \frac{1}{f-a_j}) \\ + \frac{2}{k+1}\overline{N}_{k)}(r, \frac{1}{f-a_1}) + S(r) \\ \le \frac{4}{k+1}\{T(r,f) + T(r,g)\} - \frac{2}{k+1}(1 - \frac{3}{2k})\{T(r,f) + T(r,g)\} \\ + \frac{2}{k+1}\overline{N}_{k)}(r, \frac{1}{f-a_1}) + S(r) \\ \le \frac{2k+3}{k(k+1)}\{T(r,f) + T(r,g)\} \\ + \frac{2}{k+1}\overline{N}_{k)}(r, \frac{1}{f-a_1}) + S(r). \end{aligned}$$

Notice that

$$\overline{N}_{k)}(r, \frac{1}{f-a_1}) \le N(r, \frac{1}{H_1}) \le N(r, H_1) + S(r),$$

so we have

$$\overline{N}_{k)}(r, \frac{1}{f-a_1}) \le \frac{2k+3}{k(k-1)}\{T(r,f) + T(r,g)\} + S(r). \tag{8}$$

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Noting that

$$\overline{N}_{k)}(r, a_j, f) = \overline{N}_{k)}(r, a_j, g), \quad (j = 1, \cdots, 5)$$

hence we get

$$(9)\quad \overline{N}_{k)}(r, \frac{1}{f-a_1}) + \overline{N}_{k)}(r, \frac{1}{g-a_1}) \leq \frac{2(2k+3)}{k(k-1)}\{T(r,f)+T(r,g)\}+S(r).$$

Similarly, we set

$$H_2 := \frac{L(f,1,0)(f-g)L(g,0,b)}{(f-1)fg(g-b)} - \frac{L(g,1,0)(f-g)L(f,0,b)}{(g-1)gf(f-b)};$$
$$H_3 := \frac{L(f,0,a)(f-g)L(g,a,b)}{f(f-a)(g-a)(g-b)} - \frac{L(g,0,a)(f-g)L(f,a,b)}{g(g-a)(f-a)(f-b)};$$
$$H_4 := \frac{L(f,1,b)(f-g)L(g,b,a)}{(f-1)(f-b)(g-b)(g-a)} - \frac{L(g,1,b)(f-g)L(f,b,a)}{(g-1)(g-b)(f-b)(f-a)}.$$

And if $H_j \not\equiv 0 \ \ (j = 2, 3, 4)$, we also have

$$(10)\quad \overline{N}_{k)}(r, \frac{1}{f-a_j}) + \overline{N}_{k)}(r, \frac{1}{g-a_j}) \leq \frac{2(2k+3)}{k(k-1)}\{T(r,f)+T(r,g)\}+S(r). \quad (j = 2, 3, 4)$$

Set

$$F(z) = \frac{1}{f(z)}, \quad G(z) = \frac{1}{g(z)}, \quad a_1^*(z) = \frac{1}{a_1(z)} = \frac{1}{b(z)},$$

$$a_2^*(z) = \frac{1}{a_2(z)} = \frac{1}{a(z)}, \quad a_3^*(z) = \frac{1}{a_3(z)} = 1, \quad a_4^*(z) = \infty, \quad a_5^*(z) = 0.$$

Clearly, $a_j^*(z)$ $(j = 1, \cdots, 5)$ are all small functions of $F(z)$ and $G(z)$, and

$$\overline{E}_{k)}(a_j^*(z), F) = \overline{E}_{k)}(a_j^*(z), G), \quad (j = 1, \cdots, 5)$$

where $k(\geq 22)$ is a positive integer.

Furthermore, from Nevanlinna first fundamental theorem we have

$$T(r, F) = T(r, f) + O(1), \quad T(r, G) = T(r, g) + O(1),$$

$$S(r, F) = S(r, f) = S(r), \quad S(r, G) = S(r, g) = S(r).$$

Set

$$H_5 := \frac{L(F,1,b^{-1})(F-G)L(G,a^{-1},b^{-1})}{(F-1)(F-b^{-1})(G-b^{-1})(G-a^{-1})} - \frac{L(G,1,b^{-1})(F-G)L(F,a^{-1},b^{-1})}{(G-1)(G-b^{-1})(F-b^{-1})(F-a^{-1})},$$

If $H_5 \not\equiv 0$, from Lemma 2 and (3) we have

$$m(r,H_5) = S(r,F) = S(r),$$

and

$$\begin{aligned}
N(r,H_5) &\leq \sum_{j=1}^{4} \overline{N}_{(k+1}(r,\frac{1}{F-a_j^*}) + \sum_{j=1}^{4} \overline{N}_{(k+1}(r,\frac{1}{G-a_j^*}) + S(r,F) \\
&= \sum_{j=1}^{4} \overline{N}_{(k+1}(r,\frac{1}{f-a_j}) + \sum_{j=1}^{4} \overline{N}_{(k+1}(r,\frac{1}{g-a_j}) + S(r,f) \\
&\leq \frac{4}{k+1}\{T(r,f)+T(r,g)\} - \frac{2}{k+1}\sum_{j=1}^{5} \overline{N}_{k)}(r,\frac{1}{f-a_j}) \\
&\qquad + \frac{2}{k+1}\overline{N}_{k)}(r,\frac{1}{f-a_5}) + S(r) \\
&\leq \frac{4}{k+1}\{T(r,f)+T(r,g)\} - \frac{2}{k+1}(1-\frac{3}{2k})\{T(r,f)+T(r,g)\} \\
&\qquad + \frac{2}{k+1}\overline{N}_{k)}(r,\frac{1}{f-a_1}) + S(r) \\
&\leq \frac{2k+3}{k(k+1)}\{T(r,f)+T(r,g)\} + \frac{2}{k+1}\overline{N}_{k)}(r,\frac{1}{f-a_5}) + S(r).
\end{aligned}$$

Notice that

$$\overline{N}_{k)}(r,\frac{1}{f-a_5}) = \overline{N}_{k)}(r,\frac{1}{F}) + S(r) \leq N(r,\frac{1}{H_5}) \leq N(r,H_5) + S(r),$$

so we have

$$\overline{N}_{k)}(r,\frac{1}{f-a_5}) \leq \frac{2k+3}{k(k-1)}\{T(r,f)+T(r,g)\} + S(r). \tag{11}$$

Noting that

$$\overline{N}_{k)}(r,a_j,f) = \overline{N}_{k)}(r,a_j,g), \quad (j=1,\cdots,5)$$

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hence we get

$$(12)\quad \overline{N}_{k)}(r,\frac{1}{f-a_5})+\overline{N}_{k)}(r,\frac{1}{g-a_5}) \le \frac{2(2k+3)}{k(k-1)}\{T(r,f)+T(r,g)\}+S(r).$$

While from (2), i.e.

$$(2-\frac{3}{k})\{T(r,f)+T(r,g)\} \le \sum_{j=1}^{5}\{\overline{N}_{k)}(r,a_j,f)+\overline{N}_{k)}(r,a_j,g)\}+S(r),$$

we know that there exist at least two of the five $\overline{N}_{k)}(r,\frac{1}{f-a_j})+\overline{N}_{k)}(r,\frac{1}{g-a_j})$, $(j=1,\cdots,5)$, without loss of generality, we suppose $j=2,3$, such that

$$(13)\qquad \overline{N}_{k)}(r,\frac{1}{f-a_2})+\overline{N}_{k)}(r,\frac{1}{g-a_2}) \ge \{\frac{1}{4}(1-\frac{3}{k})+o(1)\}\{T(r,f)+T(r,g)\},\quad (r\in I),$$

and

$$(14)\qquad \overline{N}_{k)}(r,\frac{1}{f-a_3})+\overline{N}_{k)}(r,\frac{1}{g-a_3}) \ge \{\frac{1}{4}(1-\frac{3}{k})+o(1)\}\{T(r,f)+T(r,g)\}\quad (r\in I),$$

where $I\subseteq R^+$, and $\mathrm{mes}I=+\infty$.

If $H_2\not\equiv 0$, then from (10) and (13) we have

$$\frac{1}{4}(1-\frac{3}{k}) \le \frac{2(2k+3)}{k(k-1)}.$$

This contradicts $k\ge 22$. So we have $H_2\equiv 0$. Similarly we can prove $H_3\equiv 0$.

From $H_2\equiv 0$ we have

$$(15)\qquad \frac{L(f,1,0)L(g,0,b)}{(f-1)(g-b)} \equiv \frac{L(g,1,0)L(f,0,b)}{(g-1)(f-b)}.$$

From $H_3\equiv 0$ we have

$$(16)\qquad \frac{L(f,0,a)L(g,a,b)}{f(g-b)} \equiv \frac{L(g,0,a)L(f,a,b)}{g(f-b)}.$$

Case 1.

(i) If $a'(z) \not\equiv 0$ suppose z_b is a zero of $g-b$, but not a zero of $f-b$, and $z_b \notin S_0$, $a'(z_b) \neq 0$ then z_b is not a pole of $f-b$. In fact, if z_b is a pole of $f-b$, z_b is a pole of the right sides of (15) and (16) with multyplicity 1, while z_b is a pole of the left sides of (15) and (16) with multyplicity 2. This is impossible. Hence the right sides of (15) and (16) are analytic at z_b. So we have $f(z_b) \neq 0$, $f(z_b) - 1 \neq 0$, but $L(f,1,0)/_{z=z_b} = 0$, $L(f,0,a)/_{z=z_b} = 0$. Thus $f'(z_b) = 0$, and $f(z_b)a'(z_b) - f'(z_b)a(z_b) = 0$. So we get $f(z_b)a'(z_b) = 0$. This contradicts $a'(z_b) \neq 0$ and $f(z_b) \neq 0$. Hence $f(z_b) - b(z_b) = 0$.

Then we have

$$g - b = 0 \implies f - b = 0. \qquad (r \notin E)$$

By symmetry, we have

$$f - b = 0 \implies g - b = 0. \qquad (r \notin E)$$

Here $E \subseteq R^+$, and $\mathrm{mes}E < +\infty$. This means f, g share b IM$''$.

(ii) If $a'(z) \equiv 0$, then $a(z) \equiv$ constant. From Lemma 3, we know that (16) is equivalent to $(16)'$:

$$(16)' \qquad \frac{L(f,a,0)L(g,0,b)}{(f-a)(g-b)} \equiv \frac{L(g,a,0)L(f,0,b)}{(g-a)(f-b)}.$$

Since $a(z) \equiv$constant, we have $L(f,a,0) \equiv kL(f,1,0)$, $L(g,a,0) \equiv kL(g,1,0)$, where $k = a \neq 0$. Substituting them to $(16)'$, and combining with (15), we have

$$\frac{f-a}{f-1} \equiv \frac{g-a}{g-1}.$$

So we get $f \equiv g$.

Case 2.

From Lemma 3, we know that (15) is equivalent to $(15)'$:

$$(15)' \qquad \frac{L(f,0,1)L(g,1,b)}{f(g-b)} \equiv \frac{L(g,0,1)L(f,1,b)}{g(f-b)},$$

(i) If $(b(z)-1)a'(z) - (a(z)-1)b'(z) \not\equiv 0$, suppose z_0 is a zero of f, but not a zero of g, and $z_0 \notin S_0$, $(b(z_0)-1)a'(z_0) - (a(z_0)-1)b'(z_0) \neq 0$, then z_0 is not a pole of g. In fact, if z_0 is a pole of g, z_0 is a pole of the right

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sides of $(15)'$ and (16) with multyplicity 1, while z_0 is a pole of the left sides of $(15)'$ and (16) with multyplicity 2. This is impossible. Hence the right sides of (16) and $(15)'$ are analytic at z_0. So we have $g(z_0) - b(z_0) \neq 0$, but $L(g, 1, b)/_{z=z_0} = 0$, $L(g, a, b)/_{z=z_0} = 0$. Thus

$$\begin{vmatrix} g(z_0) - b(z_0) & g'(z_0) - b'(z_0) \\ b(z_0) - 1 & b'(z_0) \end{vmatrix} = 0, \begin{vmatrix} g(z_0) - b(z_0) & g'(z_0) - b'(z_0) \\ b(z_0) - a(z_0) & b'(z_0) - a'(z_0) \end{vmatrix} = 0,$$

Since $g(z_0) - b(z_0) \neq 0$, we have

$$\begin{vmatrix} b(z_0) - 1 & b'(z_0) \\ b(z_0) - a(z_0) & b'(z_0) - a'(z_0) \end{vmatrix} = 0.$$

That is $(b(z_0) - 1)a'(z_0) - (a(z_0) - 1)b'(z_0) = 0$, which contradicts $(b(z_0) - 1)a'(z_0) - (a(z_0) - 1)b'(z_0) \neq 0$. Hence $g(z_0) = 0$.

Then we have

$$f = 0 \implies g = 0. \qquad (r \notin E)$$

By symmetry, we have

$$g = 0 \implies f = 0. \qquad (r \notin E)$$

Here $E \subseteq R^+$ and mes$E < +\infty$. This means f, g share 0 IM$''$.

From (15), (16) and using Lemma 3, we have

$$\frac{L(f, 0, 1)L(g, 1, a)}{f(g - a)} \equiv \frac{L(g, 0, 1)L(f, 1, a)}{g(f - a)}. \tag{17}$$

(ii) If $(b(z)-1)a'(z)-(a(z)-1)b'(z) \equiv 0$, then $a-1 = k(b-1)$, $a' = kb'$, where $k(\neq 0)$ is a constant. So we have

$$L(g, 1, a) \equiv \begin{vmatrix} g - 1 & g' \\ k(b - 1) & kb' \end{vmatrix} \equiv kL(g, 1, b),$$

$$L(f, 1, a) \equiv \begin{vmatrix} f - 1 & f' \\ k(b - 1) & kb' \end{vmatrix} \equiv kL(f, 1, b).$$

Substituting them to (17), and combining with $(15)'$, we have

$$\frac{f - a}{f - b} \equiv \frac{g - a}{g - b}.$$

So we get $f \equiv g$.

Case 3.

From (15) and $(16)'$ we have

$$\frac{L(f,a,0)L(g,0,1)}{(f-a)(g-1)} \equiv \frac{L(g,a,0)L(f,0,1)}{(g-a)(f-1)}. \tag{18}$$

(i) If $a(z)b'(z) - a'(z)b(z) \not\equiv 0$, suppose z_1 is a zero of $g-1$, but not a zero of $f-1$, and $z_1 \notin S_0$, $a(z_1)b'(z_1) - a'(z_1)b(z_1) \neq 0$, then z_1 is not a pole of $f-1$. In fact, if z_1 is a pole of $f-1$, z_1 is a pole of the left side of (15) and the right side of (18) with multyplicity 1, while z_1 is a pole of the right side of (15) and the left side of (18) with multyplicity 2. This is impossible. Hence the left side of (15) and the right side of (18) are analytic at z_1. So we have $f(z_1) - b(z_1) \neq 0$, $f(z_1) - a(z_1) \neq 0$, but $L(f,a,0)/_{z=z_1} = 0$, $L(f,0,b)/_{z=z_1} = 0$.
Thus

$$\begin{vmatrix} f(z_1) & f'(z_1) \\ a(z_1) & a'(z_1) \end{vmatrix} = 0, \begin{vmatrix} f(z_1) & f'(z_1) \\ b(z_1) & b'(z_1) \end{vmatrix} = 0.$$

Since $g(z_1) - 1 = 0$, f, g share 0 IM$''$, we have $f(z_1) \neq 0$. Hence we get

$$\begin{vmatrix} a(z_1) & a'(z_1) \\ b(z_1) & b'(z_1) \end{vmatrix} = 0,$$

That is $a(z_1)b'(z_1) - a'(z_1)b(z_1) = 0$, which contradicts $a(z_1)b'(z_1) - a'(z_1)b(z_1) \neq 0$. Hence $f(z_1) - 1 = 0$.

Then we have

$$g - 1 = 0 \implies f - 1 = 0. \qquad (r \notin E)$$

By symmetry, we have

$$f - 1 = 0 \implies g - 1 = 0. \qquad (r \notin E)$$

Here $E \subseteq R^+$ and $\text{mes}E < +\infty$. This means f, g share 1 IM$''$.

(ii) If $a(z)b'(z) - a'(z)b(z) \equiv 0$, then $a = kb$, $a' = kb'$, where $k(\neq 0)$ is a constant. So we have

$$L(g,a,0) \equiv \begin{vmatrix} g & g' \\ kb & kb' \end{vmatrix} \equiv -kL(g,0,b),$$

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$$L(f,a,0) \equiv \begin{vmatrix} f & f' \\ kb & kb' \end{vmatrix} \equiv -kL(f,0,b),$$

Substituting them to (18), and combining with (15), we have

$$\frac{f-a}{f-b} \equiv \frac{g-a}{g-b}.$$

So we get $f \equiv g$.

Case 4.

(i) If $b'(z) \not\equiv 0$, suppose z_a is a zero of $f-a$, but not a zero of $g-a$, and $z_a \notin S_0$, $b'(z_a) \neq 0$, then z_a is not a pole of $g-a$. In fact, if z_a is a pole of $g-a$, z_a is a pole of the right sides of $(16)'$ and (18) with multyplicity 1, while z_a is a pole of the left sides of $(16)'$ and (18) with multyplicity 2. This is impossible. Hence the right sides of $(16)'$ and (18) are analytic at z_a. So we have $f(z_a)-b(z_a) \neq 0$, $f(z_a)-1 \neq 0$, but $L(g,0,1)/_{z=z_a} = 0$, $L(g,0,b)/_{z=z_a} = 0$. Thus $g'(z_a) = 0$, and $g(z_a)b'(z_a) - g'(z_a)b(z_a) = 0$. So we get $g(z_a)b'(z_a) = 0$. Since $b'(z_a) \neq 0$, we have $g(z_a) = 0$. This contradicts $f,\ g$ share 0 IM$''$.

Then we have

$$f-a=0 \implies g-a=0. \qquad (r \notin E)$$

By symmetry, we have

$$g-a=0 \implies f-a=0. \qquad (r \notin E)$$

Here $E \subseteq R^+$ and mes$E < +\infty$. This means $f,\ g$ share a IM$''$.

(ii) If $b'(z) \equiv 0$, then $b(z) \equiv$ constant. So we have $L(f,0,b) \equiv kL(f,0,1)$, $L(g,0,b) \equiv kL(g,0,1)$, where $k = b \neq 0$. Substituting them to $(16)'$, and combining with (18), we have

$$\frac{f-b}{f-1} \equiv \frac{g-b}{g-1}.$$

So we get $f \equiv g$.

Case 5.

From Lemma 4 we know that

$$H_2 \equiv 0 \iff \tilde{H}_2 \equiv 0,$$

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and

$$H_3 \equiv 0 \Longleftrightarrow \tilde{H}_3 \equiv 0.$$

Here

$$\tilde{H}_2 = \frac{L(F,1,0)(F-G)L(G,0,b^{-1})}{(F-1)FG(G-b^{-1})} - \frac{L(G,1,0)(F-G)L(F,0,b^{-1})}{(G-1)GF(F-b^{-1})},$$

$$\tilde{H}_3 = \frac{L(F,0,a^{-1})(F-G)L(G,a^{-1},b^{-1})}{F(F-a^{-1})(G-a^{-1})(G-b^{-1})} - \frac{L(G,0,a^{-1})(F-G)L(F,a^{-1},b^{-1})}{G(G-a^{-1})(F-a^{-1})(F-b^{-1})}.$$

From $\tilde{H}_2 \equiv 0$ and $\tilde{H}_3 \equiv 0$ we have

$$\frac{L(F,1,0)L(G,0,b^{-1})}{(F-1)(G-b^{-1})} \equiv \frac{L(G,1,0)L(F,0,b^{-1})}{(G-1)(F-b^{-1})}, \tag{19}$$

and

$$\frac{L(F,0,a^{-1})L(G,a^{-1},b^{-1})}{F(G-b^{-1})} \equiv \frac{L(G,0,a^{-1})L(F,a^{-1},b^{-1})}{G(F-b^{-1})}. \tag{20}$$

Similar to the Case 2, we can prove that F, G share 0 IM$''$ or $F \equiv G$. So we get that f, g share ∞ IM$''$ or $f \equiv g$.

Hence we know that either $f(z) \equiv g(z)$, or f, g share $b(z)$, 0, 1, $a(z)$ and ∞ IM$''$. From Theorem B$'$ we get $f(z) \equiv g(z)$.

This completes the proof of Theorem 1.

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