ON THE GENERALIZED JOSEPHUS PROBLEM

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1. Introduction and statement of the problem. The problem of Josephus and the forty Jews is well known [**1, 3**]. In its most general form, this problem is equivalent to the problem of *m-enumeration* of a set, as described below.

Define the ordered set

$$Z_n = \{1, 2, \ldots, n\}.$$

We choose and remove cyclically, from left to right, each mth element of Z_n until the set is exhausted. The chosen elements are ordered into a new ordered set

$$Z_n^{(m)} = \{a_1, a_2, \ldots, a_n\},$$

which is therefore a permutation of Z_n, obtained by what we call m-enumeration of the set Z_n.

The following two questions arise:

(i) For any $i (1 \leqq i \leqq n)$, which position in $Z_n^{(m)}$ is occupied by i? In other words, when is the ith element of Z_n removed?

(ii) For any $k (1 \leqq k \leqq n)$, which element of Z_n occupies the kth position in $Z_n^{(m)}$? In other words, which element of Z_n is the kth to be removed?

The solution presented here is, as far as I am aware, essentially different from those of other authors. However, Rankin [**2**] used a somewhat similar method to answer the particular question: " Which element of Z_n is the nth to be removed? ".

2. Notation and method of solution. We introduce the following notations:

$$[x] = \text{integer part of } x, \quad \{x\} = -[-x],$$

$$e_\nu = \left[\frac{n_\nu}{m}\right], \quad E_\nu = \sum_{i=0}^{\nu} e_i, \quad r_\nu = R(n_\nu, m) = n_\nu - m\left[\frac{n_\nu}{m}\right],$$

where $R(n_\nu, m)$ denotes the remainder on dividing n_ν by m.

Let $n = n_0$, and denote the sequence of numbers in Z_n by $\mathscr{S}_0$. We assume that $n > m$. We have

$$\mathscr{S}_0 : 1, 2, \ldots, e_0 m + r_0 = n_0.$$

From the sequence $\mathscr{S}_0$ we remove the numbers m, $2m, \ldots, e_0 m$; they will form the 0th class in $Z_n^{(m)}$, and the sequence $\mathscr{S}_0$ becomes the sequence $\mathscr{S}'_0$.

$$\mathscr{S}'_0 : 1, 2, \ldots, m-1, m+1, \ldots, 2m-1, 2m+1, \ldots, e_0 m + r_0.$$

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We renumber the terms of the sequence $\mathscr{S}'_0$ consecutively from r_0+1 to obtain the sequence $\mathscr{S}_1$.

$$\mathscr{S}_1 : r_0+1, r_0+2, \ldots, e_1 m + r_1 = n_1.$$

From the sequence $\mathscr{S}_1$ we remove the numbers $m, 2m, \ldots, e_1 m$; they will form the first class in $Z_n^{(m)}$, and the sequence $\mathscr{S}_1$ becomes the sequence $\mathscr{S}'_1$.

$$\mathscr{S}'_1 = r_0+1, r_0+2, \ldots, m-1, m+1, \ldots, 2m-1, 2m+1, \ldots, e_1 m + r_1.$$

We renumber the terms of the sequence $\mathscr{S}'_1$ consecutively from r_1+1 to obtain the sequence $\mathscr{S}_2$.

$$\mathscr{S}_2 = r_1+1, r_1+2, \ldots, e_2 m + r_2 = n_2.$$

We proceed thus until we obtain the inequality

$$n_t < m.$$

The last sequence $\mathscr{S}_t$ will be

$$\mathscr{S}_t = r_{t-1}+1, r_{t-1}+2, \ldots, n_t,$$

which contains $n_t - r_{t-1} = u$ numbers. Among them there is no number of type τm. Since $n_t < m$, we have $u < m$.

Further m-enumeration leads to the sequence $\mathscr{S}_t$ being permuted to yield the sequence

$$\mathscr{C} = \{c_1, c_2, \ldots, c_u\},$$

say, and this will be the tth and last class in $Z_n^{(m)}$.

The following relation is easy to verify:

$$n_{v+1} = n_v - e_v + r_v - r_{v-1} \quad (v = 0, 1, \ldots, t-1).$$

Now, for an integer i $(1 \leqq i \leqq n)$, we let i_v (respectively i'_v) denote the value of that integer in $\mathscr{S}_v$ (respectively $\mathscr{S}'_v$) which has been derived from $i = i_0$ in $\mathscr{S}_0$ $(1 \leqq v \leqq t-1)$, provided that such an integer exists. Then

$$i_{v+1} = \left\{\frac{m-1}{m} i'_v\right\} + r_v - r_{v-1},$$

and conversely

$$i'_v = \left[\frac{m(i_{v+1} - r_v + r_{v-1}) - 1}{m-1}\right].$$

Therefore the sequences $\mathscr{S}_v$, $\mathscr{S}_{v+1}$ are related by

$$i_{v+1} = \left\{\frac{m-1}{m} i_v\right\} + r_v - r_{v-1}, \tag{1}$$

and

$$i_v = \left[\frac{m(i_{v+1} - r_v + r_{v-1}) - 1}{m-1}\right], \tag{2}$$

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where $i_\nu \neq \tau m$ for any integer τ, and

$$r_{\nu-1}+1 \leqq i_\nu \leqq n_\nu \quad \text{for} \quad \nu = 0, 1, 2, \dots \ .$$

The equivalence of the relations (1), (2) follows from the following easily proved lemma.

LEMMA. *If* $m \nmid b$, *then* $a = \left\{\frac{m-1}{m} b\right\}$ *if and only if* $b = \left[\frac{ma-1}{m-1}\right]$.

We shall now provide algorithms to answer the two questions posed in §1. For this purpose, we require to construct a table of values of the parameters n_ν, e_ν, E_ν, r_ν, $r_\nu - r_{\nu-1}$ for $\nu = 0, 1, 2, \dots, t$.

Question 1. Given $i \in Z_n$, put $i = i_0 = \tau_0 m + \rho_0$ $(0 \leqq \rho_0 \leqq m-1)$. If $\rho_0 = 0$, then $i_0 = \tau_0 m$ and i_0 belongs to the 0th class in $Z_n^{(m)}$ at the τ_0th position. Therefore $a_{\tau_0} = i_0$. If $\rho_0 > 0$, the number i_0 will go over to the sequence $\mathscr{S}'_0$ and thence to the sequence $\mathscr{S}_1$, in which it will assume the value

$$i_1 = \left\{\frac{m-1}{m} i_0\right\} + r_0.$$

Let $i_1 = \tau_1 m + \rho_1$. If $\rho_1 = 0$, then $i_1 = \tau_1 m$ and the number i_0 will occupy the τ_1th place of the first class of $Z_n^{(m)}$. Hence $a_{E_0+\tau_1} = i_0$. If $\rho_1 > 0$, i_1 will go over to the sequence $\mathscr{S}'_1$ and thence to the sequence $\mathscr{S}_2$, in which it will assume the value

$$i_2 = \left\{\frac{m-1}{m} i_1\right\} + r_1 - r_0.$$

Let $i_2 = \tau_2 m + \rho_2$.

Proceeding in this way we have: Let $i_\nu = \tau_\nu m + \rho_\nu$. If $\rho_\nu = 0$, then $i_\nu = \tau_\nu m$ and i_0 occupies the τ_νth place of the νth class in $Z_n^{(m)}$. Thus

$$i_0 = a_{e_0+e_1+\dots+e_{\nu-1}+\tau_\nu} = a_{E_{\nu-1}+\tau_\nu}.$$

If $\rho_\nu > 0$, i_ν will go over to the sequence $\mathscr{S}'_\nu$ and thence to the sequence $\mathscr{S}_{\nu+1}$.

If finally, the number i_0 goes over to the last sequence $\mathscr{S}_t$, it will be one of the numbers $c_1, \dots, c_u$. If, then, $i_0 \to i_t = c_j$, we have

$$i_0 = a_{E_{t-1}+j}.$$

We thus obtain the following rule for finding the position in $Z_n^{(m)}$ of i.

Rule. If $i = i_0 \to i_1 \to \dots \to i_\nu = \tau_\nu m$, then $i = a_{E_{\nu-1}+\tau_\nu} (\nu < t)$.

If $i = i_0 \to i_1 \to \dots \to i_t = c_j$, then $i = a_{E_{t-1}+j}$.

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Question 2. We apply a procedure converse to the former. We consider two cases.

(i) If $k > E_{t-1}$, then let $k = E_{t-1}+j$, where $j \leqq u$. To the number a_k there will correspond the number $c_j = i_t$. Applying formula (2) several times, we get:

$$i_{t-1} = \left[\frac{m(i_t - r_{t-1} + r_{t-2}) - 1}{m-1}\right],$$

$$i_{t-2} = \left[\frac{m(i_{t-1} - r_{t-2} + r_{t-3}) - 1}{m-1}\right],$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$i_0 = \left[\frac{m(i_1 - r_0) - 1}{m-1}\right].$$

Therefore $a_k = i_0$. In this way, we can find a_n, the last element of Z_n to be removed.

(ii) If $k \leqq E_{t-1}$, we choose v so that $E_{v-1} < k \leqq E_v$, and put $k = E_{v-1} + \tau_v$, where $\tau_v \leqq e_v$, $v \leqq t-1$. In this case, the number a_k will occupy the τ_vth place in the vth class of $Z_n^{(m)}$. Then $i_v = \tau_v m$, and applying formula (2) several times we get

$$i_{v-1} = \left[\frac{m(i_v - r_{v-1} + r_{v-2}) - 1}{m-1}\right],$$

$$i_{v-2} = \left[\frac{m(i_{v-1} - r_{v-2} + r_{v-3}) - 1}{m-1}\right],$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$i_0 = \left[\frac{m(i_1 - r_0) - 1}{m-1}\right].$$

Therefore $a_k = i_0$.

3. Illustration by an example. Suppose that $n = 117$, $m = 6$. We draw up the table of parameters n_v, e_v, E_v, r_v, $r_v - r_{v-1}$ for $v = 0, 1, \ldots, n - E_{t-1} = 117 - 113 = 4$, so that the tth class in $Z_{117}^{(6)}$ is $\{c_1, c_2, c_3, c_4\}$, obtained by 6-enumeration of $\{r_{17}+1, r_{17}+2, r_{17}+3, r_{17}+4\} = (1, 2, 3, 4)$, and so

$$\{c_1, c_2, c_3, c_4\} = \{2, 1, 4, 3\}.$$

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ν	n_ν	e_ν	E_ν	r_ν	$r_\nu - r_{\nu-1}$
0	117	19	19	3	3
1	101	16	35	5	2
2	87	14	49	3	−2
3	71	11	60	5	2
4	62	10	70	2	−3
5	49	8	78	1	−1
6	40	6	84	4	3
7	37	6	90	1	−3
8	28	4	94	4	3
9	27	4	98	3	−1
10	22	3	101	4	1
11	20	3	104	2	−2
12	15	2	106	3	1
13	14	2	108	2	−1
14	11	1	109	5	3
15	13	2	111	1	−4
16	7	1	112	1	0
17	6	1	113	0	−1
18	4	0	—	4	4

Examples of Question 1.

(a) When is the number 64 removed?

We have

$$i_0 = 64 \to 57 \to 50 \to 40 \to 36 = 6 \times 6$$

as

$$\nu \to \ 0 \to \ 1 \to \ 2 \to \ 3 \to \ 4,$$

so that $64 = a_{E_3+6} = a_{66}$; i.e., 64 is the 66th element of Z_{117} to be removed.

(b) When is the number 80 removed?

$$i_0 = 80 \to 70 \to 61 \to 49 \to 43 \to 33 \to 27 \to 26 \to 19 \to 19 \to 15 \to 14 \to 10 \to 10 \to 8 \to 10 \to 5 \to 5 \to 4 = c_3$$

as

$$\nu = 0 \to 1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 8 \to 9 \to 10 \to 11 \to 12 \to 13 \to 14 \to 15 \to 16 \to 17 \to 18,$$ so that

$80 = a_{E_{17}+3} = a_{116}$; i.e., 80 is the 116th element to be removed.

Examples of Question 2.

(a) What is the 46th element to be removed?

$$46 = E_1 + 11,$$

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so that 46 occupies 11th place in the 2nd class of $Z_{117}^{(6)}$. Applying formula (2), we get

$$i_\nu = 66 \to 76 \to 87$$

as

$$\nu = 2 \to 1 \to 0;$$

i.e., 87 is the 46th element to be removed.

(b) What is the last (117th) element to be removed?

$$117 = E_{17}+4,\ a_{117} \to c_3 = 4.$$

$$\begin{aligned} i_\nu = {} & 3 \to 4 \to 4 \to 9 \to 7 \to 9 \to 9 \to 13 \to 14 \to 17 \\ & \to 16 \to 22 \to 22 \to 27 \to 35 \to 39 \to 49 \to 56 \to 63 \end{aligned}$$

as

$$\begin{aligned} \nu = {} & 18 \to 17 \to 16 \to 15 \to 14 \to 13 \to 12 \to 11 \to 10 \to 9 \\ & \to 8 \to 7 \to 6 \to 5 \to 4 \to 3 \to 2 \to 1 \to 0; \end{aligned}$$

i.e., 63 is the last element to be removed.

REMARK. In the last step of the above method, we have to carry out the m-enumeration of the set $\{r_{t-1}+1, r_{t-1}+2, \ldots, n_t\}$, containing $u = r_t$ elements, where $r_t < m$.

If m is small, this operation may be carried out directly. However, if m is relatively large, we may, as in the general case of $n \leqq m$, use, for example, the method of increasing divisors, which is based on the following principle:

If $a_{s,n}$ is the sth element from the right in $Z_n^{(m)}$, where $s < n$, then $a_{s,n+1} = R(a_{s,n}+m, n+1)$ is the sth element from the right in $Z_{n+1}^{(m)}$. For $s = n$ we have $a_{s,s} \doteq R(m, s)$.

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