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EXTENDED OPTIMA AND EQUILIBRIA FOR CONTINUOUS GAMES. II. A CLASS OF ECONOMIC MODELS

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Abstract

For the class of continuous games where σ_i and $f_i\{\sigma_i, \phi(\sigma_1, \ldots, \sigma_N)\}$ are the strategy of and payoff to player i for $i = 1, \ldots, N$, it is proved that the set of weak type I optima defined in Paper I coincide with the set of solutions of a matrix condition. The latter condition restricts the equilibrium solutions of an adjustment process. Numerical results for $N = 2$ and $N = 3$ indicate that the set of all equilibrium solutions coincides with the above sets. The optima of types I to IV from Paper I are described fairly completely for the given class of games.

1. Introduction

In the preceding paper [5] we considered general continuous games between N players, where player i chooses a real number σ_i and receives a payoff $J_i(\boldsymbol{\sigma})$ depending on the σ's of all the players, $(\sigma_1, \sigma_2, \ldots, \sigma_N) = \boldsymbol{\sigma}$. In an economics context, this describes quantity-variation competition between N firms marketing the same product, where firm i produces a quantity σ_i per business period and makes a profit J_i per business period. We defined some new optima for this competition and showed that they are closely related to the equilibrium solutions of the adjustment process of [3] in which firms with no knowledge of the J_i's vary their outputs over successive business periods in an attempt to maximize their individual profits.

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In the present paper this relationship is shown to be much closer for a particular class of profit functions. These functions, which have special economic relevance, are

$$J_i(\boldsymbol{\sigma}) = f_i\{\sigma_i, \phi(\boldsymbol{\sigma})\} \tag{1.1}$$

for $i = 1, 2, \ldots, N$. (In the case of 2 firms, this is no restriction on the J_i's). Often the function $\phi(\boldsymbol{\sigma})$ could be regarded as the common price which the consumers pay for the product, and might depend only on the total output $\sigma_T \equiv \sigma_1 + \sigma_2 + \cdots + \sigma_N$ of the firms, perhaps a decreasing function of σ_T. Then a commonly studied version of (1.1) is $J_i = \sigma_i \phi - c_i$, where $c_i(\sigma_i)$ is the cost to firm i of producing the quantity σ_i.

It is plausible however that the price received by firm i has an additional dependence on σ_i, since specific buyers may go to specific firms, due to some attractive feature or personal preference. Similarly, the costs of firm i may have an additional dependence on σ_T, since the price of raw materials may depend on the total demand of all the firms, which may in turn depend on σ_T. All of these possibilities are included in the form (1.1) of the profit functions.

Besides their economic relevance, payoffs of the form (1.1), by a fortunate coincidence, enable one to specify fairly completely and explicitly both the optima of Paper I and the equilibrium solutions of the adjustment process of [3] and to prove various correspondences. The results are given in Sections 3 and 4, while their proofs are contained in Sections 6 to 9. Section 5 outlines various simulations of the adjustment process for 3 players.

2. Recapitulation of concepts and definitions

We suppose throughout that the functions f_i and ϕ in (1.1) are differentiable in all their variables, so that the J_i's themselves are differentiable. We normally think of the profits and outputs as applying to one business period (a month perhaps). Firm i has direct control only over its own output σ_i. We repeat some definitions from Paper I.

DEFINITION 1. *The market is said to be in* state $\boldsymbol{\sigma}$ *when the outputs are given by* $\boldsymbol{\sigma}$.

DEFINITION 2. *A* coalition *is a subset of the set* $\Lambda = (1, 2, \ldots, N)$ *of firms.*

We write J_{ij} for $\partial J_i / \partial \sigma_j$, and Δ_S for the determinant of the submatrix J_{ij} for $i, j \in S$, where $S \subseteq \Lambda$ is any subset of the firms. Thus $\Delta_i = J_{ii}$ and Δ_Λ is the determinant of the whole J_{ij} matrix.

Definition 3. *Firm i is* weakly disciplinable *by coalition C if*

$$\Delta_i \Delta_C \Delta_{C \cup i} < 0. \tag{2.1}$$

The set of such states σ is denoted by $D_{i,C}^W$.

As explained at length in Paper I, the condition $\Delta_i \Delta_C \Delta_{C \cup i} < 0$ is essentially equivalent to the condition that firm i can be *disciplined* by a coalition C in the sense that: either firm i *cannot* make an adjustment (an infinitesimal change in its output σ_i), which increases its own profit, or if it *does* make a profit increasing adjustment then some coalition C of the other firms can conspire to make adjustments which restore their own individual profits while leaving firm i with a net reduction in profit.

Weak optima O_I^W to O_{IV}^W of types I to IV are defined by sets in σ-space.

O_I^W: $\sigma \in D_{i,C}^W$ for at least one $i \in \Lambda$ and at least one $C \subseteq \Lambda - i$.

O_{II}^W: $\sigma \in D_{i,\Lambda-i}^W$ for at least one $i \in \Lambda$.

O_{III}^W: $\sigma \in D_{i,C}^W$ for every $i \in \Lambda$ and at least one $C \subseteq \Lambda - i$ for each i.

O_{IV}^W: $\sigma \in D_{i,\Lambda-i}^W$ for every $i \in \Lambda$.

In these optima, various firms are (weakly) disciplinable by various coalitions if they make any adjustments. Consequently, firms tend not to make adjustments, and so the states are *optimal* in this sense. The type IV optimum is the most stable or acceptable to all firms, because here *every* firm is not only reluctant to move through fear of being disciplined, but also belongs to *every* disciplining coalition and is consequently always protected. The other optima I, II and III apply when firms have less information. In this paper we shall refer only briefly to the *strong* optima and the original unqualified optima of Paper I.

The optima I to IV include various standard optima and game theory *solutions*, such as the Pareto optimum, as shown in Paper I. A major feature of our new optima is their connection with the *equilibrium states E* of the adjustment process of [3]. These states were proved to satisfy the new condition: there exists a (symmetric) non-zero, positive-semi-definite matrix A_{ij} such that

$$\sum_{j \in \Lambda} A_{ij} J_{ij} = 0 \quad \text{for all } i \in \Lambda. \tag{2.2}$$

This condition does not appear to be of any standard type nor amenable to any standard technique, algebraic or numerical. Our main result, Theorem 1, gives all solutions of (2.2) for profit functions of the form (1.1).

3. The main results

It is clear that

$$O_{IV}^W \subseteq O_{II}^W \subseteq O_I^W \tag{3.1}$$

and

$$O^W_{IV} \subseteq O^W_{III} \subseteq O^W_I. \tag{3.2}$$

In Paper I (Theorem 5) we proved that for general J_i's,

$$O^W_I \subseteq M, \tag{3.3}$$

where M is the set of all solutions $\boldsymbol{\sigma}$ of (2.2). We proved also (Theorem 6) that for 2 firms ($N = 2$),

$$O^W_I = O^W_{II} = O^W_{III} = O^W_{IV} = M, \tag{3.4}$$

so we need not deal further with this case. Our main new result now is

THEOREM 1. *For the profit functions* (1.1), $O^W_I = M$.

The proof is given in Section 8. Since O^W_I is given through (2.1) in terms of the J_i's alone, we consequently have all the solutions M of (2.2), if not in explicit form. These solutions and the other optima are illustrated more explicitly in the following section.

THEOREM 2. *For the profit functions* (1.1) *and the case of* 3 *firms* ($N = 3$),

$$O^W_I = O^W_{II} \cup O^W_{III}, \tag{3.5}$$

but not necessarily for 4 *or more firms.*

The main part of the proof is implied by Table 1 below, while the remaining part is given in Section 9. This theorem shows that, for 3 firms, a type I optimum is more stable or acceptable than was apparent at first. Any firm in such a state $\boldsymbol{\sigma}$ can either be disciplined ($\boldsymbol{\sigma} \in O^W_{III}$) or is protected by belonging to a disciplining coalition ($\boldsymbol{\sigma} \in O^W_{II}$).

The main significance of Theorems 1 and 2 arises from their relationship with the adjustment process of [3]. It was proved there (Theorem 1) that every equilibrium state of that process satisfies the matrix condition (2.2), under fairly weak conditions. Thus

$$E \subseteq M, \tag{3.6}$$

where E is the set of equilibrium solutions of the adjustment process. It follows from (3.4) that, for $N = 2$, every equilibrium state is an optimum of the equivalent types I to IV, as discussed already in Paper I. From Theorems 1 and 2 we have the new result

COROLLARY 1. *For the profit functions* (1.1),

$$\text{(a) } E \subseteq O^W_I \quad \textit{for all } N \tag{3.7}$$

and

$$\text{(b) } E \subseteq O_{II}^{W} \cup O_{III}^{W} \quad \textit{for } N = 3. \tag{3.8}$$

Thus equilibrium states of the adjustment process are always optima of one type or another. We think this rather remarkable, mathematically; in the adjustment process each firm's sole action is to maximize a crude estimate of its profit function, taking no account of other firms, while the optima seem to imply greater knowledge and a bargaining between the firms. This was discussed further in Paper I, although we have no deep understanding of the paradox. On the other hand, Corollary 1 supports the kind of belief, implicit in much of economic theory (for example, [1]), that firms acting independently with little knowledge of the market structure (the J_i's) can arrive at an optimum solution of the underlying market game.

Convergence of the adjustment process was proved in [3] only for a one-parameter family of initial conditions and for J_i's of the form

$$J_i = a_i\sigma_i\left(1 - \sum_{j=1}^{N} b_{ij}\sigma_j\right), \tag{3.9}$$

where the a_i and b_{ij} are constants (Theorem 4, Appendix A of that paper). Thus E is known to be non-empty in this case. Other cases where convergence can be proved are given in [4] and [2].

However, in the case of 2 firms with J_i of the form (3.9), numerical simulation of the adjustment process indicates that convergence is achieved under a wide range of sensible conditions and, further, that every solution of the matrix conditions (2.2) is a possible equilibrium state, that is,

$$E = M \quad \text{for } N = 2. \tag{3.10}$$

The simulation described in Section 5 for 3 firms strongly indicates that $E = M$ for J_i's of the form

$$J_i = a_i\sigma_i(1 - b_i\sigma_i - c_i\sigma_T), \tag{3.11}$$

where

$$\sigma_T = \sigma_1 + \sigma_2 + \cdots + \sigma_N. \tag{3.12}$$

It is our conjecture that $E = M$ for all N if the J_i's have this form. Given this conjecture, all equilibrium states are known, and by Theorems 1 and 2 every O_I^W state, and for $N = 3$ very O_{II}^W state and O_{III}^W state, is attainable as an equilibrium state of the adjustment process. The optima then specify the equilibrium states completely.

4. Description of the optima

The optima can be found more explicitly here than previously. We define the new variables

$$y_i = -h_i\phi_i/g_i \quad \text{for } i = 1, \ldots, N, \tag{4.1}$$

for $g_i \neq 0$, where

$$g_i(\boldsymbol{\sigma}) = G_i\{\sigma_i, \phi(\boldsymbol{\sigma})\}, \qquad h_i(\boldsymbol{\sigma}) = H_i\{\sigma_i, \phi(\boldsymbol{\sigma})\}, \qquad \phi_i(\boldsymbol{\sigma}) = \partial\phi(\boldsymbol{\sigma})/\partial\sigma_i,$$
$$G_i(u, v) = \partial f_i(u, v)/\partial u, \qquad H_i(u, v) = \partial f_i(u, v)/\partial v, \tag{4.2}$$

and f_i and ϕ are the functions defining J_i in (1.1).

We shall show that our disciplining set $D_{i,C}^W$ is essentially equivalent to the condition

$$(1 - y_i)(1 - y_C)(1 - y_{C \cup i}) < 0, \tag{4.3}$$

where

$$y_S = \sum_{j \in S} y_j. \tag{4.4}$$

This simple form makes the optima relatively simple to specify in the y-space rather than the $\boldsymbol{\sigma}$-space. To make the equivalence precise however, one needs to deal with the cases where $g_i = 0$, for which (4.1) does not apply. In simple practical examples it is usually clear how to deal with these cases, but the general problem requires some care. Curiously, the condition $g_i = 0$ does not seem to have any compelling game-theoretic or economic significance. It corresponds to a set of zero measure in the N-dimensional $\boldsymbol{\sigma}$ or $\mathbf{y}$ spaces.

We define the set

$$\Gamma_S \equiv \{\boldsymbol{\sigma}\colon g_j \neq 0 \text{ for all } j \in S\}, \tag{4.5}$$

so that (4.3) is meaningful for all $\boldsymbol{\sigma} \in \Gamma_{C \cup i}$. To deal with the remaining cases we define the sets

$$Z_{i,S} = \{\boldsymbol{\sigma}\colon g_i = 0, h_i\phi_i \neq 0, g_j \neq 0 \text{ for all } j \in S - i\} \tag{4.6}$$

for $i \in S$. The remaining set of $\boldsymbol{\sigma}$'s conditioning g_j and $h_j\phi_j$ for $j \in S$ is denoted by E_S.

Theorem 3. *If J_i has the form* (1.1) *then $D_{i,C}^W$ is equivalent to the set defined by*:

$$\begin{aligned} &(4.3) \text{ holds if } \boldsymbol{\sigma} \in \Gamma_{C \cup i}, \\ &y_C \geq 1 \text{ if } \boldsymbol{\sigma} \in Z_{i, C \cup i}, \\ &y_i \geq 1 \text{ if } \boldsymbol{\sigma} \in Z_{j, C \cup i}, \text{ where } j \in C, \\ &\qquad \text{or } \boldsymbol{\sigma} \in E_{C \cup i}. \end{aligned} \tag{4.7}$$

We repeat that this is simply a more complete version of (4.3). The proof of Theorem 3 is given in Section 6. We denote the set (4.7) by $\Xi_{i,C}$. We can now replace $D_{i,C}^W$ everywhere by $\Xi_{i,C}$ in the definitions of the optima O_I^W to O_{IV}^W when the J_i have the form (1.1). More formally, one can write

$$O_I^W = \bigcup_{i\in\Lambda} \bigcup_{C\subseteq\Lambda - i} \Xi_{i,C}, \quad O_{II}^W = \bigcup_{i\in\Lambda} \Xi_{i,\Lambda-i}, \tag{4.8}$$

and so on. Thus the optima are known explicitly in the y-space. For example, if $N = 2$ all the optima reduce to essentially

$$(1 - y_1)(1 - y_2)(1 - y_1 - y_2) \leqslant 0, \tag{4.9}$$

which admits only the solutions

$$y_1 \leqslant 1, y_2 \leqslant 1 \text{ and } y_1 + y_2 \geqslant 1,$$
$$y_1 \leqslant 1, y_2 \geqslant 1 \text{ and } y_1 + y_2 \leqslant 1,$$
$$y_1 \geqslant 1, y_2 \leqslant 1 \text{ and } y_1 + y_2 \leqslant 1$$

and

$$y_1 \geqslant 1, y_2 \geqslant 1 \text{ and } y_1 + y_2 \geqslant 1, \tag{4.10}$$

illustrated by the shaded regions in Figure 1.

More generally, we define the sets

$$B_0 = \{y_S < 1 \text{ for all } S \subseteq \Lambda\}, \tag{4.11}$$

$$B_i = \{y_S > 1 \text{ for all } S \ni i \text{ and } y_S < 1 \text{ for all } S \not\ni i\}, \tag{4.12}$$

$$C_0 = \{y_i \geqslant 1 \text{ for all } i \in \Lambda\}, \tag{4.13}$$

$$C_S = \{y_i \leqslant 0 \text{ and } y_{\Lambda-i} \geqslant 1 \text{ if } i \in S, y_j \geqslant 1 \text{ if } j \notin S, y_\Lambda \leqslant 1\}, \tag{4.14}$$

for all $S \subset \Lambda$, and

$$C_\Lambda = \{0 \leqslant y_i < 1, y_{\Lambda-i} \leqslant 1 \text{ for all } i \in \Lambda, y_\Lambda \geqslant 1\}. \tag{4.15}$$

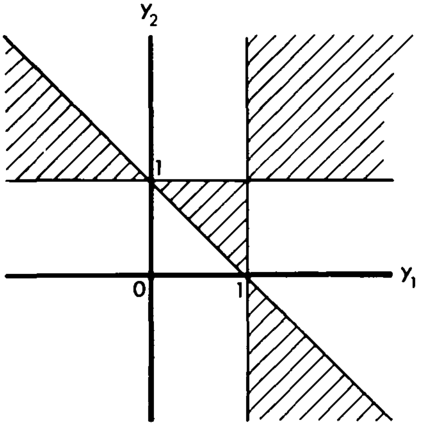

Figure 1. All the weak optima under profit functions (1.1) for the case of two firms.

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We find that, for J_i's of the form (1.1), the *disallowed* regions in y-space for type I optima are given essentially by

$$B_0 \cup B_1 \cup B_2 \cup \cdots \cup B_N = \mathfrak{B}, \text{ say,} \tag{4.16}$$

and the *allowed* regions for type IV optima are given essentially by

$$C_0 \cup \bigcup_{S \subset \Lambda} C_S \cup C_\Lambda = \mathcal{C}, \text{ say.} \tag{4.17}$$

Regions corresponding to $\mathrm{O}_{\mathrm{II}}^W$ and $\mathrm{O}_{\mathrm{III}}^W$ are rather laborious to specify for general N. The reader may find them by following the methods of Section 7. To give precise versions of (4.16) and (4.17) one must, as in Theorem 3, deal with cases where the y_i are not defined. We refer to the definitions of Γ_S and $Z_{i,S}$.

THEOREM 4. *If the* J_i *have the form* (1.1) *then*
(a) $\sim \mathrm{O}_{\mathrm{I}}^W$ *is equivalent to the set defined by*

$$\begin{aligned} &\boldsymbol{\sigma} \in \mathfrak{B} \ \textit{if}\ \boldsymbol{\sigma} \in \Gamma_\Lambda, \\ &y_S < 1 \ \textit{for all}\ S \subseteq \Lambda - i \ \textit{if}\ \boldsymbol{\sigma} \in Z_{i,\Lambda}, \ \textit{where}\ i \in \Lambda, \end{aligned} \tag{4.18}$$

and no other $\boldsymbol{\sigma}$ *values.*
(b) $\mathrm{O}_{\mathrm{IV}}^W$ *is equivalent to the set defined by*

$$\begin{aligned} &\boldsymbol{\sigma} \in \mathcal{C} \ \textit{or}\ y_\Lambda = 1 \ \textit{if}\ \boldsymbol{\sigma} \in \Gamma_\Lambda, \\ &y_j \geqslant 1 \ \textit{for all}\ j \in \Lambda - i \ \textit{if}\ \boldsymbol{\sigma} \in Z_{i,\Lambda}, \ \textit{where}\ i \in \Lambda, \\ &\textit{all}\ \boldsymbol{\sigma} \in E_\Lambda. \end{aligned} \tag{4.19}$$

The proof of Theorem 4 is given in Section 7. One can make similar statements about the strong optima $\mathrm{O}_{\mathrm{I}}^S$ and $\mathrm{O}_{\mathrm{IV}}^S$ of Paper I. One essentially replaces strict inequalities by non-strict ones, and vice versa. In some ways $\mathrm{O}_{\mathrm{IV}}^S$ is more appealing than $\mathrm{O}_{\mathrm{IV}}^W$ because all of its points have n-dimensional neighbourhoods in $\mathrm{O}_{\mathrm{IV}}^S$.

For the case of 3 firms we tabulate all the weak optima as follows. We formally put

$$I_S = \begin{cases} 0 & \text{if } y_S \leqslant 1, \\ 1 & \text{if } y_S \geqslant 1, \end{cases} \tag{4.20}$$

so that $I_S = 0$ and 1 both admit $y_S = 1$. Then a region in y-space is specified by the string

$$I_1 I_2 I_3 I_{23} I_{31} I_{12} I_{123}.$$

All the strings which are not self-contradictory are listed in Table 1, and the optimal sets $\mathrm{O}_{\mathrm{I}}^W$ to $\mathrm{O}_{\mathrm{IV}}^W$ to which they belong, if any, are indicated. Permutations of 1, 2 and 3 always belong to the same sets by the symmetry of the definitions, so such permutations are listed on the same line.

TABLE 1

All the weak optima for the case of three firms.

*	000 000 0			
*	000 000 1			I II III IV
*	000 001 0,	000 010 0,	000 100 0	I II III
*	000 001 1,	000 010 1,	000 100 1	I II
*	000 011 1,	000 101 1,	000 110 1	I III
*	000 111 1,			I III
	001 000 0,	100 000 0,	010 000 0	I II III
	001 010 0,	100 001 0,	010 100 0	I II
	001 010 1,	100 001 1,	010 100 1	I II III
	001 100 0,	100 010 0,	010 001 0	I II
	001 100 1,	100 010 1,	010 001 1	I II III
	001 110 0,	100 011 0,	010 101 0	I II III IV
*	001 110 1,	100 011 1,	010 101 1	
*	001 111 1,	100 111 1,	010 111 1	I II III
	011 100 0,	101 010 0,	110 001 0	I II III IV
	011 100 1,	101 010 1,	110 001 1	I III
	011 101 1,	101 110 1,	110 011 1	I II III
	011 110 1,	101 011 1,	110 101 1	I II III
*	011 111 1,	101 111 1,	110 111 1	I II
*	111 111 1			I II III IV

For 2 firms, where all the optima and the matrix condition (2.2) are equivalent, the optima are readily illustrated. We take

$$J_i = \sigma_i(1 - \sigma_i - 0.3\sigma_T), \tag{4.21}$$

with $i = 1, 2$ and $\sigma_T = \sigma_1 + \sigma_2$, and apply Theorem 3. As shown in [3], the optima reduce to the shaded region in Figure 2.

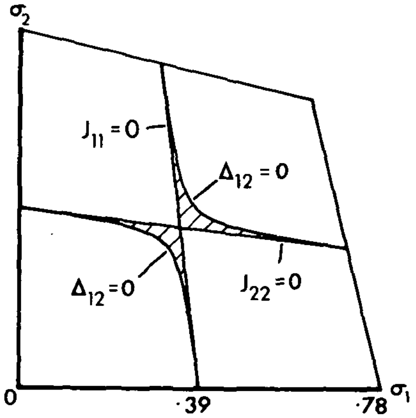

Figure 2. All the weak optima under profit functions (4.21) for the case of two firms.

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For 3 firms, let us first consider the special case

$$J_i = \sigma_i(1 - \sigma_T). \tag{4.22}$$

Here (4.1) reduces to

$$y_i = \sigma_i / (1 - \sigma_T). \tag{4.23}$$

Since the σ_i's are to be all non-negative, it follows that the y_i are all of the same sign. Since all optima satisfy the matrix condition (2.2), we have, on substituting (4.22) and (4.23) in (2.2),

$$y_i = A_{ii} / \left(\sum_j A_{ij} \right). \tag{4.24}$$

Since A_{ij} is positive-semi-definite we can put $A_{ij} = \mathbf{c}_i \cdot \mathbf{c}_j$ giving

$$\sum_i |\mathbf{c}_i|^2 / y_i = \left(\sum_i \mathbf{c}_i \right) \cdot \left(\sum_i \mathbf{c}_i \right) \geqslant 0, \tag{4.25}$$

so that $y_i \geqslant 0$ for all i. This restricts the possible strings to those labelled with a star in Table 1. Thus the type I optimum *excludes* only the strings 000 000 0 and the 3 permutations of 001 110 1. These readily transform into excluded regions in $\boldsymbol{\sigma}$-space, namely the tetrahedron below $\sigma_T = \frac{1}{2}$, and the 3 tetrahedra

$$(2\sigma_T - \sigma_i < 1 \text{ and } \sigma_T + \sigma_i > 1) \tag{4.26}$$

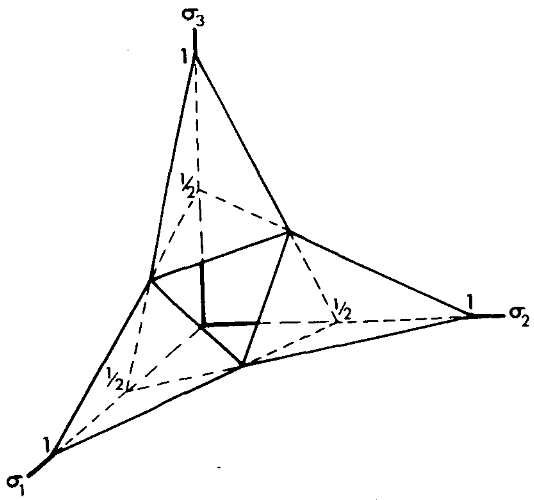

***Figure* 3. Excluded regions for outputs for type I optima under profit functions (4.22) for the case of three firms.**

for $i = 1, 2, 3$. These are illustrated in Figure 3. Similarly the *included* regions for type IV optima comprise the 2 strings 000 000 1 and 111 111 1, which transform into the 2 tetrahedra

$$\left(2\sigma_T - \sigma_i \leqslant 1 \text{ for } i = 1, 2, 3 \text{ and } \sigma_T \geqslant \tfrac{1}{2}\right), \tag{4.27}$$

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and

$$(\sigma_T + \sigma_i \geqslant 1 \text{ for } i = 1, 2, 3 \text{ and } \sigma_T \leqslant 1), \tag{4.28}$$

where $\sigma_i \geqslant 0$ and $J_i \geqslant 0$ (that is, $\sigma_T \leqslant 1$) for $i = 1, 2, 3$. These tetrahedra are illustrated in Figure 4. The type II and III optima can be obtained readily in similar fashion.

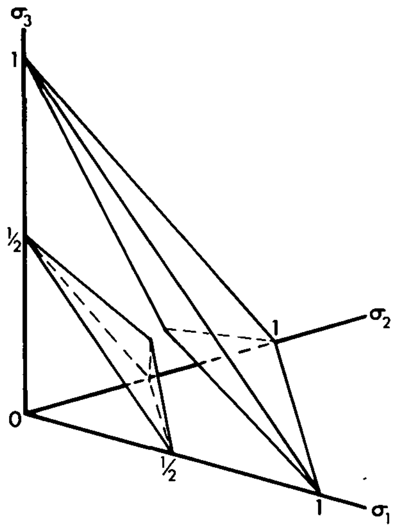

Figure 4. Included regions for outputs for type IV optima under profit functions (4.22) for the case of three firms.

For 3 firms with profit functions

$$J_i = \sigma_i(1 - \sigma_i - \gamma\sigma_T) \tag{4.29}$$

we must include $y_i < 0$, so that all of Table 1 is required in order to give the regions in σ-space for each optimum. These regions are rather difficult to illustrate for (4.29): they are bounded by 3 planes $\Delta_i = 0$, 3 hyperbolae $\Delta_{ij} = 0$ of 2 sheets, and a cubic surface $\Delta_{123} = 0$ of 3 sheets. The regions are, by and large, distorted versions of those obtained for $J_i = \sigma_i(1 - \sigma_T)$. For example, the regions excluded by O_I^W comprise (a) a "tetrahedron" with faces $\sigma_1 = 0$, $\sigma_2 = 0$, $\sigma_3 = 0$ and the sheet of $\Delta_{123} = 0$ nearest to $\boldsymbol{\sigma} = \mathbf{0}$, (b) 3 "tetrahedra" T_i with faces $\sigma_j = 0$, $\sigma_k = 0$, $J_{ii} = 0$ and the sheet of $\Delta_{jk} = 0$ nearest the σ_i axis, and (c) a region above the upper sheets of the surfaces $\Delta_{12} = 0$, $\Delta_{23} = 0$, $\Delta_{31} = 0$ and $\Delta_{123} = 0$. These regions are truncated by the surfaces $J_i = 0$.

For payoff functions of the form (4.29) the regions corresponding to the various optima are all quite small if γ is much less than 1. For example, if $\gamma = 0.3$ the largest of the regions, type I, occupies only about 9.7% of the volume (≈ 0.25) of the output space ($\sigma_i \geqslant 0$, $J_i \geqslant 0$, for $i = 1, 2, 3$). This is because the surfaces $\Delta_i = 0$, $\Delta_{ij} = 0$ and $\Delta_{123} = 0$ are bowed in towards the Cournot optimum, $J_{ii} = 0$ for all i, as in the 2 firm case shown in Figure 2. The strongest optimum, type IV, occupies only about 1.3% of the volume of the output space. In fact all the optima coalesce into the surfaces $J_{ii} = 0$ as $\gamma \to 0$, as can be seen easily from the matrix condition (2.2); it is also fairly clear from the definitions of the optima.

5. Numerical results for 3 firms

We take the profit functions

$$J_i = \sigma_i m_i, \tag{5.1}$$

where

$$m_i(\sigma) = a_i(1 - \sigma_i - 0.3\sigma_T) \tag{5.2}$$

for $i = 1, 2, 3$, where the a_i are positive constants which can be set equal to 1 without loss of generality. The m_i are the profits per good or so-called *average profits*. The adjustment process derived in [3] gives a new output $\sigma_i(t + 1)$ for each firm in business period $t + 1$, in terms of all the preceding outputs $\boldsymbol{\sigma}(1)$, $\boldsymbol{\sigma}(2), \ldots, \boldsymbol{\sigma}(t)$, according to the relations

$$\sigma_i(t + 1) = \frac{1}{2} \cdot \frac{A_i(t)D_i(t) - B_i(t)C_i(t)}{D_i(t) - A_i(t)B_i(t)} \tag{5.3}$$

where

$$A_i(t) = \frac{1}{t} \sum_{\tau=1}^{t} \sigma_i(\tau), \tag{5.4}$$

$$B_i(t) = \frac{1}{t} \sum_{\tau=1}^{t} m_i\{\boldsymbol{\sigma}(\tau)\}, \tag{5.5}$$

$$C_i(t) = \frac{1}{t} \sum_{\tau=1}^{t} \sigma_i(\tau)^2 \tag{5.6}$$

and

$$D_i(t) = \frac{1}{t} \sum_{\tau=1}^{t} J_i\{\boldsymbol{\sigma}(\tau)\}. \tag{5.7}$$

This is just equation (2.6) of [3] with equal *weights* and *confidence factor* unity. It represents a model of the accounting procedures of firms whereby each firm independently maximizes its own estimate of its profit function which it obtains by a least-squares linear fit of its previous average profits.

As mentioned in Section I, [3, Theorem 1] proves that the equilibrium solutions σ of (5.3) satisfy the matrix condition (2.2) (for m_i's more general than (5.2), in fact). The set E of equilibrium solutions was proved (Theorem 4 therein) to contain at least the $N - 1$ dimensional set where $\Delta_{123} = 0$. We now look for the whole set E by numerical iteration of (5.3).

We chose 211 pairs ($\sigma(1)$, $\sigma(2)$) randomly and uniformly from the region ($\sigma_i > 0$, $J_i \geqslant 0$ for all i) in σ-space. For each pair we performed 50 iterations of (5.3) corresponding to 50 adjustments by every firm over 50 business periods. Of the 211 cases only 7 had not converged or were doubtful after 50 iterations. The equilibrium points of the 204 which converged were *all* of type I and were scattered widely through all the corresponding regions given in Table 1 (with the exception of 001 110 0 and its permutations, which correspond to very small regions in σ-space). Among these 204 type I points, 178 were of type II, 144 were of type III and 38 were of type IV. Thus no single one of the stronger optima II, III, IV completely contains the equilibrium solutions. Corollary 1 part (a) seems therefore to be the best possible result, in that all type I optima seem to be possible equilibria, that is, $O_I^W = E$.

Since O_{IV}^W contains 38 of the 204 simulated O_I^W optima while occupying only 13% of the O_I^W volume in σ-space, it has an average density of simulation points which is about 1.4 times that in O_I^W. Thus the Type IV optimum is somewhat favoured for equilibrium points of the adjustment process on a per area basis. This fact may be related to the stability of O_{IV}^W mentioned in Section 2.

Following Theorem 6 of Paper I, we gave a rationalization of the observation that with 2 firms the adjustment process converges to the optima. But now, with 3 firms, 60 of the equilibrium points σ^* obtained are not of types III or IV, and hence at least one firm is not disciplinable for such σ's. Why does the adjustment process not allow such a firm to make a profit increasing move? Presumably the firm's estimate (the $\tilde{J}_i$ of [3]) fails to make such a prediction. We have no real insight into this mechanism and consider it an important unsolved problem for this work.

6. Proof of Theorem 3

It is convenient to begin with Theorem 3 since it clarifies properties of the y_i variables which are used in proofs of the other theorems. With the definitions of Section 4 we have

$$J_{ij} = \begin{cases} g_i + h_i\phi_i & \text{if } i = j, \\ h_i\phi_j & \text{otherwise,} \end{cases} \tag{6.1}$$

which yields

$$\Delta_C = \begin{cases} (1 - y_C) \prod_{i \in C} g_i \text{ for } \boldsymbol{\sigma} \in \Gamma_C, \\ h_j \phi_j \prod_{i \in C - j} g_i \text{ for } \boldsymbol{\sigma} \in Z_{j,C}, j \in C, \\ 0 \text{ otherwise, that is, for } \boldsymbol{\sigma} \in E_C. \end{cases} \tag{6.2}$$

We deduce that

$$\Delta_i \Delta_C \Delta_{C \cup i} = (1 - y_i)(1 - y_C)(1 - y_{C \cup i}) \prod_{j \in C \cup i} g_j^2 \tag{6.3}$$

if $g_j \neq 0$ for all $j \in C \cup i$, so that (2.1) is equivalent to (4.3). If $g_i = 0$, $h_i \phi_i \neq 0$ and $g_j \neq 0$ for all $j \in C$, then

$$\Delta_i \Delta_C \Delta_{C \cup i} = (1 - y_C) h_i^2 \phi_i^2 \prod_{j \in C} g_j^2, \tag{6.4}$$

so that (2.1) is equivalent to $y_C \geqslant 1$. If $g_i \neq 0$ and $g_j = 0$, $h_j \phi_j \neq 0$ for exactly one $j \in C$, then

$$\Delta_i \Delta_C \Delta_{C \cup i} = (1 - y_i) h_j^2 \phi_j^2 \prod_{k \in C \cup i - j} g_k^2, \tag{6.5}$$

so that (2.1) is equivalent to $y_i \geqslant 1$. In the remaining cases

$$\Delta_i \Delta_C \Delta_{C \cup i} = 0, \tag{6.6}$$

which already satisfies (2.1). Combining these cases gives the statement of Theorem 3.

7. Proof of Theorem 4

We prove Theorem 4 next, because the first part, (a), is needed in our proof of Theorem 1. We partition the $\boldsymbol{\sigma}$-space into $2N + 2$ parts:

$$P = \{\boldsymbol{\sigma}: y_i < 1 \text{ for all } i \in \Lambda\},$$
$$Q_i = \{\boldsymbol{\sigma}: y_i > 1, y_j < 1 \text{ for } j \in \Lambda - i\}, \qquad i \in \Lambda,$$
$$Q_i' = \{\boldsymbol{\sigma}: y_i = 1, y_j < 1 \text{ for } j \in \Lambda - i\}, \qquad i \in \Lambda,$$

and

$$R = \{\boldsymbol{\sigma}: y_i \geqslant 1 \text{ for at least two } i \in \Lambda\}. \tag{7.1}$$

Suppose $\boldsymbol{\sigma} \in \Gamma_\Lambda$ for the moment.

LEMMA 1. *Let* $\boldsymbol{\sigma} \in P$. *Then* $\boldsymbol{\sigma} \in \sim \mathrm{O}_{\mathrm{I}}^{W}$ *if and only if* $y_S < 1$ *for all* $S \subseteq \Lambda$.

PROOF. Suppose $\boldsymbol{\sigma} \in P$. If $y_S < 1$ for all $S \subseteq \Lambda$ then

$$(1 - y_i)(1 - y_C)(1 - y_{C \cup i}) > 0 \tag{7.2}$$

for all $i \in \Lambda$ and $C \subseteq \Lambda - i$, and hence $\boldsymbol{\sigma} \in \sim \mathrm{O}_{\mathrm{I}}^{W}$ by Theorem 3. Conversely, if $\boldsymbol{\sigma} \in \sim \mathrm{O}_{\mathrm{I}}^{W}$, then

$$(1 - y_C)(1 - y_{C \cup i}) > 0 \tag{7.3}$$

for all $i \in \Lambda$ and $C \subseteq \Lambda - i$. We write an arbitrary S as $(i_1, i_2, \ldots, i_k)$ and put $S_n = (i_1, \ldots, i_n)$ for $n \leqslant k$. Then $y_{S_n} < 1$ and (7.3) imply $y_{S_{n+1}} < 1$ for $n < k$, while $y_{i_1} < 1$ since $\boldsymbol{\sigma} \in P$. Thus $y_S < 1$ by induction.

LEMMA 2. *Let $\boldsymbol{\sigma} \in Q_i$. Then $\boldsymbol{\sigma} \in \sim \mathrm{O}_{\mathrm{I}}^{W}$ if and only if*

$$y_S > 1 \text{ for all } S \ni i \text{ and } y_S < 1 \text{ for all } S \not\ni i. \tag{7.4}$$

PROOF. Suppose $\boldsymbol{\sigma} \in Q_i$. If $\boldsymbol{\sigma} \in \sim \mathrm{O}_{\mathrm{I}}^{W}$ then

$$(1 - y_C)(1 - y_{C \cup i}) < 0 \tag{7.5}$$

for all $C \subseteq \Lambda - i$. If $y_C > 1$ then $y_{C \cup i} > 2$ so that (7.5) is violated. If $y_{C \cup i} < 1$ then $y_C < 0$ so that (7.5) is again violated. Thus (7.4) must hold. Conversely, if $y_S > 1$ for all $S \ni i$ and $y_S < 1$ for all $S \not\ni i$ then

$$(1 - y_C)(1 - y_{C \cup i}) < 0 \tag{7.6}$$

for all $C \subseteq \Lambda - i$. Since $\boldsymbol{\sigma} \in Q_i$, we further have

$$(1 - y_i)(1 - y_C)(1 - y_{C \cup i}) > 0, \tag{7.7}$$

so that $\boldsymbol{\sigma} \in \sim \mathrm{O}_{\mathrm{I}}^{W}$.

If $\boldsymbol{\sigma} \in Q_i'$, then

$$(1 - y_i)(1 - y_C)(1 - y_{C \cup i}) = 0, \tag{7.8}$$

so that $\boldsymbol{\sigma} \in \mathrm{O}_{\mathrm{I}}^{W}$. For $\boldsymbol{\sigma} \in R$, suppose $y_i \geqslant 1$ and $y_j \geqslant 1$. Then

$$(1 - y_i)(1 - y_j)(1 - y_i - y_j) \leqslant 0, \tag{7.9}$$

so that again $\boldsymbol{\sigma} \in \mathrm{O}_{\mathrm{I}}^{W}$.

Combining these results with Lemmas 1 and 2 gives

$$(\sim \mathrm{O}_{\mathrm{I}}^{W}) \cap \Gamma_\Lambda = \mathfrak{B} \cap \Gamma_\Lambda, \tag{7.10}$$

which is almost part (a) of Theorem 4.

Suppose now that $\boldsymbol{\sigma} \in Z_{i,\Lambda}$. As in (6.4) to (6.5) there are 3 cases of $\Delta_j \Delta_C \Delta_{C \cup j} > 0$:

$$\begin{aligned} (1 - y_C) > 0 \text{ if } j = i \text{ and } C \subseteq \Lambda - i, \\ (1 - y_j) > 0 \text{ if } j \neq i \text{ and } i \in C \end{aligned} \tag{7.11}$$

and

$$(1 - y_j)(1 - y_C)(1 - y_{C \cup j}) > 0 \text{ if } j \in \Lambda - i \text{ and } C \subseteq \Lambda - i - j. \tag{7.12}$$

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If $y_C < 1$ for all $C \subseteq \Lambda - i$, then all three inequalities are satisfied. Conversely, if all 3 inequalities hold, then the first implies the reverse implication. This completes the proof of part (a) of Theorem 4.

To prove part (b) we note that

$$O_{IV}^W \cap \Gamma_\Lambda = \{\sigma: (1 - y_i)(1 - y_{\Lambda-i})(1 - y_\Lambda) \leqslant 0 \text{ for all } i \in \Lambda\} \cap \Gamma_\Lambda. \tag{7.13}$$

LEMMA 3. *Suppose* $\sigma \in O_{IV}^W \cap \Gamma_\Lambda$.
(a) *If* $y_i > 1$ *then either* $y_{\Lambda-i} \geqslant 1$ *or* $y_\Lambda \leqslant 1$.
(b) *If* $0 \leqslant y_i < 1$ *then* $y_{\Lambda-i} \leqslant 1 \leqslant y_\Lambda$.
(c) *If* $y_i \leqslant 0$ *then* $y_\Lambda \leqslant 1 \leqslant y_{\Lambda-i}$.

The proof is immediate from (7.13). We look at various subsets of Γ_Λ.

(i) $$\{\sigma: y_i < 0 \text{ for all } i \in \Lambda\} \cap \Gamma_\Lambda \subset (\sim O_{IV}^W) \cap \Gamma_\Lambda, \tag{7.14}$$
directly from (7.13).

(ii) $$\{\sigma: y_i \geqslant 1 \text{ for all } i \in \Lambda\} \cap \Gamma_\Lambda \subset O_{IV}^W \cap \Gamma_\Lambda, \tag{7.15}$$
directly from (7.13).

(iii) If $0 \leqslant y_i < 1$ for all $i \in \Lambda$ then, by (b), $\sigma \in O_{IV}^W$ if and only if $y_{\Lambda-i} \leqslant 1 \leqslant y_\Lambda$ for all $i \in \Lambda$.

(iv) Suppose that $0 \leqslant y_i < 1$ and $y_j < 0$ for some j. then, by (b) and (c), only $y_\Lambda = 1$ admits $\sigma \in O_{IV}^W$.

(v) Suppose that $0 \leqslant y_i < 1$ and $y_j > 1$ for some j. Then by (a) and (b), only $y_\Lambda = 1$ admits $\sigma \in O_{IV}^W$.

(vi) By (a) and (c), a combination of $y_i \geqslant 1$ for all $i \in S$ and $y_i \leqslant 0$ for all $i \in \Lambda - S$ gives $\sigma \in O_{IV}^W$ if and only if $y_\Lambda \leqslant 1$ and $y_{\Lambda-i} \geqslant 1$ for all $i \in S$.
Combining (i) to (vi) proves that $\sigma \in \mathcal{C}$ or $y_\Lambda = 1$ if $\sigma \in \Gamma_\Lambda$.

If $\sigma \in Z_{i,\Lambda}$ then, as at (7.11), O_{IV}^W is equivalent to $y_{\Lambda-i} \geqslant 1$ and $y_j \geqslant 1$ for $j \neq i$, which are equivalent to $y_j \geqslant 1$ for $j \neq i$. This completes the proof of Theorem 4(b).

8. Proof of Theorem 1

Given that the J_i's have the form (1.1), we establish that, if Ω is the σ-space less some minor points, then

$$(\sim O_I^W) \cap \Omega \subseteq (\sim M) \cap \Omega. \tag{8.1}$$

We use the expression (4.18) for $\sim O_I^W$ obtained in Theorem 4. The reverse implication, $O_I^W \subseteq M$, is given in Theorem 5 of Paper I. One readily shows also

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that $O_I^W \cap (\sim\Omega) = M \cap (\sim\Omega)$. Our method of proof is therefore more roundabout and less transparent than the corresponding proof of Theorem 6 in Paper I for the $N = 2$ case.

Initially, we consider σ confined to the set

$$\Omega = \{\sigma\colon g_i, h_i, \phi_i \text{ all nonzero for all } i \in \Lambda\}, \tag{8.2}$$

where the y_i of (4.1) are all finite and nonzero. Clearly, $\Omega \subset \Gamma_\Lambda$. Let $L = \{i\colon A_{ii} \neq 0\}$, which is not empty. Label the elements of L as $1, \ldots, \nu$, where $\nu < N$.

Given (1.1), multiplying the ith matrix equation of (2.2) by ϕ_i^2, for all i, gives

$$(y_i - 1)C_{ii} + y_i \sum_{j \in L - i} C_{ij} = 0 \tag{8.3}$$

for $i \in L$, where

$$C_{ij} = \phi_i A_{ij} \phi_j \tag{8.4}$$

and y_i is defined by (4.1). The equations (2.2) for $i \in \Lambda - L$ vanish since, in a positive-semi-definite matrix, $A_{ii} = 0$ implies $A_{ij} = 0$ for all $j \in \Lambda$. Since C_{ij} is also positive-semi-definite, we can find ν-vectors $\mathbf{b}_1, \ldots, \mathbf{b}_\nu$, all nonzero, such that $C_{ij} = \mathbf{b}_i \cdot \mathbf{b}_j$, a scalar product. Write

$$\mathbf{b} = \mathbf{b}_1 + \cdots + \mathbf{b}_\nu, \tag{8.5}$$

so that $\sum_{j \in L} C_{ij} = \mathbf{b}_i \cdot \mathbf{b}$. If $\sigma \in M \cap \Omega$, then, because $\mathbf{b}_i \neq 0$ for $i \in L$, it is clear from (8.3) that $\mathbf{b}_i \cdot \mathbf{b} \neq 0$ and thus, also from (8.3),

$$y_i = |b_i|^2/\mathbf{b} \cdot \mathbf{b}_i \quad \text{for } i \in L. \tag{8.6}$$

We define a generalization of (8.6):

$$Y_s = |b_S|^2/\mathbf{b} \cdot \mathbf{b}_S, \tag{8.7}$$

where

$$\mathbf{b}_S = \sum_{i \in S} \mathbf{b}_i \tag{8.8}$$

for any $S \subseteq L$ provided that the denominator is not zero. Cases where $\mathbf{b} \cdot \mathbf{b}_S$ is possibly zero will be handled separately in due course. Examples of Y's are $Y_i = y_i$ and $Y_L = 1$. The following lemmas are needed, in which we assume all Y's are *a priori* well defined.

LEMMA 4. *For any disjoint R and S in L,*

$$\operatorname{sgn}(Y_R + Y_S - Y_{R \cup S}) = \operatorname{sgn} Y_R \cdot \operatorname{sgn} Y_S \cdot \operatorname{sgn} Y_{R \cup S}. \tag{8.9}$$

LEMMA 5. *For any $S \subset L$*

$$\operatorname{sgn}\{(1 - Y_S)(1 - Y_{L-S})\} = \operatorname{sgn} Y_S \cdot \operatorname{sgn} Y_{L-S}. \tag{8.10}$$

PROOFS. Lemma 4 follows from the relation

$$Y_R + Y_S - Y_{R\cup S} = \frac{(\mathbf{b}_R\cdot\mathbf{b})(\mathbf{b}_S\cdot\mathbf{b})}{(\mathbf{b}_R+\mathbf{b}_S)\cdot\mathbf{b}}\left|\frac{\mathbf{b}_R}{\mathbf{b}_R\cdot\mathbf{b}} - \frac{\mathbf{b}_S}{\mathbf{b}_S\cdot\mathbf{b}}\right|^2 \tag{8.11}$$

and Lemma 5 from the relation

$$(1 - Y_S)(1 - Y_{L-S}) = (\mathbf{b}_S\cdot\mathbf{b}_{L-S})^2/\{(\mathbf{b}_S\cdot\mathbf{b})(\mathbf{b}_{L-S}\cdot\mathbf{b})\}. \tag{8.12}$$

Now we look at the sets B_0 and $B_1, \ldots, B_N$, which, by Theorem 4, comprise $\sim O_I^W$ when $\sigma \in \Omega$. For $\sigma \in M \cap \Omega$ we note first from (8.6) that (4.25) holds here, so that $y_i > 0$ for at least one $i \in L$. Thus M may be divided into the following parts, which are compared separately with B_0.

(i) $M_1 = M \cap \{\sigma\colon y_i > 0$ for exactly one $i \in L\}$. For $\sigma \in M_1$, (8.6) gives $\mathbf{b}\cdot\mathbf{b}_j < 0$ for all $j \neq i$ and hence $Y_{L-i} < 0$; Y_{L-i} is well-defined since the denominator is strictly negative, being the sum of strictly negative quantities. Then Lemma 5 gives $(1 - y_i)(1 - Y_{L-i}) < 0$, which implies $y_i > 1$. But $y_i < 1$ for $\sigma \in B_0$, so that $M_1 \cap B_0 = \varnothing$.

(ii) $M_S = M \cap \{\sigma\colon y_i > 0$ for all $i \in S$, $y_i < 0$ for all $i \in L - S\}$. For $\sigma \in M_S$, (8.6) gives $\mathbf{b}\cdot\mathbf{b}_i > 0$ for $i \in S$ and $\mathbf{b}\cdot\mathbf{b}_i < 0$ for $i \in L - S$, so that $Y_S > 0$ and $Y_{L-S} < 0$; again, both are well-defined. Thus, by Lemma 5, $(1 - Y_S)(1 - Y_{L-S}) < 0$ which in turn gives $Y_S > 1$. But Lemma 4 gives $y_i + y_K > y_{K\cup i}$ for $i \in S$ and $K \subseteq S - i$, so that, by induction, $y_S > Y_S$, and consequently $y_S > 1$. But $y_S < 1$ for $\sigma \in B_0$, so that $M_S \cap B_0 = \varnothing$.

Noting that the conclusions of (i) and (ii) are independent of L, we see they jointly imply that, for $\sigma \in \Omega$,

$$B_0 \cap M = \varnothing. \tag{8.13}$$

The part of $M \cap \Omega$ where $y_1 < 1$ has no intersection with B_1. The remaining part can be subdivided into the regions

$$\mu_R = M \cap \{\sigma\colon y_1 > 1, y_i > 0 \text{ for all } i \in R, y_i < 0 \text{ for all } i \in L - 1 - R\}, \tag{8.14}$$

where $R \not\ni 1$; recall that, by definition, $1 \in L$. For $\sigma \in \mu_R$ we have, by Lemma 4 as above,

$$y_R > Y_R > 0 \tag{8.15}$$

and

$$y_{L-R-1} < Y_{L-R-1} < 0, \tag{8.16}$$

both Y's being well-defined. We are now interested in Y_{L-R}, which may not be well-defined, so we must consider three possibilites.

(a) Suppose $Y_{L-R} < 0$ is well-defined. Then, by Lemma 5, $(1 - Y_R)(1 - Y_{L-R}) < 0$, so that $Y_R > 1$, and, by (8.15), $y_R > 1$.

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(b) Suppose $Y_{L-R} > 0$ is well-defined. Then, by Lemma 5, $(1 - Y_R)(1 - Y_{L-R}) > 0$, so that either $Y_R > 1$ and $Y_{L-R} > 1$ or $0 < Y_R < 1$ and $0 < Y_{L-R} < 1$. In the former case, $y_R > 1$ by (8.15). In the latter case, Lemma 4 gives $y_1 + Y_{L-R-1} < Y_{L-R} < 1$ which, with (8.16), implies $y_{L-R} < 1$.
(c) Suppose $\mathbf{b} \cdot \mathbf{b}_{L-R} = 0$. Then

$$0 = \mathbf{b} \cdot \mathbf{b}_{L-R} = |\mathbf{b}_{L-R}|^2 + \mathbf{b}_R \cdot \mathbf{b}_{L-R},$$

whence $\mathbf{b}_R \cdot \mathbf{b}_{L-R} \leqslant 0$. So, from (8.7),

$$Y_R = |\mathbf{b}_R|^2 / \left\{|\mathbf{b}_R|^2 + \mathbf{b}_R \cdot \mathbf{b}_{L-R}\right\} > 1$$

and, by (8.15), $y_R > 1$.

Combining (a) to (c), we see that, if $\boldsymbol{\sigma} \in \mu_R$, then either $y_R > 1$ or $y_{L-R} < 1$. Since $R \not\ni 1$ and $L - R \ni 1$, both conditions exclude B_1, so that $B_1 \cap \mu_R = \varnothing$. It follows that $B_1 \cap M = \varnothing$ and, more generally, for $\boldsymbol{\sigma} \in \Omega$,

$$B_i \cap M = \varnothing \quad \text{for all } i \in L. \tag{8.17}$$

Again, this conclusion depends on L only *via* $i \in L$. If $i \in \Lambda - L$, note that, by definition, $B_i \subseteq \{y_S < 1 \text{ for all } S \subseteq L\} = B_0(L)$, say. It is easy to see, however, that the proofs in (i) and (ii) establish that, for $\boldsymbol{\sigma} \in \Omega$, $B_0(L) \cap M = \varnothing$, whence, for $\boldsymbol{\sigma} \in \Omega$ and any L,

$$B_i \cap M = \varnothing \quad \text{for } i \in \Lambda - L. \tag{8.18}$$

Since (8.17) and (8.18) are true for any L we have, for $\boldsymbol{\sigma} \in \Omega$,

$$B_i \cap M = \varnothing \quad \text{for } i \in \Lambda. \tag{8.19}$$

Combining (8.13) and (8.19) gives

$$(\sim O_1^W) \cap \Omega \subseteq (\sim M) \cap \Omega. \tag{8.20}$$

It remains to deal with the complement of Ω, which can be written as

$$\sim \Omega = \bigcup_1^5 W_i, \tag{8.21}$$

where

$W_1 = \{\boldsymbol{\sigma}: h_i = 0 \text{ for at least one } i \in \Lambda, \phi_i \text{ and } g_i \text{ non-zero for all } i \in \Lambda\}$,

$W_2 = \{\boldsymbol{\sigma}: \phi_i = 0 \text{ for at least one } i \in \Lambda, g_i \neq 0 \text{ for all } i \in \Lambda, (h\text{'s arbitrary})\}$,

$W_3 = \{\boldsymbol{\sigma}: g_i = 0 \text{ for exactly one } i, h_i\phi_i \neq 0 \text{ (other } h\text{'s and } \phi\text{'s arbitrary)}\}$,

$W_4 = \{\boldsymbol{\sigma}: g_i = 0, h_i\phi_i \neq 0 \text{ for at least two } i \in \Lambda\}$

and

$$W_5 = \{\boldsymbol{\sigma}: g_i = 0, h_i\phi_i = 0 \text{ for at least one } i \in \Lambda\}. \tag{8.22}$$

If $\boldsymbol{\sigma} \in W_1$, let $h_i \neq 0$ for $i \in L_1 \subset \Lambda$, say, and $h_i = 0$ for $i \in \Lambda - L_1$. It is clear from (2.2) and (4.1) that we now have the equations (8.3) only for $i, j \in L_1$

with, of course, $y_i = 0$ for $i \in \Lambda - L_1$. The previous proof covers precisely this reduction of (8.3) (for a different reason), since g_i, h_i, ϕ_i are all nonzero for $i \in L_1$, and hence

$$\sim O_I^W \cap W_1 \subseteq (\sim M) \cap W_1. \tag{8.23}$$

If $\sigma \in W_2$, let $\phi_i \neq 0$ for $i \in L_2 \subset \Lambda$, say, and $\phi_i = 0$ for $i \in \Lambda - L_2$. Since $y_i = 0$ for $i \in \Lambda - L_2$, the inequalities on the y_i's specifying $\sim O_I^W$ in Theorem 4(a) involve only y_i for $i \in L_2$. Consider equations (2.2) only for $i, j \in L_2$; because g_i and ϕ_i are nonzero for $i \in L_2$ we get (8.3) for $i, j \in L_2$. These equations determine y_i for $i \in L_2$ and, if their solution set is $M(L_2)$, the previous arguments for $\sigma \in \Omega$ or W_1 now apply *verbatim* to this restricted problem and yield, for $\sigma \in W_2$,

$$(\sim O_I^W) \cap M(L_2) = \varnothing. \tag{8.24}$$

But $M \cap W_2 \subseteq M(L_2) \cap W_2$, because the additional equations involved in M can only restrict the solutions in $M(L_2)$. Thus

$$(\sim O_I^W) \cap W_2 \subseteq (\sim M) \cap W_2. \tag{8.25}$$

If $\sigma \in W_3$, assume for the moment that $h_i\phi_i \neq 0$ for *all* $i \in \Lambda$. It is clear from (2.2) and (4.1) that, under these conditions, $\mathbf{b} \cdot \mathbf{b}_i = 0$ when $g_i = 0$. By Theorem 4(a) we have

$$(\sim O_I^W) \cap W_3 = \{\sigma: y_S < 1 \text{ for all } S \subseteq \Lambda - i \text{ if } g_i = 0\} \cap W_3. \tag{8.26}$$

Consider $\sigma \in M \cap W_3$. Suppose $g_i = 0$, $L \ni i$, $y_j > 0$ for $j \in S \subseteq L$ and $y_j < 0$ for $j \in L - S - i$. Then, by Lemma 4, $y_S > Y_S > 0$ and, by Lemma 5,

$$\operatorname{sgn}\{(1 - Y_S)(1 - Y_{L-S})\} = \operatorname{sgn} Y_{L-S}. \tag{8.27}$$

By (8.7), however,

$$\operatorname{sgn} Y_{L-S} = \operatorname{sgn}(\mathbf{b}_{L-S} \cdot \mathbf{b}) = \operatorname{sgn}(\mathbf{b}_{L-S-i} \cdot \mathbf{b}) = \operatorname{sgn} Y_{L-S-i}, \tag{8.28}$$

because $\mathbf{b}_i \cdot \mathbf{b} = 0$; this, incidentally, proves also that Y_{L-S} is well defined. Thus (8.27) implies $Y_S > 1$, which implies, in turn, $y_S > 1$. If $L \not\ni i$, the same conclusion follows from (8.27) alone. Comparison with (8.26) shows that, under the stated conditions, for $\sigma \in W_3$, $(\sim O_I^W) \cap M = \varnothing$. If either h_i or ϕ_i is zero for some $i \in \Lambda$, then arguments similar to those above, for W_1 and W_2, establish that

$$(\sim O_I) \cap W_3 \subseteq (\sim M) \cap W_3. \tag{8.29}$$

Combining (8.20), (8.23), (8.25) and (8.29) gives

$$(\sim O_I^W) \cap (\Omega \cup W_1 \cup W_2 \cup W_3) \subseteq (\sim M) \cap (\Omega \cup W_1 \cup W_2 \cup W_3) \tag{8.30}$$

and, since $O_I^W \subseteq M$ by Theorem 2 of Paper I, we have

$$(\sim O_I^W) \cap (\Omega \cup W_1 \cup W_2 \cup W_3) = (\sim M) \cap (\Omega \cup W_1 \cup W_2 \cup W_3). \tag{8.31}$$

If $\sigma \in W_4$ then $\sigma \in O_I^W$ by Theorem 3 but, if $\sigma \in W_4$, we can take $g_1 = g_2 = 0$, $h_1\phi_1 \neq 0$ and $h_2\phi_2 \neq 0$ without loss of generality. Then the positive-semi-definite matrix

$$A_{ij} = \begin{cases} \phi_i\phi_j & \text{if } i = 1, 2 \text{ and } j = 1, 2, \\ 0 & \text{otherwise,} \end{cases} \tag{8.32}$$

satisfies the matrix condition (2.2), so that $\sigma \in M$, and hence

$$O_I^W \cap W_4 = M \cap W_4. \tag{8.33}$$

If $\sigma \in W_5$ then $\sigma \in O_I^W$ by Theorem 3 but, if $\sigma \in W_5$, then $\Delta_\Lambda = 0$ so that $A_{ij} = x_i x_j$ satisfies (2.2) for suitable x_i's. Thus, again, $\sigma \in M$ and

$$O_I^W \cap W_5 = M \cap W_5. \tag{8.34}$$

Combining (8.31), (8.33) and (8.34) completes the proof of Theorem 1.

9. Completion of the proof of Theorem 2

Table 1 proves the statement

$$O_I \cap \Gamma_{123} = (O_{II} \cup O_{III}) \cap \Gamma_{123} \tag{9.1}$$

with Γ's defined by (4.5). Superfices W are dropped from the notation in this section. Since $O_I \supset O_{II}, O_{III}$, we immediately have

$$O_I \cap (\sim \Gamma_{123}) \supseteq (O_{II} \cup O_{III}) \cap (\sim \Gamma_{123}), \tag{9.2}$$

and it remains to prove the reverse implication. Throughout this section $\Lambda = (1, 2, 3)$.

We define

$$\hat{D}_{i,C} = D_{i,C} \cap (\sim \Gamma_{i \cup C}). \tag{9.3}$$

Then we define $\hat{O}_k$ in terms of $\hat{D}_{i,C}$ just as O_k is defined in terms of $D_{i,C}$ for $k = \text{I, II, III}$. Defining

$$H_{ij,k} = (\sim \Gamma_{ij,k}) \cap \Gamma_{ij} = \{\sigma\colon g_k = 0, g_i \neq 0, g_j \neq 0\}, \tag{9.4}$$

we have

$$O_I \cap (\sim \Gamma_{123}) = \hat{O}_I \cup \bigcup_{(i,j)} (D_{i,j} \cap H_{ij,k}), \tag{9.5}$$

$$O_I \cap (\sim \Gamma_{123}) = \hat{O}_{II} \tag{9.6}$$

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and

$$\mathrm{O}_{\mathrm{III}} \cap (\sim \Gamma_{123}) = \bigcap_i \left\{ \bigcup_{C \subsetneq \Lambda - i} \hat{D}_{i,C} \bigcup_{j \in \Lambda - i} (D_{i,j} \cap H_{ij,k}) \right\} \supseteq \hat{\mathrm{O}}_{\mathrm{III}}. \quad (9.7)$$

LEMMA 6. $\hat{\mathrm{O}}_{\mathrm{I}} \subseteq \hat{\mathrm{O}}_{\mathrm{II}}$.

LEMMA 7. $D_{i,j} \cap H_{ij,k} \subseteq \mathrm{O}_{\mathrm{II}} \cap H_{ij,k}$.

Lemma 6 with (9.6) and (9.7) gives

$$\hat{\mathrm{O}}_{\mathrm{I}} \subseteq \hat{\mathrm{O}}_{\mathrm{II}} \cup \hat{\mathrm{O}}_{\mathrm{III}} \subseteq (\mathrm{O}_{\mathrm{II}} \cup \mathrm{O}_{\mathrm{III}}) \cap (\sim \Gamma_{123}),$$

which, with (9.5), gives

$$\mathrm{O}_{\mathrm{I}} \cap (\sim \Gamma_{123}) \subseteq \{(\mathrm{O}_{\mathrm{II}} \cup \mathrm{O}_{\mathrm{III}}) \cap (\sim \Gamma_{123})\} \cup \bigcup_{i,j} (D_{i,j} \cap H_{ij,k}).$$

With Lemma 7, this gives the desired reverse of (9.2). It remains to prove the Lemmas.

PROOF OF LEMMA 6. From (4.7), $\hat{D}_{1,2} = (T_2 \cap Z_{1,12}) \cup (T_1 \cap Z_{2,12}) \cup E_{12}$ and $\hat{D}_{1,23} \supseteq \{T_1 \cap (Z_{2,123} \cup Z_{3,123})\} \cup E_{123}$, where $T_i = \{y_i > 1\}$. Writing out $\sim E_{12}$ and $\sim E_{123}$ explicitly one readily finds that $\sim E_{123} \subseteq \sim E_{12}$, whence $E_{12} \subseteq E_{123}$. Now

$$\begin{aligned} T_1 \cap Z_{2,12} &= T_1 \cap \left[Z_{2,123} \cup \{ g_2 = g_3 = 0, h_2\phi_2 \neq 0 \} \right] \\ &\subseteq T_1 \cap Z_{2,123} \cup E_{123} \subseteq \hat{D}_{1,23}. \end{aligned}$$

Similarly, $T_2 \cap Z_{1,12} \subseteq \hat{D}_{2,13}$, so that $\hat{D}_{1,2} \subseteq \hat{D}_{1,23} \cup \hat{D}_{2,13}$. Writing out $\hat{\mathrm{O}}_{\mathrm{I}}$ and $\hat{\mathrm{O}}_{\mathrm{II}}$ explicitly in terms of $\hat{D}$'s we see that Lemma 6 is proved.

PROOF OF LEMMA 7.

$$\begin{aligned} D_{i,j} \cap H_{ij,k} &= \{(1 - y_i)(1 - y_j)(1 - y_{ij}) \leqslant 0\} \cap H_{ij,k} \\ &\subseteq (\{y_i \geqslant 1\} \cup \{y_j \geqslant 1\} \cup \{y_{ij} \geqslant 1\}) \cap H_{ij,k} \\ &\subseteq \left[(\{y_i \geqslant 1\} \cup \{y_j \geqslant 1\} \cup \{y_{ij} \geqslant 1\}) \cap Z_{k,ijk} \right] \\ &\quad \cup (H_{ij,k} \cap \{h_k\phi_k = 0\}) \\ &= (D_{i,jk} \cup D_{j,ki} \cup D_{k,ij}) \cap H_{ij,k} \\ &= \mathrm{O}_{\mathrm{II}} \cap H_{ij,k}. \end{aligned}$$

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