Proceedings of the Edinburgh Mathematical Society (1995) **38**, 121–131 ©

THE CO-LOCALIZATION OF AN ARTINIAN MODULE

by LEIF MELKERSSON and PETER SCHENZEL*

(Received 1st June 1993)

For a multiplicative set S of a commutative ring R we define the co-localization functor $\operatorname{Hom}_R(R_S,\cdot)$. It is a functor on the category of R-modules to the category of R_S-modules. It is shown to be exact on the category of Artinian R-modules. While the co-localization of an Artinian module is almost never an Artinian R_S-module it inherits many good properties of A, e.g. it has a secondary representation. The construction is applied to the dual of a result of Bourbaki, a description of asymptotic prime divisors and the co-support of an Artinian module.

1991 *Mathematics subject classification*: 13C05.

1. Introduction

In this paper we define the co-localization $\operatorname{Hom}_R(R_S,A)$ of an Artinian R-module A with respect to a multiplicative set S in R. Here R always is a commutative ring, but not necessarily Noetherian. Our first hope was that this construction would give an Artinian R_S-module, but in fact this is very seldom the case. Nevertheless the co-localization of an Artinian R-module A has many good properties inherited from A. For example it always has a secondary representation. In fact starting from a minimal secondary representation of A we can construct a minimal secondary representation of $\operatorname{Hom}_R(R_S,A)$. This will give a description of the attached prime ideals of the R_S-module $\operatorname{Hom}_R(R_S,A)$.

The technique of co-localization is first applied to describe the attached prime ideals of a tensor product. This description is dual to the one of Bourbaki of $\operatorname{Ass}_R \operatorname{Hom}_R(N,M)$ where N and M are modules over a Noetherian ring R, N being in addition finitely generated. For this purpose a natural isomorphism is established. Next the co-localization is used in order to show a result of Taherizadeh [12], concerning a prime in the difference set $\operatorname{At}(I,A)\backslash\operatorname{Bt}(I,A)$, where

$$\operatorname{At}(I,A)=\operatorname{Att}_R 0:_A I^n \quad\text{and}\quad \operatorname{Bt}(I,A)=\operatorname{Att}_R 0:_A I^{n+1}/0:_A I^n$$

for all large n. That these two sequences of sets of prime ideals both become eventually constant has been shown by Sharp [10]; see also [7] for a proof. We conclude with a

*During the preparation of this note the second named author was supported by the United Kingdom SERC (Grant No. F 55584).

few comments on the co-support of a module defined by the co-localization in a way similar to the support.

A preliminary version of this paper contained an error. Due to a misunderstanding of the R_S-module structure on $\mathrm{Hom}_R(R_S, A)$ we claimed that the co-localization for an Artinian R-module A is an Artinian R_S-module. The authors are grateful to R. Y. Sharp for pointing out the incorrectness of our argument.

2. Definition and exactness of the co-localization

Definition 2.1. The co-localization of an R-module X with respect to the multiplicative set S in R is the R_S-module $\mathrm{Hom}_R(R_S, X)$.

For an Artinian R-module A let $S(A)$ denote the S-component of A, $S(A)=\bigcap_{s\in S} sA$, which is in fact equal to sA for some $s\in S$. Observe that if $I=\bigcup_{s\in S} 0:_R s$, which is an ideal of R, then $IS(A)=0$, so $S(A)$ can be considered to be a module over $\bar{R}=R/I$. To see this take $a\in I$ and $x\in S(A)$. Then $sa=0$ for some $s\in S$, and since $x\in sA$, $ax=0$. Let $\bar{S}$ be the image of S in $\bar{R}$. Hence $\bar{S}$ is a multiplicative set in $\bar{R}$ consisting of nonzero divisors. Define a relation $\leqq$ on $\bar{S}$ by $\sigma\leqq\tau$ if there is an $\alpha\in\bar{R}$ such that $\tau=\alpha\sigma$. Then $(\bar{S}, \leqq)$ becomes a directed ordered set.

Let us define a direct system $\{R_\sigma, f_\tau^\sigma\}$ of R-modules over $\bar{S}$ by $R_\sigma=\bar{R}$ and $f_\tau^\sigma: R_\sigma\to R_\tau$ as multiplication by α, where $\tau=\alpha\sigma$. The direct limit of this direct system is isomorphic to $\bar{R}_{\bar{S}}\simeq R_S$, see [7, p. 36]. By virtue of [9, Theorem 2.27], for each R-module X there is a natural isomorphism

$$\mathrm{Hom}_R(\varinjlim \{R_\sigma, f_\tau^\sigma\}, X)\simeq \varprojlim \{\mathrm{Hom}_R(R_\sigma, X), g_\sigma^\tau\},$$

where $g_\sigma^\tau=\mathrm{Hom}_R(f_\tau^\sigma, X)$. So the following result is shown.

Proposition 2.2. *There is a natural isomophism*

$$\mathrm{Hom}_R(R_S, X)\simeq \varprojlim \{\mathrm{Hom}_R(R_\sigma, X), g_\sigma^\tau\}$$

for any R-module X.

In case S consists of nonzero divisors, one may think of the co-localization of X with respect to S as the following R-module

$$\left\{(x_s)_{s\in S}\in \prod_{s\in S} X_s : x_s=tx_{st} \quad \text{for all} \quad s,t\in S\right\}.$$

Note that the isomorphism is given by $x_s=\varphi(\frac{1}{s})$, $\varphi\in\mathrm{Hom}_R(R_S, X)$.

Let I be a directed ordered set and $\varphi_i: \{X_i, f_i^j\}\to\{Y_i, g_i^j\}$ a morphism of inverse systems of R-modules, indexed by I. Even if for each i, φ_i is surjective, the induced map $\varprojlim \varphi_j$ is not in general surjective. However, this is true under a certain additional condition.

Lemma 2.3. *If $\varphi_i:\{X_i, f_i^j\} \to \{Y_i, g_i^j\}$ is an inverse system of R-modules such that for each i, φ_i is surjective and* $\operatorname{Ker}\varphi_i$ *is an Artinian R-module, then the induced map*

$$\varprojlim \varphi_i : \varprojlim \{X_i, f_i^j\} \to \varprojlim \{Y_i, g_i^j\}$$

is surjective.

Proof. Let $y=(y_i)\in \varprojlim Y_i$ and put $W_i=\varphi_i^{-1}(\{y_i\})$. This is a coset of the Artinian submodule $\operatorname{Ker}\varphi_i$ in X_i. Clearly $f_i^j(W_j)\subset W_i$ for all $i\leq j$ and we get an induced inverse system $\{W_i\}$ of sets, whose limit we have to show is nonempty. For each $i\in I$ let $\mathcal{S}_i$ be the collection of subsets of W_i, which are cosets of submodules of $\operatorname{Ker}\varphi_i$ together with the empty set. Then it is easily seen that the family $\{\mathcal{S}_i\}_{i\in I}$ satisfies the conditions in the theorem on p. 85 in [1]. So we conclude that $\varprojlim W_i$ is not empty. □

Proposition 2.4. *Let $0\to A'\to A\to A''\to 0$ be a short exact sequence of Artinian R-modules. Then the derived sequence*

$$0\to \operatorname{Hom}_R(R_S, A')\to \operatorname{Hom}_R(R_S, A)\to \operatorname{Hom}_R(R_S, A'')\to 0$$

is also exact.

Proof. The sequence $0\to \operatorname{Hom}_R(R_S, A')\to \operatorname{Hom}_R(R_S, A)\to \operatorname{Hom}_R(R_S, A'')$ is obviously exact. It remains to prove that $\operatorname{Hom}_R(R_S, A)\to \operatorname{Hom}_R(R_S, A'')$ is surjective. Now $\operatorname{Hom}_R(R_S, A)=\operatorname{Hom}_R(R_S, S(A))$ for if $s\in S$ and $x\in R_S$ we can find $y\in R_S$ such that $x=sy$ so for $f\in \operatorname{Hom}_R(R_S, A), f(x)=sf(y)\in sA$. Also if $g: A\to A''$ denotes the surjective map considered, then $g(S(A))=S(A'')$. We can namely choose $s\in S$ such that $S(A)=sA$ and $S(A'')=sA''$ and evidently $g(sA)=sA''$. Putting this together in order to show the surjectivity of $\operatorname{Hom}_R(R_S, A)\to \operatorname{Hom}_R(R_S, A'')$ we may therefore assume that A and A'' are $\bar{R}$-modules. But then $\operatorname{Hom}_R(R_\sigma, A)\simeq A$ and similarly for A''. So we have a surjective map of inverse systems of Artinian modules

$$\{\operatorname{Hom}_R(R_\sigma, A), g_\sigma^\tau\}\to \{\operatorname{Hom}_R(R_\sigma, A''), g_\sigma^\tau\},$$

where g_σ^τ denotes the corresponding multiplication map. Now the claim follows because the induced map of the corresponding inverse limits is surjective by (2.3). □

3. Secondary representation of the co-localization

Now it is time to recall basic facts concerning a secondary module resp. a secondary representation of a module, for the details see [3], [5], and [8]. An R-module $X\neq 0$ is called secondary if for each $r\in R$ multiplication by r on X is either surjective or nilpotent. Then $P=\operatorname{Rad}\operatorname{Ann}_R X$ is a prime ideal and X is called P-secondary. We say that X has a secondary representation if there is a finite number of secondary submodules $X_1,\ldots,X_k$ such that $X=X_1+\cdots+X_k$. One may assume that the prime ideals $P_i=\operatorname{Rad}\operatorname{Ann}_R X_i, i=1,\ldots,k$, are all distinct and, by omitting redundant summands, that the representation is minimal. Then the set of prime ideals $\{P_1,\ldots,P_k\}$ depends only on X and not on the minimal representation, see [**5**, (2.2)]. This set is called the set of attached prime ideals $\operatorname{Att}_R X$. If $Y\subset X$ both have a secondary representation, then so has X/Y and

$$\operatorname{Att}_R(X/Y)\subset \operatorname{Att}_R X\subset \operatorname{Att}_R Y\cup \operatorname{Att}_R(X/Y),$$

see [**5**, (4.1)]. By [**5**, (5.2)], any Artinian R-module A has a secondary representation. See also the Appendix to Section 6 in [**6**], where a short account of the theory of secondary representation is found. By [**11**, (2.6)], for an Artinian R-module A it follows that $P\in\operatorname{Att}_R A$ if and only if there is a homomorphic image B of A with $P=\operatorname{Ann}_R B$.

Theorem 3.1. *Let B be an Artinian P-secondary R-module. Let S denote a multiplicative set in R.*

(1) *If $S\cap P\neq\emptyset$, then* $\operatorname{Hom}_R(R_S,B)=0$.

(2) *If $S\cap P=\emptyset$, then the canonical map* $\operatorname{Hom}_R(R_S,B)\to B, f\mapsto f(1)$, *is surjective. In this case,* $\operatorname{Hom}_R(R_S,B)$, *considered as an R_S-module, is PR_S-secondary.*

Proof. Let $s\in S\cap P$. Because B is P-secondary the induced multiplication map $\operatorname{Hom}_R(R_S,B)\xrightarrow{s}\operatorname{Hom}_R(R_S,B)$ is nilpotent and bijective, i.e., $\operatorname{Hom}_R(R_S,B)=0$. Now suppose $S\cap P=\emptyset$. Because $S\subset R\backslash P$ it implies $B=sB$ for all $s\in S$, $B=S(B)$ and therefore $IB=0$, where $I=\bigcup_{s\in S}0:_R s$. Hence B is an R/I-module. Thus we may assume that S consists of nonzero divisors. Then

$$\operatorname{Hom}_R(R_S,B)\simeq\varprojlim\{B_s,g_s^t\},$$

where $B_s=B$ for each $s\in S$ and $g_s^t\colon B_t\to B_s$ is multiplication by a if $t=as$, see (2.2). We have an exact sequence of inverse systems

$$0\to\{0:_B s,g_s^t\}\to\{B_s,g_s^t\}\to\{B/0:_B s,g_s^t\}\to 0,$$

where $B_s=B$ for all $s\in S$ and the corresponding maps on $0:_B s, B_s,$ resp. $B/0:_B s$ are multiplication by a, if $t=as$. By (2.3), the homomorphism

$$\varprojlim\{B_s,g_s^t\}\to\varprojlim\{B/0:_B s,g_s^t\}$$

is surjective. Now $g_s^t\colon B/0:_B t\to B/0:_B s$ is easily seen to be an isomorphism for all $s,t\in S$ with $t=as$. Note that multiplication by a on B is surjective since B is P-secondary and $a\in R\backslash P$. Therefore $\varprojlim\{B/0:_B s,g_s^t\}\to B$ is an isomorphism and it follows that $\operatorname{Hom}_R(R_S,B)\to B$ is surjective. In particular, $\operatorname{Hom}_R(R_S,B)\neq 0$. Furthermore, multiplication by $\frac{r}{s}\in R_S$ on $\operatorname{Hom}_R(R_S, B)$ is either surjective or nilpotent. Because $\operatorname{Rad}\operatorname{Ann}_{R_S}\operatorname{Hom}_R(R_S,B)=PR_S$ it is a PR_S-secondary module. □

Now we prove the main technical tool relating the secondary representation of A to that of its co-localization $\operatorname{Hom}_R(R_S, A)$.

Theorem 3.2. *Let A denote an Artinian R-module with $A=A_1+\cdots+A_n$ a minimal secondary representation. Let $P_i=\operatorname{Rad}\operatorname{Ann}_R A_i, i=1,\dots,n$, and $S\cap P_i=\emptyset$ for $i=1,\dots,m$, resp. $S\cap P_i\neq\emptyset$ for $i=m+1,\dots,n$. Then*

$$\operatorname{Hom}_R(R_S,A)=\operatorname{Hom}_R(R_S,A_1)+\cdots+\operatorname{Hom}_R(R_S,A_m)$$

is a minimal secondary representation of $\operatorname{Hom}_R(R_S,A)$. *In particular, we have that*

$$\operatorname{Att}_{R_S}\operatorname{Hom}_R(R_S,A)=\{PR_S : P\in\operatorname{Att}_R A, P\cap S=\emptyset\}.$$

Proof. First note that $\operatorname{Hom}_R(R_S,A_1\cap A_2)=\operatorname{Hom}_R(R_S,A_1)\cap\operatorname{Hom}_R(R_S,A_2)$ for two submodules A_1, A_2 of A. Next use the exactness of $\operatorname{Hom}_R(R_S,\cdot)$ on the category of Artinian R-modules in order to show that

$$\operatorname{Hom}_R(R_S,A_i/A_1\cap A_2)\simeq\operatorname{Hom}_R(R_S,A_i)/\operatorname{Hom}_R(R_S,A_1)\cap\operatorname{Hom}_R(R_S,A_2),$$

$i=1,2$. Then take the short exact sequence

$$0\to A_1\cap A_2\to A_1+A_2\to A_1/A_1\cap A_2\oplus A_2/A_1\cap A_2\to 0$$

in order to conclude that $\operatorname{Hom}_R(R_S,A_1+A_2)=\operatorname{Hom}_R(R_S,A_1)+\operatorname{Hom}_R(R_S,A_2)$. To this end note that $\operatorname{Hom}_R(R_S,\cdot)$ is exact on the category of Artinian R-modules and commutes with direct sums.

Therefore, if $A=A_1+\cdots+A_n$ is a minimal secondary representation, then

$$\operatorname{Hom}_R(R_S,A)=\operatorname{Hom}_R(R_S,A_1)+\cdots+\operatorname{Hom}_R(R_S,A_m)$$

is a secondary representation of $\operatorname{Hom}_R(R_S,A)$, see (3.1). Suppose $\operatorname{Hom}_R(R_S,A_i)\subset\operatorname{Hom}_R(R_S,\sum_{j\neq i}A_j)$ for a certain $1\leqq i\leqq m$. Let $a\in A_i$ be an arbitrary element. Since A_i is P_i-secondary there is an $f_a\in\operatorname{Hom}_R(R_S,A_i)$ such that $f_a(1)=a$, see (3.1). But then $f_a\in\operatorname{Hom}_R(R_S,\sum_{j\neq i}A_j)$ and $A_i\subset\sum_{j\neq i}A_j$, a contradiction to the minimality of the secondary representation of A. □

In particular, for an Artinian R-module A, (3.2) shows that $P\in\operatorname{Att}_R A$ if and only if $PR_P\in\operatorname{Att}_{R_P}\operatorname{Hom}_R(R_P,A)$.

Corollary 3.3. *The image of the natural map* $\operatorname{Hom}_R(R_S,A)\to A$ *is $S(A)$, the S-component of A.*

Proof. This follows from (3.1), (3.2) and [5, (3.1)]. □

4. About the non-Artinianness of the co-localization

The co-localization $\mathrm{Hom}_R(R_S, A)$ of an Artinian R-module A is an R_S-module which has a secondary representation. However it is almost never an Artinian R_S-module. In fact if E is the injective hull of the residue field of a local Noetherian ring R with maximal ideal M and if P is a prime ideal in R, which is not minimal and which is in addition distinct from M, then E is an Artinian R-module, but $\mathrm{Hom}_R(R_P, E)$ is not an Artinian R_P-module. In order to show this we first determine the associated prime ideals of the R_P-module $\mathrm{Hom}_R(R_P, E)$.

Lemma 4.1. (See [13, Folgerlung 4.7].) *If P is a non-maximal prime ideal in a local Noetherian ring (R, M), and E is the injective hull of R/M, then*

$$\mathrm{Ass}_R \mathrm{Hom}_R(R_P, E) = \{Q \in \mathrm{Spec}\, R : Q \subset P\}.$$

Proof. Clearly the left hand side is included in the right hand side. For a prime ideal $Q \subset P$ there is the natural isomorphism $\mathrm{Hom}_R(R_P/QR_P, E) \simeq \mathrm{Hom}_{R/Q}(R_P/QR_P, 0:_E Q)$. So these modules are isomorphic to a submodule of $\mathrm{Hom}_R(R_S, E)$. Now $0:_E Q$ is the injective hull of the residue field of R/Q. Therefore we may assume that R is a domain, i.e., we have to show that (0) is associated to $\mathrm{Hom}_R(R_P, E)$. Let $x \in M \backslash P$ and let Q be a prime ideal minimal over xR, so ht $Q = 1$, by Krull's principal ideal theorem. If $s \in R \backslash P$ and $t \in R \backslash Q$, then $(s,t)R$ is neither contained in P nor Q, so $(s,t)R \subsetneq P \cup Q$, i.e. $u = as + bt \in (R \backslash P) \cap (R \backslash Q)$ for some $a, b \in R$. Then

$$\frac{1}{st} = \frac{u}{stu} = \frac{a}{tu} + \frac{b}{su} \in R_P + R_Q.$$

This shows that $R_P + R_Q$ is a subring of K, the quotient field of R. If it is a proper subring of K, then it possesses a nonzero prime ideal N. Then $0 \neq N \cap R \subset Q$ and ht $Q = 1$, so $N \cap R = Q$ and since $N \cap R \subset PR_P \cap R = P$ we would get the contradiction $x \in P$. Thus $R_P + R_Q = K$, and therefore

$$R_P / R_P \cap R_Q \simeq K/R_Q \neq 0$$

If $af = 0$, $a \neq 0$, where $f \in \mathrm{Hom}_R(K/R_Q, E)$, then

$$0 = af(K/R_Q) = f(aK/R_Q) = f(K/R_Q),$$

so $f = 0$. Consequently

$$(0) \in \mathrm{Ass}_R \mathrm{Hom}_R(R_P/R_P \cap R_Q, E) \subset \mathrm{Ass}_R \mathrm{Hom}(R_P, E). \qquad \square$$

Suppose now that $\mathrm{Hom}_R(R_P, E)$, where $P \neq M$, is an Artinian R_P-module. Then $\mathrm{Ass}_{R_P} \mathrm{Hom}_R(R_P, E) = \{PR_P\}$ and therefore $\mathrm{Ass}_R \mathrm{Hom}_R(R_P, E) = \{P\}$, so P must be minimal by (4.1).

But even if P is a minimal prime ideal, $\mathrm{Hom}_R(R_P, E)$ is not necessarily an Artinian R_P-module. Let namely R be a local Noetherian domain with quotient field K, such that the completion $\hat{R}$ has a nonminimal prime ideal Q contracting to (0) in R. Since $Q^{(n)}$ is Q-primary, $A_n = 0:_E Q^{(n)}$ is a Q-secondary $\hat{R}$-module, hence a (0)-secondary R-module. Since $Q^{(n)} \neq Q^{(n+1)}$ it follows that $A_n \subsetneq A_{n+1}$ and A_{n+1}/A_n is (0)-secondary for all n. Suppose $\mathrm{Hom}_R(K, E)$ is an Artinian K-module, i.e., a K-vector space of finite dimension, say d. By (2.4) we see the exactness of the sequence

$$0 \to \mathrm{Hom}_R(K, A_n) \to \mathrm{Hom}_R(K, E) \to \mathrm{Hom}_R(K, E/A_n) \to 0.$$

Therefore $d_n := \dim_K \mathrm{Hom}_R(K, A_n) \leqq d$. By (2.4) there is the short exact sequence

$$0 \to \mathrm{Hom}_R(K, A_n) \to \mathrm{Hom}_R(K, A_{n+1}) \to \mathrm{Hom}_R(K, A_{n+1}/A_n) \to 0,$$

which provides $d_{n+1} - d_n = \dim_K \mathrm{Hom}_R(K, A_{n+1}/A_n) > 0$, see (3.1). Se we have arrived at a contradiction.

5. Application I: dual of a theorem of Bourbaki

First note that if A is an Artinian R-module and N is a finitely generated R-module, then $A \otimes_R N$ is also an Artinian R-module. One would like to describe its attached prime ideals. This is done in case N is in addition finitely presented, and the result is a dual form of a theorem by Bourbaki, [2, p. 138]. That theorem states that if N is a finitely generated and M any module over a Noetherian ring R, then

$$\mathrm{Ass}_R \mathrm{Hom}(N, M) = \mathrm{Ass}_R M \cap \mathrm{Supp}_R N.$$

In order to apply the technique of co-localization, we first construct a canonical isomorphism.

Lemma 5.1. *If A is an Artinian R-module, N a finitely presented R-module and S is a multiplicative set in R, then there is a natural isomorphism*

$$\mathrm{Hom}_{R_S}(R_S, A) \otimes_{R_S} N_S \to \mathrm{Hom}_R(R_S, A \otimes_R N).$$

Proof. For each R-module X there are natural homomorphisms

$$\mathrm{Hom}_R(R_S, A) \otimes_{R_S} X_S \to \mathrm{Hom}_R(R_S, A) \otimes_R X \to$$
$$\mathrm{Hom}_R(R_S, A) \otimes_R \mathrm{Hom}_R(R, X) \to$$
$$\mathrm{Hom}_R(R_S \otimes_R R, A \otimes_R X) \to \mathrm{Hom}_R(R_S, A \otimes_R X).$$

Thus we have constructed a natural transformation η_X between the two functors

$$X \mapsto \mathrm{Hom}_R(R_S, A) \otimes_R X_S \quad \text{and} \quad X \mapsto \mathrm{Hom}_R(R_S, A \otimes_R X).$$

As a consequence of the exactness of the tensorproduct and the exactness of the co-localization on the category of Artinian modules, see (2.4), both functors are right exact on the category of finitely generated R-modules. Since η_R is an isomorphism, it is a general fact, shown by means of diagram-chasing, that η_N is an isomorphism for any finitely presented R-module N. □

Proposition 5.2. *Let R be a commutative ring, A an Artinian R-module and N a finitely presented R-module. Then* $\mathrm{Att}_R\, A \otimes_R N = \mathrm{Att}_R\, A \cap \mathrm{Supp}_R\, N$.

Proof. Since N is a homomorphic image of R^m for some m, $A \otimes_R N$ is a homomorphic image of A^m for some m. Hence $\mathrm{Att}_R\, A \otimes_R N \subset \mathrm{Att}_R\, A^m = \mathrm{Att}_R\, A$. Since $\mathrm{Ann}\, N \subset \mathrm{Ann}\, A \otimes_R N$, any $P \in \mathrm{Att}_R\, A \otimes_R N$ must contain $\mathrm{Ann}\, N$, i.e. $P \in \mathrm{Supp}_R\, N$.

Now let $P \in \mathrm{Att}_R\, A \cap \mathrm{Supp}_R\, N$. Then A has a P-secondary quotient B such that $P = \mathrm{Ann}\, B$, see [**11**, (2.6)]. Multiplication by an element $a \in R$ on $B \otimes_R N$ is zero if $a \in P$ and surjective if $a \in R \backslash P$. Provided $B \otimes_R N \neq 0$ it follows that $B \otimes_R N$ is P-secondary, and since it is a homomorphic image of $A \otimes_R N$ thus $P \in \mathrm{Att}_R\, A \otimes_R N$. Now, by (5.1) we see that

$$\mathrm{Hom}_R(R_P, B \otimes_R N) \simeq \mathrm{Hom}_R(R_P, B) \otimes_{R_P} N_P \simeq \mathrm{Hom}_R(R_P, B) \otimes_{R_P/PR_P} N_P/PN_P$$

since $PR_P\, \mathrm{Hom}_R(R_P, B) = 0$. Now $\mathrm{Hom}_R(R_P, B)$ and N_P/PN_P are nonzero vectorspaces over the field R_P/PR_P. It follows that $\mathrm{Hom}_R(R_P, B \otimes_R N) \neq 0$ and therefore also $B \otimes_R N \neq 0$. □

In order to show the inclusion $\mathrm{Att}_R\, A \otimes_R N \subset \mathrm{Att}_R\, A \cap \mathrm{Supp}_R\, N$ we thus needed merely that N is a finitely generated R-module.

6. Application II: obtaining a result of Taherizadeh

As another application of the co-localization we deduce the following result due to Taherizadeh. To this end let A denote an Artinian R-module. For a prime ideal P of R let $S_P(A)$ denote the S_P-component of A, where $S_P = R \backslash P$.

Proposition 6.1. (See [**12**, Theorem 3.4].) *Let I denote an ideal of R. Let $P \in \mathrm{At}(I, A) \backslash \mathrm{Bt}(I, A)$. Then there is an integer k and a P-secondary submodule B of A, in fact $B = S_P(0 :_A P^k)$, such that B is part of some minimal secondary representation of $0 :_A I^n$ for all $n \geqq k$.*

Proof. Let S denote the multiplicative set $R \backslash \bigcup_{i=1}^{t} P_i$, where

$$\{P_1,\dots,P_t\}=\{Q\in \operatorname{Att}(I,A): Q \subsetneqq P\}.$$

Take $s\in P\cap S$. Since s is not an element of any prime ideal Q contained in P for $Q\in \operatorname{Att}(0:_A I^{n+1}/0:_A I^n)$, n large enough, it follows that

$$\operatorname{Hom}_R(R_P, 0:_A I^{n+1}/0:_A I^n)\otimes_R R/sR=0$$

by view of the secondary representation. Applying the functor $\operatorname{Hom}_R(R_P,\cdot)$ to the exact sequence

$$(0:_A I^n)\otimes_R R/sR\to(0:_A I^{n+1})\otimes_R R/sR\to$$
$$(0:_A I^{n+1}/0:_A I^n R)\otimes_R R/sR\to 0$$

and composing surjections, we deduce that there is an integer l such that the homomorphism

$$\operatorname{Hom}_R(R_P, 0:_A I^l)\otimes_R R/sR\to \operatorname{Hom}_R(R_P, 0:_A I^n)\otimes_R R/sR$$

is surjective for all $n\geqq l$ and $\{Q\in \operatorname{Att}(0:_A I^n): Q\subset P\}=\{P,P_1,\dots,P_t\}$ for $n\geqq l$. This implies

$$\operatorname{Hom}_R(R_P, 0:_A I^l)=\operatorname{Hom}_R(R_P,(0:_A I^l+s(0:_A I^n)))\quad\text{for all } n\geqq l.$$

By iteration $S_P(0:_A I^n)=S_P(0:_A I^l)+s^r S_P(0:_A I^n)$ for $n\geqq l$ and all $r\geqq 1$. But for a given $n\geqq l$ there is an r such that $s^r S_P(0:_A I^n)=S(0:_A I^n)$. Hence

$$S_P(0:_A I^n)=S_P(0:_A I^l)+S(0:_A I^n)\quad\text{for all } n\geqq l.$$

Now we have $S_P(0:_A I^l)=B'+S(0:_A I^l)$ for some P-secondary module B'. Choose $k\geqq l$ such that $P^k B'=0$. Then $B'\subset S_P(0:_A P^k)\subset S_P(0:_A I^n), n\geqq k$. Putting $B=S_P(0:_A P^k)$ we finally get

$$S_P(0:_A I^n)=B+S(0:_A I^n)\quad\text{for all } n\geqq k.$$

Hence for all $n\geqq k$ it follows that B is a P-secondary module which is part of a minimal secondary representation of $0:_A I^n$. □

7. The co-support of a module

Related to the support of a module defined in terms of the localization one may define a co-support of a module in terms of the concept of the co-localization.

Definition 7.1. For an R-module X let

$$\operatorname{Cos}_R X = \{P \in \operatorname{Spec} R : \operatorname{Hom}_R(R_P, X) \neq 0\}$$

denote the co-support of X.

It is easy to see that $\operatorname{Cos}_R X$ is a subset of $\operatorname{Spec} R$ stable with respect to specialization. To this end let $P \subset Q$ be two prime ideals of R. Then $\operatorname{Hom}_R(R_P, X) \simeq \operatorname{Hom}_{R_Q}(R_P, \operatorname{Hom}_R(R_Q, X))$, which proves the claim.

Lemma 7.2. (1) *Let A be an Artinian R-module. Then any prime ideal of R containing an element of* $\operatorname{Att}_R A$ *belongs to* $\operatorname{Cos}_R A$.

(2) *Any prime ideal of* $\operatorname{Cos}_R A$ *contains an element of* $\operatorname{Att}_R A$.

Proof. Let $A = A_1 + \cdots + A_n$ be a minimal secondary representation of A, where A_i is P_i-secondary, $i = 1, \ldots, n$. Given a prime ideal P we may assume that $P_i \subset P$ if and only if $1 \leqq i \leqq m$ for a certain integer m. Then $\operatorname{Hom}_R(R_P, A) = \sum_{i=1}^{m} \operatorname{Hom}_R(R_P, A_i)$ by (3.2). Now $\operatorname{Hom}_R(R_P, A_i) \neq 0$ if and only if $P_i \subset P$ by (3.1). The conclusions follow from this. □

For an arbitrary R-module X it is known that $X \neq 0$ if and only if $\operatorname{Supp}_R X \neq \emptyset$. For an Artinian R-module A we have that $A \neq 0$ if and only if $\operatorname{Cos}_R A \neq \emptyset$. But this does not hold in general. To this end let p denote a prime number of $\mathbb{Z}$, the integers, or 0. Then it follows that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}_{(p)}, \mathbb{Z}) = 0$ for all p, while $\mathbb{Z} \neq 0$. This is the reason to restrict the concept of the co-support to Artinian R-modules. It is not clear to the authors how to define a more satisfactory concept of a co-support which should coincide with $\operatorname{Cos}_R A$ for an Artinian R-module A. In connection with this handicap one may add a comment to a result of Macdonald, see [**5**, (4.5)].

Lemma 7.3. (1) *An Artinian R-module A has a composition series*

$$0 = A_r \subset A_{r-1} \subset \cdots \subset A_1 \subset A_0 = A$$

in which each quotient A_{i-1}/A_i is secondary.

(2) *In each such composition series, if $P_i = \operatorname{Rad} \operatorname{Ann}_R A_{i-1}/A_i$, for $1 \leqq i \leqq r$, then*

$$\operatorname{Att}_R A \subset \{P_1, \ldots, P_r\} \subset \operatorname{Cos}_R A.$$

(3) *In* (2) *all of these three sets have the same minimal elements equal to the set of minimal prime ideals containing* $\operatorname{Ann}_R A$.

Proof. (1) and part of (2) are shown in [**5**, (4.5)], for the more general situation of an R-module admitting a secondary representation. The rest of (2) follows because $\operatorname{Cos}_R A = \operatorname{Cos}_R A' \cup \operatorname{Cos}_R A''$ for Artinian R-modules in a short exact sequence $0 \to A' \to A \to A'' \to 0$, see (2.4). Finally (3) is true by virtue of [**5**, (2.7)], and (7.2). □

In particular, for an Artinian R-module A, (7.3) implies that $\operatorname{Cos}_R A = V(\operatorname{Ann}_R A)$, i.e., it is a closed subset of Spec R. Let us conclude with a characterization when an Artinian module is of finite length.

Proposition 7.4. *Let (R, M) denote a quasi-local ring. For an Artinian R-module $A \neq 0$ the following conditions are equivalent:*

(i) $\mathrm{Cos}_R A = \{M\}$. (ii) $\mathrm{Att}_R A = \{M\}$. (iii) *A is an R-module of finite length.*

Proof. By the previous results, see (7.2) and (3.1), it is enough to show that (ii) implies the condition (iii). To this end it suffices to prove that $M^k A = 0$ for a certain integer k. Since A is an Artinian R-module there is a finitely generated ideal $I \subset M$ such that $0:_A I^n = 0:_A M^n$, for all $n \geqq 1$, see [3]. Because A is M-secondary it follows now that $I^k A = 0$ for a certain integer k. □

REFERENCES

1. N. Bourbaki, *Théorie des ensembles. Chap 3. sec. ed.* (Hermann, Paris, 1963).

2. N. Bourbaki, *Algèbre commutative. Chap 3. et 4.* (Hermann, Paris, 1967).

3. D. Kirby, Coprimary decomposition of Artinian modules, *J. London Math. Soc.* (2) **6** (1973), 571–576.

4. J. P. Lafon, *Algèbre commutative: Langages géométrique et algébrique* (Hermann, Paris, 1977).

5. I. G. Macdonald, Secondary representation of modules over a commutative ring, *Sympos. Math.* **11** (1973), 23–43.

6. H. Matsumura, *Commutative ring theory* (Cambridge University Press, Cambridge, 1986).

7. L. Melkersson, On asymptotic stability for sets of prime ideals connected with the powers of an ideal, *Math. Cambridge Philos. Soc.* **107** (1990), 267–271.

8. D. G. Northcott, Generalized Koszul complexes and Artinian modules, *Quart. J. Math. Oxford* (2) **23** (1972), 289–297.

9. J. Rotman, *An introduction to homological algebra* (Academic Press, New York, San Francisco, London, 1979).

10. R. Y. Sharp, Asymptotic behaviour of certain sets of attached prime ideals, *J. London Math. Soc.* (2) **34** (1986), 212–218.

11. R. Y. Sharp, A method for the study of Artinian modules, with an application to asymptotic behavior, in: *Commutative Algebra* (Math. Sciences Research Inst. Publ., No. 15, Springer-Verlag, 1989), 443–465.

12. A.-J. Taherizadeh, On asymptotic values of certain sets of attached prime ideals, *Glasgow Math. J.* **30** (1988), 293–300.

13. H. Zöschinger, Über koassoziierte Primideale, *Math. Scand.* **63** (1988), 196–211.

University of Lund
Department of Mathematics
Box 118, S-221 00 Lund
Sweden

Max-Planck-Gesellschaft
AG Algebraische Geometrie und Zahlentheorie
Jägerstr. 10-11
D-10 117 Berlin
Germany