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Involutions Fixing $F^n \cup \{\text{Indecomposable}\}$

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Abstract. Let M^m be an m-dimensional, closed and smooth manifold, equipped with a smooth involution $T\colon M^m \to M^m$ whose fixed point set has the form $F^n \cup F^j$, where F^n and F^j are submanifolds with dimensions n and j, F^j is indecomposable and $n > j$. Write $n - j = 2^p q$, where $q \geq 1$ is odd and $p \geq 0$, and set $m(n-j) = 2n + p - q + 1$ if $p \leq q + 1$ and $m(n-j) = 2n + 2^{p-q}$ if $p \geq q$. In this paper we show that $m \leq m(n-j) + 2j + 1$. Further, we show that this bound is *almost* best possible, by exhibiting examples $(M^{m(n-j)+2j}, T)$ where the fixed point set of T has the form $F^n \cup F^j$ described above, for every $2 \leq j < n$ and j not of the form $2^t - 1$ (for $j = 0$ and 2, it has been previously shown that $m(n-j) + 2j$ is the best possible bound). The existence of these bounds is guaranteed by the famous 5/2-theorem of J. Boardman, which establishes that under the above hypotheses $m \leq \frac{5}{2}n$.

1 Introduction

Suppose M^m is a smooth and closed m-dimensional manifold and $T\colon M^m \to M^m$ is a smooth involution defined on M^m. The fixed point set of T, F, is a disjoint union of closed submanifolds of M^m, $F = \bigcup_{j=0}^{n} F^j$, where F^j denotes the union of those components of F having dimension j and thus n is the dimension of the components of F of largest dimension. If the involution pair (M^m, T) is not an equivariant boundary, then n cannot be too small with respect to m. This intriguing fact was first evidenced from an 1964 result of P. E. Conner and E. E. Floyd [4, Theorem 27.1] which states that for each natural number n, there exists a number $\varphi(n)$ with the property that if (M^m, T) is an involution fixing $F = \bigcup_{j=0}^{n} F^j$ and if $m > \varphi(n)$, then (M^m, T) bounds equivariantly. Later, this was explicitly confirmed by the well-known $\frac{5}{2}$-Theorem of J. Boardman [1]: if (M^m, T) fixes $F = \bigcup_{j=0}^{n} F^j$ and M^m is non-bounding, then $m \leq \frac{5}{2}n$. A strengthened version of this fact was obtained by R. E. Stong and C. Kosniowski [7]: if (M^m, T) is a non-bounding involution, which is equivalent to the fact that the normal bundle of F in M^m is not a boundary (see [4]), then $m \leq \frac{5}{2}n$. In particular, if F is non-bounding (which means that at least one F^j is non-bounding), then $m \leq \frac{5}{2}n$. The generality of this result, which is valid for every $n \geq 1$, allows the possibility that fixed components of all dimensions j, $0 \leq j \leq n$, occur; in this way, it is natural to ask whether there exist better bounds for m when we omit some components of F and restrict the set of involved dimensions n. This question is inspired by the following results from the literature:

(1) (R. E. Stong and C. Kosniowski [7]): if (M^m, T) is an involution whose fixed point set has constant dimension n, and if $m > 2n$, then (M^m, T) bounds equivariantly. In particular, if $F = F^n$ with constant dimension n is non-bounding, and if

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(M^m, T) fixes F, then $m \le 2n$. This bound is best possible, as can be seen by taking the involution $(F^n \times F^n, T)$, where F^n is any non-bounding n-dimensional manifold (with the exception of $n = 1$ and $n = 3$) and T switches coordinates; that is, one has in this case an improvement for the Boardman bound by omitting the j-dimensional components of F with $j < n$ and excluding $n = 1$ and 3.

(2) (D. C. Royster [14]): In this case, the result in question refers to an intriguing improvement for the Boardman bound, given by n odd and the omission of the j-dimensional components of F with $0 < j < n$. Let (M^m, T) be an involution whose fixed point set has the form $F = F^n \cup \{\text{point}\}$. Then in this case the bound for the codimension of the top dimensional component of F is constant and quite small: $m \le n + 1$. Evidently, this bound is best possible, and is realized by the involution (RP^{n+1}, T), where RP^{n+1} is the $(n + 1)$-dimensional real projective space and $T[x_0, x_1, \dots, x_{n+1}] = [-x_0, x_1, \dots, x_{n+1}]$, with n odd.

This class of problems was introduced by P. Pergher [8], where the above Royster result was enlarged in the following way: if (M^m, T) is an involution fixing $F = F^n \cup \{\text{point}\}$, where $n = 2p$ with p odd, then $m \le n + p + 3$. This case ($F = F^n \cup \{\text{point}\}$) was completed by R. Stong and P. Pergher[9]: for each natural number n, write $n = 2^p q$, where $p \ge 0$ and q is odd, and set

$$m(n) = \begin{cases} 2n + p - q + 1 & \text{if } p \le q + 1, \\ 2n + 2^{p-q} & \text{if } p \ge q. \end{cases}$$

Then if (M^m, T) is an involution fixing $F = F^n \cup \{\text{point}\}$, $m \le m(n)$; further, there are involutions with $m = m(n)$ fixing a point and some F^n, which shows that these bounds are best possible.

Once the cases $F = F^n$ and $F = F^n \cup \{\text{point}\}$ are established, the next natural step is to consider fixed point sets of the form $F = F^n \cup F^j$, $0 < j < n$. Recently, some advances have been obtained in this direction. Specifically, we find best possible bounds for $j = 1$ in [5,6], $j = 2$ in [10,11,13], and $j = n-1$ in [12]. For $F = F^n \cup F^1$, this bound is $m = m(n - 1) + 1$ if n is odd, and $m = m(n - 1) + 2$ if n is even. For $F = F^n \cup F^2$, this bound is $m = m(n - 2) + 4$, and for $F = F^n \cup F^{n-1}$ it is $m = 2n$ (which coincides with $m = m(n - (n - 1)) + 2(n - 1)$). We remark that the method used in the case $F = F^n \cup F^{n-1}$ does not work for $F = F^n \cup F^{n-2}$; on the other hand, the arguments used in the cases $j = 0$, 1, and 2 become an unpleasant mess for $j > 2$. In other words, the general case $F = F^n \cup F^j$, $n > j$, is difficult. In this paper we contribute to this general case by supposing that F^j is an indecomposable j-dimensional manifold; we recall that a closed manifold is called *indecomposable* if its unoriented cobordism class cannot be expressed as a sum of products of lower dimensional cobordism classes. This hypothesis is not so restrictive, since in a certain sense half of the manifolds have this property (if j is not of the form $2^t - 1$, then half of the elements of the unoriented cobordism group $\mathcal{N}_j$ are indecomposable). The result to be proved is the following.

Theorem 1 *Let (M^m, T) be an involution having fixed point set of the form $F = F^n \cup F^j$, where F^j is indecomposable and $n > j$. Then $m \le m(n - j) + 2j + 1$.*

The crucial point of our method will be the combination of two very special characteristic classes. One of them, called $\mathcal{X}$, was introduced by R. E. Stong and P. Pergher [9]. It was also used to find bounds in [5, 11, 13]. The other class, associated with line bundles over closed manifolds, is built with the use of the splitting principle, and is well related with the standard class that detects indecomposability.

In addition, we will also give examples of involutions $(M^{m(n-j)+2j}, T)$ having fixed point set F of the form $F = F^n \cup F^j$, where F^j is indecomposable and $n > j$, for every $n \geq 3$ and $j \geq 2$ not of the form $2^t - 1$ (we recall that indecomposable j-dimensional manifolds occur only for these values of j), thus showing that the bound $m \leq m(n-j) + 2j + 1$ is *almost* best possible.

Note that if the pair (n, j) satisfies $n - j = 2^p$ for some $p \geq 0$, then

$$m(n-j)+2j+1 = 2(n-j)+2^{p-1}+2j+1 = 2(n-j)+\frac{n-j}{2}+2j+1 = \frac{5}{2}n+1-\frac{j}{2}.$$

Therefore, our result is redundant for these pairs if in addition $j = 0$ or 2 (as previously mentioned, in these cases $m(n-j) + 2j$ is the best possible bound). However, for the remaining pairs (n, j), the result improves the Boardman bound. The best possible improvement in this case occurs when $n - j$ is odd: $m \leq n + j + 2$. Again, this characterizes an intriguing small codimension phenomenon: the maximal codimension in this case $(= j + 2)$ does not depend on n.

The question of either improving the bound $m \leq m(n-j) + 2j + 1$ to $m \leq m(n-j) + 2j$ or finding a maximal example $(M^{m(n-j)+2j+1}, T)$ will be left open.

2 Proof of the Result and Almost Maximal Examples

First we give some preliminaries and establish the notations to be used in the proof of the result announced in Section 1. Consider an involution (M^m, T) with fixed point set of the form $F^n \cup F^j$, where F^n is any n-dimensional closed manifold and F^j is an indecomposable j-dimensional manifold with $n > j$. Denote by $\eta \to F^n$ and $\mu \to F^j$ the normal bundles of F^n and F^j in M^m, and write

$$\begin{aligned} W(F^n) &= 1 + w_1(F^n) + \cdots + w_n(F^n) = 1 + \theta_1 + \cdots + \theta_n, \\ W(\eta) &= 1 + u_1 + \cdots + u_k, \\ W(F^j) &= 1 + w_1 + \cdots + w_j, \\ W(\mu) &= 1 + v_1 + \cdots + v_l, \end{aligned}$$

for the Stiefel–Whitney classes of F^n, η, F^j, and μ, respectively; here, $m = j + l = n + k$. The following fact from [4] will be needed to prove our result: the projective space bundles $RP(\eta)$ and $RP(\mu)$, with the standard line bundles $\lambda \to RP(\eta)$ and $\nu \to RP(\mu)$, are cobordant as elements of the cobordism group of manifolds with line bundles, $\mathcal{N}_{m-1}(BO(1))$. This implies that any class of dimension $m - 1$, given by a product of the characteristic classes $w_i(RP(\eta))$ and $w_1(\lambda)$, evaluated on the fundamental homology class $[RP(\eta)]$, gives the same characteristic number as the one obtained by the corresponding product of the classes $w_i(RP(\mu))$ and $w_1(\nu)$, evaluated on $[RP(\mu)]$. The key point is the choice of suitable classes; as mentioned in Section 1, this will be made by combining two very special classes. The Stiefel–Whitney classes $W(RP(\eta))$ and $W(RP(\mu))$ were determined in [3, p. 517]: setting $W(\lambda) = 1 + c$, $W(\nu) = 1 + d$, one has

$$W(RP(\eta)) = (1 + \theta_1 + \cdots + \theta_n)\{(1 + c)^k + (1 + c)^{k-1}u_1 + \cdots + (1 + c)u_{k-1} + u_k\},$$

$$W(RP(\mu)) = (1 + w_1 + \cdots + w_j)\{(1 + d)^l + (1 + d)^{l-1}v_1 + \cdots + (1 + d)v_{l-1} + v_l\},$$

where we are suppressing bundle maps.

Now we describe the class $\mathcal{X}$ of Stong and Pergher mentioned in Section 1; this class is associated with line bundles over projective space bundles, hence sometimes we use the notation $\mathcal{X}(\lambda \to RP(\eta))$ to specify the line bundle. For any integer r, one lets

$$W[r] = \frac{W(RP(\eta))}{(1 + c)^{k-r}}.$$

Note that each class $W[r]_j$ is a polynomial in the classes $w_i(RP(\eta))$ and c. Further, these classes satisfy the following special properties (see [9, §2]:

$$W[r]_{2r} = \theta_r c^r + \text{ terms with smaller powers of } c,$$

$$W[r]_{2r+1} = (\theta_{r+1} + u_{r+1})c^r + \text{ terms with smaller powers of } c.$$

Write $n - j = 2^p q$, where $p \geq 1$ and q is odd, and suppose first that $p < q + 1$. In this case, the class $\mathcal{X}$ is

$$\mathcal{X}(\lambda \to RP(\eta)) = W[2^p - 1]^{q+1-p}_{2^{p+1}-1}.W[r_1]_{2r_1}.W[r_2]_{2r_2} \ldots W[r_p]_{2r_p},$$

where $r_i = 2^p - 2^{p-i}$ for $1 \leq i \leq p$. If $p \geq q + 1$, $\mathcal{X}$ is

$$\mathcal{X}(\lambda \to RP(\eta)) = W[r_1]_{2r_1}.W[r_2]_{2r_2} \ldots . W[r_{q+1}]_{2r_{q+1}},$$

where $r_i = 2^p - 2^{p-i}$ for $1 \leq i \leq q + 1$. An easy calculation shows that $\mathcal{X}$ has dimension $m(n - j)$; also, by using the properties of the classes $W[r]_j$ listed above, it can be proved that $\mathcal{X}$ has the form

$$\mathcal{X}(\lambda \to RP(\eta)) = A_t.c^{m(n-j)-t} + \text{ terms with smaller powers of } c,$$

where A_t is a cohomology class of dimension $t \geq n - j + 1$ and comes from the cohomology of F^n (see [9, 11].

The next step is to describe a special class associated with line bundles over closed manifolds, which is well related with the standard class that detects indecomposability. First we recall that R. Thom [15, p. 79] showed that the geometric concept of indecomposability is recognized in the following algebraic way: identify $w_i(F^j)$ with the i-th elementary symmetric function on one-dimensional variables $t_1, t_2, \ldots, t_j$, and next express the symmetric function $t_1^j + t_2^j + \cdots + t_j^j$ as a j-dimensional polynomial $s_j(F^j)$ in $w_i{'s}(F^j)$. Then F^j is indecomposable if and only if the characteristic

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number $s_j(F^j)[F^j]$ is nonzero. Now consider an arbitrary line bundle $\lambda \to N$, where N is a closed $(m-1)$-dimensional manifold, and take the polynomial on degree-one variables $x_1, x_2, \dots, x_{m-1}, c$ given by

$$\mathcal{S}_{2j+1}(x_1, x_2, \dots, x_{m-1}, c) = x_1^j(x_1 + c)^{j+1} + x_2^j(x_2 + c)^{j+1} + \cdots + x_{m-1}^j(x_{m-1} + c)^{j+1}.$$

This polynomial is symmetric in the variables $x_1, x_2, \dots, x_{m-1}$. As before, we then identify $w_1(\lambda)$ with c and each $w_i(N)$ to the i-th elementary symmetric function in the variables $x_1, x_2, \dots, x_{m-1}$; next we express the above polynomial as a polynomial of dimension $2j+1$ in the $w_{i's}(N)$ and $w_1(\lambda)$. This class will be denoted by $\mathcal{S}_{2j+1}(\lambda \to N)$. Our interest is to analyze the behavior of $\mathcal{S}_{2j+1}$ with respect to line bundles over projective space bundles; to do this, we will use the *splitting principle*, which allows writing the Stiefel–Whitney class of any e-dimensional vector bundle ξ formally as

$$W(\xi) = 1 + w_1(\xi) + w_2(\xi) + \cdots + w_e(\xi) = (1 + x_1)(1 + x_2) \cdots (1 + x_e),$$

where each x_i has degree one. Consider an e-dimensional vector bundle $\xi \to Q$, where Q is a closed s-dimensional manifold, and let $\lambda \to RP(\xi)$ be the standard line bundle. Using the splitting principle, write

$$\begin{aligned} W(Q) &= (1 + x_1)(1 + x_2) \cdots (1 + x_s), \\ W(\xi) &= (1 + y_1)(1 + y_2) \cdots (1 + y_e), \end{aligned}$$

and set $w_1(\lambda) = c$. Then

$$W(RP(\xi)) = (1 + x_1)(1 + x_2) \cdots (1 + x_s)(1 + c + y_1)(1 + c + y_2) \cdots (1 + c + y_e).$$

It follows that

$$\begin{aligned} \mathcal{S}_{2j+1}(\lambda \to RP(\xi)) &= x_1^j(x_1 + c)^{j+1} + x_2^j(x_2 + c)^{j+1} + \cdots + x_s^j(x_s + c)^{j+1} \\ &\quad + y_1^{j+1}(y_1 + c)^j + y_2^{j+1}(y_2 + c)^j + \cdots + y_e^{j+1}(y_e + c)^j \\ &= x_1^j\Big(c^{j+1} + \sum_{i=0}^{j} \binom{j+1}{i} x_1^{j+1-i} c^i\Big) \\ &\quad + x_2^j\Big(c^{j+1} + \sum_{i=0}^{j} \binom{j+1}{i} x_2^{j+1-i} c^i\Big) \\ &\quad + \cdots + x_s^j\Big(c^{j+1} + \sum_{i=0}^{j} \binom{j+1}{i} x_s^{j+1-i} c^i\Big) + y_1^{j+1}(y_1 + c)^j \\ &\quad + y_2^{j+1}(y_2 + c)^j + \cdots + y_e^{j+1}(y_e + c)^j \\ &= (x_1^j + x_2^j + \cdots + x_s^j)c^{j+1} + \text{ terms with smaller powers of } c \\ &= s_j(Q)c^{j+1} + \text{ terms with smaller powers of } c. \end{aligned}$$

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Now we return to the line bundles over the projective space bundles coming from the fixed data of (M^m, T). The fact that

$$\mathcal{X}(\lambda \to RP(\eta)) = A_t.c^{m(n-j)-t} + \text{ terms with smaller powers of } c,$$

where A_t is a cohomology class of dimension $t \geq n-j+1$ and comes from the cohomology of F^n, says that each term of $\mathcal{X}(\lambda \to RP(\eta))$ has a factor of dimension at least $n-j+1$ from the cohomology of F^n. On the other hand, the fact that

$$\mathcal{S}_{2j+1}(\lambda \to RP(\eta)) = s_j(F^n)c^{j+1} + \text{ terms with smaller powers of } c,$$

says that every term of $\mathcal{S}_{2j+1}(\lambda \to RP(\eta))$ has a factor of dimension at least j from the cohomology of F^n. In this way, $\mathcal{X}(\lambda \to RP(\eta)).\mathcal{S}_{2j+1}(\lambda \to RP(\eta))$ is a class in $H^{m(n-j)+2j+1}(RP(\eta), Z_2)$ with each one of its terms having a factor of dimension at least $n+1$ from F^n, which means that

$$\mathcal{X}(\lambda \to RP(\eta)).\mathcal{S}_{2j+1}(\lambda \to RP(\eta)) = 0.$$

Suppose by contradiction that $m > m(n-j)+2j+1$. Then $m-1 \geq m(n-j)+2j+1$, and thus it makes sense to consider the class

$$\mathcal{X}(\lambda \to RP(\eta)).\mathcal{S}_{2j+1}(\lambda \to RP(\eta)).c^{m-1-m(n-j)-2j-1} \in H^{m-1}(RP(\eta), Z_2),$$

which yields the zero characteristic number

$$\mathcal{X}(\lambda \to RP(\eta)).\mathcal{S}_{2j+1}(\lambda \to RP(\eta)).c^{m-1-m(n-j)-2j-1}[RP(\eta)].$$

Our next task is to analyse the class associated with $\nu \to RP(\mu)$ and belonging to $H^{m-1}(RP(\mu), Z_2)$ which corresponds to

$$\mathcal{X}(\lambda \to RP(\eta)).\mathcal{S}_{2j+1}(\lambda \to RP(\eta)).c^{m-1-m(n-j)-2j-1}.$$

This class is

$$\mathcal{Y}(\nu \to RP(\mu)).\mathcal{S}_{2j+1}(\nu \to RP(\mu)).d^{m-1-(m(n-2)+2)},$$

where $\mathcal{Y}(\nu \to RP(\mu))$ is obtained from $\mathcal{X}(\lambda \to RP(\eta))$ by replacing each $W[r]_i$ by $W[n+r-j]_i$. One has

$$\begin{aligned}\mathcal{S}_{2j+1}(\nu \to RP(\mu)) &= s_j(F^j)d^{j+1} + \text{ terms with smaller powers of } d\\ &= s_j(F^j)d^{j+1} + \sum A_t d^s,\end{aligned}$$

where $t+s = 2j+1$, $s < j+1$ and A_t comes from F^j. Thus each A_t is zero and $\mathcal{S}_{2j+1}(\nu \to RP(\mu)) = s_j(F^j)d^{j+1}$. This implies that if $\mathscr{I}$ denotes the ideal of

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$H^*(RP(\mu), Z_2)$ generated by the classes coming from F^j and with positive dimension, then $\mathcal{S}_{2j+1}(\nu \to RP(\mu)).\theta = 0$ for each $\theta \in \mathscr{I}$. Thus, in the computation of $\mathcal{Y}$, one needs to consider only that $W(RP(\mu)) \equiv (1+d)^l = (1+d)^{n+k-j} \bmod \mathscr{I}$ and, for each integer t, $W[t] \equiv (1+d)^t \bmod \mathscr{I}$. For $r_i = 2^p - 2^{p-i}$, $i = 1, 2, \dots, p$, set $t_i = n + r_i - 2 = 2^p q + 2 + 2^p - 2^{p-i} - 2 = 2^p q + 2^p - 2^{p-i}$. Then

$$W[t_i]_{2r_i} \equiv \binom{2^p q + 2^p - 2^{p-i}}{2^{p+1} - 2^{p-i+1}} d^{2r_i} \bmod \mathscr{I}.$$

Also, if $r = 2^p - 1$,

$$t = n + r - 2 = 2^p q + 2^p - 1,$$
$$W[t]_{2r+1} \equiv \binom{2^p q + 2^p - 1}{2^{p+1} - 1} d^{2r+1} \bmod \mathscr{I}.$$

The lesser term of the 2-adic expansion of $2^p q + 2^p$ is 2^{p+1}. Using the fact that a binomial coefficient $\binom{a}{b}$ is nonzero modulo 2 if and only if the 2-adic expansion of b is a subset of the 2-adic expansion of a, we conclude that the above binomial coefficients are nonzero modulo 2. It follows that all classes $W[r]_i$ occurring in $\mathcal{Y}$ satisfy $W[r]_i \equiv d^i \bmod \mathscr{I}$, which implies that $\mathcal{Y} \equiv d^{m(n-j)} \bmod \mathscr{I}$. Since from the Leray–Hirsch theorem (see [2, p. 129]) $H^*(RP(\mu), Z_2)$ is the free $H^*(F^j, Z_2)$-module on $1, d, d^2, \dots, d^{n+k-j-1}$, we then have

$$\begin{aligned} &\mathcal{Y}(\nu \to RP(\mu)).\mathcal{S}_{2j+1}(\nu \to RP(\mu)).d^{m-1-(m(n-2)+2)}[RP(\nu)] \\ &\quad = s_j(F^j).d^{m-j-1}[RP(\nu)] = s_j(F^j)[F^j] = 1, \end{aligned}$$

which gives the desired contradiction.

Finally, we describe the almost maximal examples mentioned in Section 1. Take $n \geq 3$ and $j \geq 2$ not of the form $2^t - 1$, with $n > j$. Choose any indecomposable j-dimensional manifold F^j. As remarked in Section 1, Pergher and Stong [9] constructed for each $n \geq 1$ a special involution $(M^{m(n)}, T_n)$ for which the fixed point set has the form $F^n \cup \{\text{point}\}$. Consider the involution $(M^{m(n-j)} \times F^j \times F^j, T)$, where $T(x, y, z) = (T_{n-j}(x), z, y)$. The fixed point set of T has the form

$$(F^{n-j} \cup \{\text{point}\}) \times F^j = (F^{n-j} \times F^j) \cup F^j,$$

which shows that $(M^{m(n-j)} \times F^j \times F^j, T)$ is the desired example.

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