SOME FINITENESS CONDITIONS CONCERNING INTERSECTIONS OF CONJUGATES OF SUBGROUPS

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1. Introduction. In [**3**], a group G was said to be a CF-group if, for every subgroup H of G, $H/\mathrm{Core}_G H$ is finite. It was shown there that a locally finite CF-group G is abelian-by-finite and that there is a bound for the indices $|H:\mathrm{Core}_G H|$ as H runs through all subgroups of G. (Groups for which such a bound exists were referred to in [**3**] as BCF-groups.) The CF-property was further investigated in [**10**], one of the main results there being that nilpotent CF-groups are (again) abelian-by-finite and BCF. In the present paper, we shall discuss the CF-property in conjunction with some related properties, which are defined as follows.

A group G has property S_1, A_1 or C_1 respectively if every subgroup, every abelian subgroup or every cyclic subgroup (respectively) of G has finite index over its core in G. A group G has property S_2, A_2 or C_2 respectively if the index $|H:H\cap H^x|$ is finite for every element x of G and every subgroup, every abelian subgroup or every cyclic subgroup H of G.

It is clear that S_1, S_2, C_1 and C_2 are inherited by homomorphic images. Not surprisingly, this is not true of properties A_1 and A_2, and a suitable example is given in Section 5 below. The class of S_1-groups is precisely that of CF-groups, and it is contained in all of the other classes. Indeed, it is clear that $S_1 \Rightarrow S_2$, $A_1 \Rightarrow A_2$ and $C_1 \Rightarrow C_2$ and that $S_i \Rightarrow A_i \Rightarrow C_i$, $i = 1, 2$. We shall see in Section 5 that there are no further implications between any of the above properties, even with the additional hypothesis of nilpotency (which, as stated above, proved quite decisive for CF-groups). Another hypothesis that it is reasonable to impose is that of finite generation. It will be shown that a finitely generated C_2-group is a C_1-group and that a finitely generated A_2-group satisfies A_1. We have not resolved the problem as to whether a finitely generated S_2-group is S_1, but such evidence as we have suggests to us that this is the case. On the other hand, for finitely generated soluble groups, the property C_2 certainly suffices to ensure that S_1 holds (see Corollary 2.5). The key to establishing the relevant properties of C_2-groups is Proposition 2.1, which shows that torsionfree elements of a C_2-group "almost commute"—a finitely generated torsionfree C_2-group is then seen to be centre-by-periodic. Clearly such groups will be centre-by-finite in many cases, and centre-by-finite groups are of course BCF.

2. Cyclic subgroups. We begin by considering the relationship between any two torsionfree elements in a C_2-group. We have the following.

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PROPOSITION 2.1. *Let G be a group such that $|\langle g\rangle : \langle g\rangle \cap \langle g\rangle^h|$ is finite for all g, h in G and let a, x be elements of infinite order in G. Then there exist nonzero integers α, β such that $[a, x^\alpha] = 1 = [x, a^\beta]$.*

Proof. For a contradiction, we may assume that no nonzero power of x centralizes a. By hypothesis, there exist nonzero integers m, n, r, s such that $(a^x)^m = a^n$, $(x^a)^r = x^s$. Set $d = (m, n)$ (the greatest common divisor) and write $m = dm_1$, $n = dn_1$. Similarly, let $r = er_1$, $s = es_1$, where $e = (r, s)$. Thus $(m_1, n_1) = 1 = (r_1, s_1)$. Let $v = a^d$, $w = x^e$ and suppose first that $|r_1| = |s_1| = 1$. Thus $r = \pm s$. If $r = -s$ then $(x^r)^a = x^{-r}$, that is, $(x^{er_1})^a = x^{-er_1}$ and so $w^a = w^{-1}$ and $w^{a^2} = w$, giving $(wa^2)^a = w^{-1}a^2$ and hence $\langle wa^2\rangle^a = \langle w^{-1}a^2\rangle$. Now if $w = a^{-2}$ then $w^a = w = w^{-1}$ and we have the contradiction $w^2 = 1$. Thus $\langle wa^2\rangle \neq 1$ and, by the C_2-property, it follows that $\langle wa^2\rangle \cap \langle w^{-1}a^2\rangle \neq 1$, that is, there exist nonzero integers γ, δ such that $(w^{-1}a^2)^\gamma = (wa^2)^\delta$. Since a^2 and w commute, we have $w^{\gamma+\delta} = a^{2(\gamma-\delta)}$, giving $(w^{\gamma+\delta})^a = w^{\gamma+\delta}$ and hence $\gamma + \delta = 0$. Thus $a^{2(\gamma-\delta)} = 1$ and $\gamma = \delta = 0$, a contradiction which shows that if $|r_1| = |s_1| = 1$ then $r = s$ and hence $[x^r, a] = 1$, another contradiction. So either $|r_1| \neq 1$ or $|s_1| \neq 1$.

Now write $m_2 = m_1^e$, $n_2 = n_1^e$, $r_2 = r_1^d$, $s_2 = s_1^d$. Then $(v^{m_2})^w = v^{n_2}$ and $(w^{r_2})^v = w^{s_2}$. Further, $(v^{m_2^{r_2}})^{w^{r_2}} = v^{n_2^{r_2}}$ and so $(v^{m_2^{r_2}})^{w^{s_2}} = (v^{m_2^{r_2}})^{(w^{r_2})^v} = v^{-1}((v^{m_2})^{r_2})^{w^{r_2}}v = v^{n_2^{r_2}}$, that is, $(v^{m_2^{r_2}})^{w^{s_2}} = (v^{m_2^{r_2}})^{w^{r_2}}$ and so $[v^{m_2^{r_2}}, w^{r_2-s_2}] = 1$. Similarly, $[w^{r_2^{m_2}}, v^{m_2-n_2}] = 1$. Now $|r_2|$ and $|s_2|$ are coprime and not both equal to 1, so $r_2 - s_2 \neq 0$. Also, $z = v^{(m_2^{r_2})(m_2-n_2)}$ is centralized by both $w^{r_2-s_2}$ and $w^{r_2^{m_2}}$ and hence by w (since $(r_2 - s_2, r_2) = 1$). Thus $z = z^w = ((v^{m_2})^w)^{m_2^{r_2-1}(m_2-n_2)} = v^{n_2 m_2^{r_2-1}(m_2-n_2)}$. By torsionfreeness we have $v^{m_2(m_2-n_2)} = v^{n_2(m_2-n_2)}$ and hence $m_2 = n_2$, which implies $|m_1| = |n_1| = 1$ and $m = \pm n$. But $m = -n$ leads to a contradiction in exactly the same way as did the hypothesis $r = -s$, so we may assume $(a^m)^x = a^m$ and thus $x^{a^m} = x$. Now $(x^r)^a = x^s$ and so $(x^{r^m})^{a^m} = x^{s^m}$. But $[x, a^m] = 1$ implies $(x^{r^m})^{a^m} = x^{r^m}$ and we have $x^{r^m} = x^{s^m}$ and hence $r = \pm s$ (again by torsionfreeness). This in turn implies $|r_1| = |s_1| = 1$, which we have seen to be impossible. This completes the proof of Proposition 2.1.

Note that the infinite dihedral group $\langle x, a : x^a = x^{-1}, a^2 = 1\rangle$ satisfies C_2 but no nonzero power of x centralizes a. Thus torsionfreeness is an essential hypothesis in the above.

The main consequence for us of Proposition 2.1 is the following.

THEOREM 2.2. *Every finitely generated C_2-group satisfies C_1.*

Proof. Let $\{g_1, \ldots, g_r\}$ be a generating set for the C_2-group G and let $x \in G$. If x has finite order then of course $\langle x\rangle/\mathrm{Core}_G\langle x\rangle$ is finite, so suppose $\langle x\rangle$ is infinite. If g_i has infinite order then, by Proposition 2.1, there is a nonzero integer α_i such that $[x^{\alpha_i}, g_i] = 1$, while if g_i has finite order k then there exist nonzero integers m_i, n_i such that $(x^{m_i})^{g_i} = x^{n_i}$ and hence $x^{m_i^k} = (x^{m_i^k})^{g_i^k} = x^{n_i^k}$ and $m_i = \pm n_i$. Let $\alpha_i = m_i$ in this case and write $\alpha = \alpha_1 \ldots \alpha_r$. Then $\langle x^\alpha\rangle \lhd G$ and the theorem is proved.

The next result is also a consequence of Proposition 2.1. Its proof is easy and is omitted.

COROLLARY 2.3. *Let G be a finitely generated torsionfree C_2-group and let Z denote the centre of G. Then G/Z is periodic.*

In the context of this corollary, it is worth remarking that Adian has constructed a finitely generated torsionfree group G, with cyclic centre Z, such that G/Z is a Tarski p-group. Every subgroup of G thus has index at most p over its core.

As we mentioned above, the infinite dihedral group satisfies C_2, although it is not centre-by-periodic. However, finitely generated groups satisfying C_2 are almost centre-by-periodic, in the following sense.

COROLLARY 2.4. *Let G be a finitely generated C_2-group. Then there is a normal subgroup C of G such that $|G/C| \leq 2$ and $C/Z(C)$ is periodic. In particular, G is abelian-by-periodic.*

Proof. By Theorem 2.2, G satisfies C_1. Let A be the subgroup generated by all infinite cyclic normal subgroups of G. As in Lemma 4.3 of [**3**], A is abelian and its centralizer C has index at most 2 in G. By the C_1-property, C/A is periodic.

Of course, for many familiar classes $\mathscr{X}$ of groups, all finitely generated periodic $\mathscr{X}$-groups are finite. A class which contains many such $\mathscr{X}$ as subclasses (for instance, the class of hyperabelian groups) is that of groups with ascending series whose factors are locally (nilpotent or finite). Denoting this class by $\acute{P}L(\mathscr{N} \cup \mathscr{F})$, we have the following.

COROLLARY 2.5. *Let G be a finitely generated C_2-group belonging to the class $\acute{P}L(\mathscr{N} \cup \mathscr{F})$. Then G is abelian-by-finite and BCF. In particular, G satisfies S_1.*

Proof. By Corollary 2.4, such a group G is abelian-by-finite. The argument of the proof of Corollary 4.4 of [**3**] now shows that G is *BCF*.

We see from Corollary 2.5 that a torsionfree, locally nilpotent C_2-group is abelian. We now record a few more results on torsionfree C_2-groups. Firstly we note that, with the notation of Corollary 2.4, if $|G:C| = 2$ then there exists $x \in G$ such that $g^x = g^{-1}$ for all g in A (see Lemma 4.3 of [**3**]), but this is impossible if G is torsionfree, since G/A is periodic. Thus $A \leq Z(G)$ and G is centre-by-periodic. Since a torsionfree centre-by-finite group is abelian, we may state the following.

COROLLARY 2.6. *Let G be a torsionfree C_2-group. If G is finitely generated and every periodic image of G is finite then G is abelian. In particular, if G is locally radical or locally (soluble-by-finite) then G is abelian.*

3. Abelian subgroups. We turn our attention now to the properties A_1 and A_2. Firstly, we recall from [**3**] that a subgroup H of a group G is said to be G-hamiltonian if every (cyclic) subgroup of H is normal in G. If x is an element of G and H is a subgroup of G, we shall say that H is x-hamiltonian if every subgroup of H is normalized by x. Now suppose that G is a C_1-group and that $N = \langle g \in G : \langle g \rangle \lhd G \text{ and } |g| = \infty \rangle$. Then N is abelian and even G-hamiltonian (see the remarks preceding Corollary 2.6), and G/N is periodic. This property of N is useful in the proof of our next main result.

THEOREM 3.1. *Let G be a finitely generated A_2-group. Then G satisfies A_1.*

Proof. Let G be as stated and assume, for a contradiction, that some abelian subgroup A has infinite index over its core C. By Theorem 2.2, G is a C_1-group and so, with the notation as above, N is G-hamiltonian. Thus $A \cap N \lhd G$ and, in particular, $A \cap N \leqslant C$ and A/C is periodic. Although the property A_2 is not inherited by homomorphic images, from now on we shall only be using the A_2-property as it applies to subgroups of A. For convenience, then, we shall assume that $C = 1$ and hence that A is periodic. Every subgroup of A of type C_{p^∞} is G-hamiltonian and so A must be reduced. If some p-component of A is infinite then we may assume that A is an elementary p-group. Otherwise, we may suppose that $A = \underset{p \in \pi}{\mathrm{Dr}} \langle a_p \rangle$, where each a_p has prime order p and the set π is infinite. In the latter case, let x be an arbitrary element of G. Then $A \cap A^x$ has finite index n, say, in A. For all primes p not dividing n we have $a_p \in A \cap A^x$. Thus $\langle a_p \rangle = \langle a_p^x \rangle$ and $x \in N_G(\langle a_p \rangle)$. Letting x run through a generating set for G, we see that $\langle a_p \rangle \lhd G$ for almost all p in π, a contradiction. Thus we may assume that A is an elementary p-group, for some prime p. Again let $x \in G$.

Claim. A is "almost x-hamiltonian", that is, there is a subgroup K of finite index in A such that $\langle a \rangle^x = \langle a \rangle$ for all a in K. Suppose that the claim is false. Since $A \cap A^{x^{-1}}$ has finite index in A, there exists a_1 in $A \cap A^{x^{-1}}$ such that $\langle a_1 \rangle \neq \langle a_1 \rangle^x$. Let $B_1 = \langle a_1, a_1^x \rangle$. Then $B_1 \leqslant A$ and $A = B_1 \times A_1$, for some A_1. Since $A_1 \cap A_1^{x^{-1}}$ has finite index in A_1 and hence in A, there exists $a_2 \in A_1 \cap A_1^{x^{-1}}$ such that $\langle a_2 \rangle \neq \langle a_2 \rangle^x$. Let $B_2 = \langle a_2, a_2^x \rangle$. Then $B_2 \leqslant A_1$ and so $\langle B_1, B_2 \rangle = B_1 \times B_2$ and $A_1 = B_2 \times A_2$, for some A_2. We may then choose a_3 in $A_2 \cap A_2^{x^{-1}}$ such that $\langle a_3 \rangle \neq \langle a_3 \rangle^x$ and we get $B_3 = \langle a_3, a_3^x \rangle \leqslant A_2$. Eventually, we obtain a subgroup B of A, where $B = B_1 \times B_2 \times \ldots, B_i = \langle a_i, a_i^x \rangle$ and $\langle a_i \rangle \neq \langle a_i \rangle^x$ for each i. Let $D = \langle a_1^x, a_2^x, \ldots \rangle$. Then $D^{x^{-1}} = \langle a_1, a_2, \ldots \rangle$ and so $D \cap D^{x^{-1}} = 1$, contradicting the A_2-property. This establishes the claim.

Now let X be a finite generating set for G. Then A is almost x-hamiltonian for each x in $X \cup X^{-1}$ and so A certainly has finite index over its core in G, a contradiction.

The theorem is thus proved.

4. Arbitrary subgroups. Our main concern in this section is with the question: Does every finitely generated S_2-group satisfy S_1? As stated in the introduction, we do not know the answer, but we have been able to effect a reduction which allows us to deal with one or two special cases and provides strong evidence (in our view) that the answer to this question is in the affirmative. We begin with a result which allows us to focus our attention on finitely generated subgroups.

LEMMA 4.1. *Let G be a finitely generated S_2-group in which every finitely generated subgroup has finite index over its core in G. Then G has the property S_1.*

Proof. Suppose that G is as given and assume, for a contradiction, that there is a subgroup H of G such that $H/\mathrm{Core}_G\, H$ is infinite. Now it is easy to see that the property on finitely generated subgroups is inherited by homomorphic images of G and so we may assume H is corefree in G. Since H is countable, there is an ascending chain $H_1 < H_2 < \ldots$ of finitely generated subgroups whose union is H. Then each H_i is corefree in G and therefore finite, and so H is locally finite. Since H is also infinite, it contains an infinite abelian subgroup [**5**] and we may thus assume that H is abelian. Theorem 3.1 now gives a contradiction.

We note that the above proof also establishes the following.

LEMMA 4.2. *Let G be a finitely generated S_2-group and H a locally finite subgroup of G. Then (every subgroup of) H is finite over its core in G.*

Our next reduction is to the case where G is periodic.

LEMMA 4.3. *Let G be a finitely generated S_2-group all of whose periodic images satisfy S_1. Then G also satisfies S_1.*

Proof. By Corollary 2.4, there is a normal subgroup C of index at most 2 in G and a G-invariant subgroup A of $Z(C)$ such that G/A is periodic. With A as defined in Corollary 2.4, we also know that every subgroup of A is normal in G (again see the remarks preceding Corollary 2.6). Let H be an arbitrary subgroup of G. In order to show that $H/\mathrm{Core}_G H$ is finite, we may of course assume that H is contained in C and that H is corefree in G. By hypothesis, there is a normal subgroup N of G such that $A \leqslant N \leqslant HA$ and HA/N is finite. Then $N = A(N \cap H)$ and $|H:N \cap H|$ is finite and we may assume that $N = AH$. We now have $N' = (AH)' = A'H'[A, H] = H'$, since $H \leqslant C$ and $A \leqslant Z(C)$. Since H is corefree and $N' \lhd G$ we see that H is abelian. Also, $H \cap A$ is normal in G and hence trivial, giving H periodic and hence of finite index over its core, by Lemma 4.2.

We have already seen that it is finitely generated subgroups of a finitely generated S_2-group that need to be considered. The following lemma provides the basis for an inductive argument. Its proof will involve an appeal to some deep results of Zel'manov and others and, although this is somewhat unsatisfactory here, we note that we are in the realm of finitely generated infinite periodic groups and that a positive solution to the question we have raised may well depend on some difficult theorems. In any case, here is the reduction.

LEMMA 4.4. *Let G be a periodic S_2-group and let H, K be finitely generated subgroups of G such that $H \leqslant K$. Suppose that $H/\mathrm{Core}_K H$ and $K/\mathrm{Core}_G K$ are finite. Then $H/\mathrm{Core}_G H$ is finite.*

Proof. We shall assume that H is corefree in G. Set $L = \mathrm{Core}_G K$, $M = \mathrm{Core}_K H$, $N = L \cap M$. Then H/N is finite and N is normal in K and so $N \lhd N^G \lhd G$. Now for all g in G we have $N^g \lhd N^G$ and $|N^gN:N| = |N^g:N^g \cap N|$, which is finite. Thus N^G/N is locally finite. Further, K/N is a finitely generated S_2-group and hence, by Lemma 4.2, every subgroup of N^G/N is finite over its core in K/N and hence over its core in N^G/N. By the main result of [3], N^G/N is abelian-by-finite. Let T be a normal subgroup of finite index t, say, in N^G such that $N \leqslant T$ and T/N is abelian, and let $P = ((N^G)^t)'$. Then $P \lhd G$ and $P \leqslant T' \leqslant N \leqslant H$. Since H is corefree, $P = 1$ and so N^t is abelian. Also, H/N^t has finite exponent. Let B be a maximal normal abelian subgroup of H containing N^t. Again since H is corefree in G, the S_2-property tells us that H is residually finite and hence that H/B is residually finite (by the maximality of B). But H/B is finitely generated and of finite exponent and so, by Zel'manov's solution to the Restricted Burnside Problem ([**11**] and [**12**]), there is a bound for the order of a finite image of H/B. It follows that H/B is finite and so H is finitely generated abelian-by-finite and hence finite, since G is periodic. The result follows.

Our final reduction is to the case where $G = \langle H, x \rangle$ (where H is finitely generated and, for a contradiction, has infinite index over its core). Probably the clearest manner in which to make our point is simply to exhibit the reduction for what it is.

LEMMA 4.5. *If there exists a finitely generated S_2-group which is not S_1, then there exists such a group G which is periodic and generated by a finitely generated subgroup H and an element x such that $H/\mathrm{Core}_G\, H$ is infinite.*

Proof. Suppose G is a finitely generated periodic S_2-group which is not S_1. By Lemma 4.1, there is a finitely generated subgroup H of G which has infinite index over its core. Then $G = \langle H, g_1, \ldots, g_r \rangle$ for some elements g_i and we may assume, as inductive hypothesis, that r is minimal subject to the pair (G, H) being a counterexample (with H finitely generated). Since $r > 1$ we may write $K = \langle H, g_1, \ldots, g_{r-1} \rangle$. The induction hypothesis and Lemma 4.4 now give the contradiction that $H/\mathrm{Core}_G\, H$ is finite. The lemma is thus proved.

Lemma 4.5 seems to provide us with quite a substantial reduction of the problem. We now consider the special case where G is a p-group. Even here we have met with only limited success. The key result is the following.

PROPOSITION 4.6. *Let G be an S_2-group and H a proper subgroup of G. If G is a p-group, for some prime p, then there exists an element g of $G \backslash H$ such that $H/\mathrm{Core}_{G_1}\, H$ is finite, where $G_1 = \langle H, g \rangle$.*

Proof. Let $x \in G\backslash H$ and write $n = |x|$, $V = H \cap H^x \cap \ldots \cap H^{x^{n-1}}$. Then $|H:V|$ is finite and V is normal in $\langle V, x \rangle$. Thus the set $\Omega = \{T \leqslant H : T \geqslant V$ and $\exists g \in G\backslash H$ such that $T/\mathrm{Core}_{\langle T,g \rangle}\, T$ is finite$\}$ is nonempty. Let M be maximal in Ω. Then there exists g in $G\backslash H$ and a normal subgroup C of $\langle M, g \rangle$ such that $C \leqslant M$ and M/C is finite. If $M = H$ then we have the result, so assume, for a contradiction, that $M < H$. So there is a subgroup B of H, with M a normal subgroup of B, such that $|B/M| = p$ and, for every $x \in G/H$, $B/\mathrm{Core}_{\langle B,x \rangle}\, B$ is infinite. Thus $B \nsubseteq \langle M, g \rangle$. Write $B = M\langle h \rangle$, with $h^p \in M$ and $S = \langle M, g \rangle \cap \langle M, g \rangle^h \cap \ldots \cap \langle M, g \rangle^{h^{r-1}}$. Clearly $M \leqslant S$ and $B \leqslant N_G(S)$. Further, $|\langle M, g \rangle : S|$ is finite. Now suppose $S \nsubseteq H$. Then $\langle S, B \rangle = SB \geqslant S$ and $|SB:S| = |B:S \cap B| = |B:M| = p$ and hence $S \lhd SB$, $|SB/S| = p$ and $SB = S\langle h \rangle$. We have $C \lhd S$ and M/C finite and, for every $i = 0, 1, \ldots, p-1$, $C^{h^i} \lhd S$, M/C^{h^i} finite. So, writing $D = C \cap C^h \cap \ldots \cap C^{h^{p-1}}$, we have $S \leqslant N_G(D)$, B/D finite and $D \lhd SB \nsubseteq H$, contradicting the maximality of M. Thus $S \subseteq H$. Therefore $|\langle M, g \rangle : \langle M, g \rangle \cap H|$ is finite and $\langle M, g \rangle / C$ is a finite p-group. Hence there exists $y \in \langle M, g \rangle \backslash H$ such that $y \in N_G(\langle M, g \rangle \cap H)$. Hence $\langle M, g \rangle \cap H = M$, since M is maximal in Ω. But now we have $y \in N_G(M)$, $B \leqslant N_G(M)$ and $M \lhd \langle B, y \rangle$, again a contradiction to the maximality of M. This completes the proof.

We now present a few positive results on S_2-groups.

THEOREM 4.7. *Let G be an S_2-group with the maximal condition on subgroups. If G is a p-group, for some prime p, then G satisfies S_1.*

THEOREM 4.8. *Let G be an S_2-group with is also a 2-group. Then G is locally nilpotent.*

COROLLARY 4.9. *If the 2-group G is a finitely generated S_2-group then G is finite (and therefore certainly an S_1-group).*

Proof of Theorem 4.7. Let G be as stated. Assuming the result false, let H be a subgroup of G which is maximal subject to $H/\mathrm{Core}_G\, H$ being infinite. By Proposition 4.6,

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$H/\text{Core}_K H$ is finite for some $K = \langle H, g \rangle > H$. By maximality, $K/\text{Core}_G K$ is finite. The result follows by Lemma 4.4.

We remark that Theorem 4.7 does not in fact require the results of Zel'manov—the proof of Lemma 4.4 is much more elementary in the case where G satisfies max, since (with the same notation) the group N^G/N is locally finite with max and hence finite, which implies $N/\text{Core}_G N$ finite and thus $H/\text{Core}_G H$ finite.

Proof of Theorem 4.8. Again with G as stated, let H be an arbitrary proper subgroup of G. We shall show that $H < N_G(H)$—the result will follow from a theorem of Plotkin (see Section 6.1 of [**8**]). By Proposition 4.6, there exists $g \in G \backslash H$ such that $H/\text{Core}_{G_1} H$ is finite, where $G_1 = \langle H, g \rangle$. Then $H/\text{Core}_{G_1} H$ is a proper finite subgroup of the 2-group $\bar{G} = G_1/\text{Core}_{G_1} H$ and is thus properly contained in its normalizer in $\bar{G}$ (see Theorem 3.15 of [**8**]). The result follows.

The existence of finitely generated, infinite p-groups all of whose proper subgroups are cyclic (see, for instance, [**7**]) shows that there is no equivalent of Theorem 4.8 for odd primes p.

We conclude this section with the obvious

CONJECTURE. *Every finitely generated S_2-group is an S_1-group.*

5. Some examples. We remarked in the introduction that there are certain very obvious inclusions among the classes of groups defined by our six properties. We now show that these are the only inclusions.

THEOREM 5.1. (a) $S_1 \Rightarrow A_1 \Rightarrow C_1$, $S_2 \Rightarrow A_2 \Rightarrow C_2$, $S_1 \Rightarrow S_2$, $A_1 \Rightarrow A_2$, $C_1 \Rightarrow C_2$.

(b) *Apart from the implications given in* (a) (*and those which are formal logical consequences of these*) *there are no further implications among the six properties, even with the additional hypothesis of nilpotency.*

Proof. Of course, only part (b) needs verifying. To do this, we exhibit three nilpotent groups G_1, G_2 and G_3 such that G_1 has S_2 but not C_1, G_2 has A_1 but not S_2 and G_3 has C_1 but not A_2.

For each prime p, let $\langle a_p \rangle$ be a cyclic group of order p^2 and set $A = \text{Dr}_p \langle a_p \rangle$. Define $z \in \text{Aut}\, A$ by $a_p^z = a_p^{p+1}$ for all p, and write $G_1 = A]\langle z \rangle$. Then $G_1' = \text{Dr}_p \langle a_p^p \rangle = Z(G_1)$ and G_1 is nil-2 (and of rank 2). Clearly $\text{Core}_{G_1}\langle z \rangle = 1$ and so G_1 is not C_1. Now let $H \leq G_1$, $x \in G_1$. We wish to show that $|H : H \cap H^x|$ is finite. Modulo $H \cap A$ (which is normal in G_1) we have H cyclic and hence $\langle H, x \rangle$ finitely generated and finite-by-cyclic. This gives $\langle H, x \rangle$ centre-by-finite (mod $H \cap A$) and the result follows.

Next, let p be an odd prime and G_2 the free nil-2, exponent-p group on generators $a_0, a_1, a_2, \ldots$. So $G_2' = Z(G_2) = \langle [a_i, a_j] : i < j \rangle = Z$, say. If A is an arbitrary abelian subgroup of G_2 then it is easy to see that $|AZ/Z| \leq p$ and so G_2 is certainly an A_1-group. Now let $H = \langle a_1, a_2, \ldots \rangle$ and write $B = H' = \langle [a_i, a_j] : i, j > 0 \rangle$, $x = a_0$. Then $H^x = \langle a_i[a_i, a_0] : i = 1, 2, \ldots \rangle$ and clearly $H \cap H^x = B$ and so $|H : H \cap H^x|$ is infinite and G_2 is not an S_2-group.

For our group G_3 we take the wreath product of an infinite elementary abelian

p-group A by a cyclic group $\langle x \rangle$ of order p. Then G_3 is nilpotent [**2**] and obviously satisfies C_1, since it is periodic. However, identifying A with the "first component subgroup" of the base group of G_3, we have that $A \cap A^x = 1$ and hence G_3 is not an A_2-group.

This completes the proof of the theorem.

The group G_2 above also provides an example of an A_1-group which has a homomorphic image not satisfying A_2—indeed, with H and B as defined, the image of H in G_2/B is abelian but has infinite index over its intersection with (the image of) H^x. Further, the fact that an A_1-group need not satisfy S_1 shows that there is no generalization of the main theorem of [**3**] along the lines of Eremin's improvement to the theorem of B. H. Neumann on groups with finite classes of conjugate subgroups (see [**6**] and [**4**]).

The property of being finitely generated has been seen to be quite a strong one with regard to the pairs (A_1, A_2) and (C_1, C_2), and at least of some influence with regard to (S_1, S_2). We may now ask whether finite generation leads to some interdependence between these pairs, other than as stated in Theorem 5.1. That there are no further implications of this kind is the import of our final result, although we recall that the situation is quite different for soluble groups (Corollary 2.5).

THEOREM 5.2. *There exist finitely generated groups G_4, G_5 such that G_4 satisfies C_1 but not A_2, while G_5 satisfies A_1 but not S_2.*

Proof. Let H be any finitely generated, infinite periodic group having an infinite abelian subgroup A (see e.g. [**9**]) and let G_4 be the wreath product of H with a cyclic group $\langle x \rangle$ of order 2. Then G_4 is finitely generated and periodic and hence a C_1-group, but $A \cap A^x = 1$ and so G_4 does not satisfy A_2.

Now let K be any finitely generated, infinite periodic group in which all abelian subgroups are finite (see [**1**]) and let $G_4 = K \operatorname{wr}\langle x \rangle$, where again x has order 2. Then all abelian subgroups of G_5 are finite and so G_5 is an A_1-group. But $K \cap K^x = 1$ and so G_5 does not satisfy S_2.

The easy constructions above rely, of course, on some highly nontrivial examples. Corollary 2.5 may provide us with some justification for this apparent extravagance.

REFERENCES

1. S. I. Adian, *The problem of Burnside and identities in groups*, (Nauka, 1975) (Russian). (Translation by J. C. Lennox and J. Wiegold, (Springer-Verlag 1979)).

2. G. Baumslag, Wreath products and p-groups, *Proc. Cambridge Phil. Soc.* **55** (1959), 224–231.

3. J. T. Buckley, J. C. Lennox, B. H. Neumann, H. Smith and J. Wiegold, Groups with all subgroups normal-by-finite, *J. Australian Math. Soc.* (to appear).

4. I. I. Eremin, Groups with finite classes of conjugate abelian subgroups, *Dokl. Akad. Nauk.* SSSR (N.S.) **118** (1958), 223–224.

5. P. Hall and C. R. Kulatilaka, A property of locally finite groups, *J. London Math. Soc.* **39** (1964), 235–239.

6. B. H. Neumann, Groups with finite classes of conjugate subgroups, *Math. Z.* **63** (1955), 76–96.

7. A. Yu. Ol'shanskii, *Geometry of defining relations in groups*, (Nauka, 1989).

8. D. J. S. Robinson, *Finiteness conditions and generalized soluble groups*, (Springer-Verlag 1972).

9. A. V. Rozhkov, On subgroups of infinite finitely generated p-groups, *Mat. Sb.* **129** (171) (1986) (=*Math. USSR Sbornik* **57** (1987), No. 2).

10. H. Smith and J. Wiegold, Locally graded groups with all subgroups normal-by-finite, *J. Austral. Math. Soc.*, to appear.

11. E. I. Zel'manov, Solution of the restricted Burnside problem for groups of odd exponent, *Izv. Akad. Nauk. SSR Ser. Mat.* **54** (1990), 42–59.

12. E. I Zel'manov, Solution of the restructed Burnside problem for 2-groups, *Mat. Sb.* **182** (1991), 568–592.

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