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A NOTE ON UNIVERSALLY ZERO-DIVISOR RINGS

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In this note we consider commutative rings with identity over which every unitary module is a zero-divisor module. We call such rings Universally Zero-divisor (UZD) rings. We show (1) a Noetherian ring R is a UZD if and only if R is semilocal and the Krull dimension of R is at most one, (2) a Prüfer domain R is a UZD if and only if R has only a finite number of maximal ideals, and (3) if a ring R has Noetherian spectrum and descending chain condition on prime ideals then R is a UZD if and only if Spec(R) is a finite set. The question of ascent and descent of the property of a ring being a UZD with respect to integral extension of rings has also been answered.

Introduction

Let R be a commutative ring with identity. Let M be a unitary R-module. Recall that M is said to be a Zero-divisor R-module if for every submodule N of M, $N \neq M$, the set of zero divisors of M/N (that is, $\{x \in R : xm \in N \text{ for some } m \in M \setminus N\}$) denoted by $Z_R(M/N)$ is the union of a finite number of prime ideals of R. R is said to be a Zero-divisor ring (Z.D. ring) if R is a Z.D. R-module [4]. In this note we study the properties of those commutative rings R with identity for which every R-module is a Z.D. R-module.

All rings considered here are assumed to be commutative and with identity. If $A \subseteq B$ are rings we assume that A and B have the same identity element. By dimension of a ring we mean the Krull dimension. Modules are assumed to be unitary. Whenever a set A is a subset of a set B and $A \neq B$ we denote this symbolically as $A \subset B$.

We begin with the following definition.

We say a ring R is a Universally Zero-divisor (UZD) ring if every R-module is a Z.D. R-module.

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Proposition 1. *Let R be a ring. Then R is a UZD if and only if the union of any family of prime ideals of R is the union of a finite number of prime ideals of R (not necessarily belonging to the same family).*

Proof: Assume that R is a UZD. Let $\{P_\alpha\}_{\alpha\in\Lambda}$ be any family of prime ideals of R. Let $M = \bigoplus_{\alpha\in\Lambda} R/P_\alpha$ (that is, direct sum of the R-modules R/P_α), It is easy to see that $Z_R(M) = \bigcup_{\alpha\in\Lambda} P_\alpha$. Since R is a UZD, M is a Z.D. R-module and so $Z_R(M) = \bigcup_{i=1}^{t} Q_i$ for some finite number of prime ideals $Q_1, \ldots, Q_t$ of R. Thus $Z_R(M) = \bigcup_{\alpha\in\Lambda} P_\alpha = \bigcup_{i=1}^{t} Q_i$.

Conversely assume that the union of any family of prime ideals of R is the union of a finite number of prime ideals of R. Let M be any R-module. Let N be a submodule of M, $N \neq M$. Notice that $R \setminus Z_R(M/N)$ is a saturated multiplicatively closed subset of R. Hence by [**1**, Exercise 7 (i), p.44] $Z_R(M/N)$ is a union of prime ideals of R. By assumption it follows that $Z_R(M/N)$ is the union of a finite number of prime ideals of R. Thus M is a Z.D. R-module. Hence we obtain that R is a UZD. ∎

Remark 2. Using the above Proposition we see that R is a UZD implies that any homomorphic image of R is a UZD and $S^{-1}R$ is a UZD for every multiplicatively closed subset S of R, $S \subseteq R \setminus \{0\}$.

Proposition 3. *Let R be an integral domain with quotient field K. Then R is a UZD if and only if K is a Z.D. R-module.*

Proof: The "only if" part is clear. The "if" part follows from [**13**, Remark 2.1] and Proposition 1. ∎

Proposition 4.

(i) *Let R be a Noetherian ring. Then R is a UZD if and only if R is semilocal and the dimension of R is at most 1.*

(ii) *A Prüfer domain R is a UZD if and only if R has only a finite number of maximal ideals.*

The proof of Proposition 4 makes use of the following results.

Lemma 5. *If a ring R is a UZD then R has only a finite number of maximal ideals.*

Proof: Let $\{M_\alpha\}_{\alpha\in\Lambda}$ be the family of all maximal ideals of R. By Proposition 1, $\bigcup_{\alpha\in\Lambda} M_\alpha = \bigcup_{i=1}^{s} Q_i$ for some finite number of prime ideals $Q_1, \ldots, Q_s$ of R. Let M_i $(i = 1, \ldots, s)$ be maximal ideals of R such that $Q_i \subseteq M_i$ (for $i = 1, \ldots, s$). Then it is clear that $\bigcup_{\alpha\in\Lambda} M_\alpha = \bigcup_{i=1}^{s} Q_i = \bigcup_{i=1}^{s} M_i$. It is now evident that distinct elements among $M_1, \ldots, M_s$ are all the maximal ideals of R. ∎

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RESULT 6. In a Noetherian ring every prime ideal has finite height [**1**, Corollary 11.12].

RESULT 7. In a Noetherian ring any prime ideal of height 2 contains an infinite number of height 1 prime ideals [**11**, Theorem 144].

RESULT 8. Let I be any ideal of a Noetherian ring R, $I \neq R$. Then the set of prime ideals of R which are minimal over I is finite.

Result 8 follows by applying [**1**, Exercise 9, p.79] to the Noetherian ring R/I.

PROOF OF PROPOSITION 4: (i) Assume that R is a Noetherian ring and R is a UZD. By Lemma 5, R is semilocal. We prove that the dimension of R is at most 1. Suppose that the dimension of R is at least 2. Then by Result 6 it follows that there exists a prime ideal p of R such that height $p = 2$.

Let $\{Q_\alpha\}_{\alpha\in\Lambda}$ be the set of all height one prime ideals of R_p. Note that $\{Q_\alpha\}_{\alpha\in\Lambda} = \{P_\alpha R_p\}_{\alpha\in\Lambda}$ where $\{P_\alpha\}_{\alpha\in\Lambda}$ are prime ideals of R such that height $P_\alpha = 1$ and $P_\alpha \subset p$ for each $\alpha \in \Lambda$. By Result 7 it follows that Λ is an infinite set. Result 8 and [**2**, Exercise 2, p.121] imply that there exists an element $y \in pR_p$ which is not in any of the minimal prime ideals of R_p. Now Result 8 implies that y can belong to only a finite number of height 1 prime ideals of R_p. Let them be $\{P_{\alpha_i} R_p\}_{i=1}^{t}$. Let $A = \Lambda \setminus \{\alpha_1, \ldots, \alpha_t\}$. Then it is easy to see that $\bigcup_{\alpha\in A} P_\alpha R_p$ cannot be equal to the union of any finite number of prime ideals of R_p. This is in contradiction to the fact that R_p is a UZD. Thus R is semilocal and the dimension of R is at most 1.

Conversely if R is semilocal and the dimension of R is at most 1 then any prime ideal of R is either a maximal ideal of R or a minimal prime ideal of R. Since the set of minimal prime ideals of a Noetherian ring is finite we obtain that R has only a finite number of prime ideals. It is then clear that R is a UZD.

(ii) In view of Lemma 5, we need only prove the "if part" of (ii). Assume that R is a Prüfer domain with only a finite number of maximal ideals $M_1, \ldots, M_t$. Let $\{P_\alpha\}_{\alpha\in\Lambda}$ be any family of prime ideals of R. Let C_i be the union of those P_α's which are contained in M_i (for $i = 1, \ldots, t$). Now R_{M_i} is a valuation ring and so in the case $C_i \neq \emptyset$, C_i is the union of some pairwise comparable prime ideals of R and hence $C_i \in \operatorname{Spec}(R)$. This is true for $i = 1, \ldots, t$. Further it is clear that $\bigcup_{\alpha\in\Lambda} P_\alpha = \bigcup_{i=1}^{t} C_i$. Hence by Proposition 1, R is a UZD. ∎

REMARK 9. We have noted in Proposition 3 that an integral domain R is a UZD if and only if the quotient field of R is a Z.D. R-module. We now mention an example which shows (for an arbitrary ring R) that "the total quotient ring of R is a Z.D. R-module" need not imply that R is a UZD. Consider $T = \mathbf{Q}(\sqrt{2})\,[[X, Y, Z]]$, the power series ring in three indeterminates X, Y, Z over $\mathbf{Q}(\sqrt{2})$ where $\mathbf{Q}$ denotes the field of rationals. Let M denote the unique maximal ideal of T. Let $S = \mathbf{Q} + M$. Notice that

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the dimension of T is 3 and T is a finite integral extension of S. Hence by [**6**, 11.8, p.106] and [**3**, Theorem 2] it follows that the dimension of S is 3 and S is Noetherian. Consider the chain of prime ideals $(0) \subset P_1 \subset P_2 \subset M$ of S where $P_1 = XT$ and $P_2 = XT + YT$. Let $R = S/(XS)$. We now show that the unique maximal ideal $M/(XS)$ of R is full of zero divisors. For an element $m \in M$, let $m + XS \in M/(XS)$. Note that $(m + XS)(\sqrt{2}X + XS) = \sqrt{2}(mX) + XS = X(\sqrt{2}m) + XS = XS$ since $\sqrt{2}m \in M \subset S$. But $\sqrt{2}X \notin XS$. For if $\sqrt{2}X \in XS$ then we obtain $\sqrt{2} \in S$ which in turn implies that $\sqrt{2} = q + y$ for some $q \in \mathbb{Q}$, $y \in M$. This implies that $\sqrt{2} - q = y \in \mathbb{Q}(\sqrt{2}) \cap M = (0)$ and so $\sqrt{2} = q \in \mathbb{Q}$ which is not true. Thus $\sqrt{2}X \notin XS$. This proves that $M/(XS)$ is full of zero divisors. Hence R equals the total quotient ring of R. Since R is a Noetherian ring, R is a Z.D. R-module. Since the dimension of R is 2, it follows from Proposition 4 (i) that R is not a UZD.

PROPOSITION 10. *Let R be a ring with Noetherian spectrum and descending chain condition on prime ideals. Then R is a UZD if and only if* Spec(R) *is a finite set.*

PROOF: Assume that R has Noetherian spectrum and has descending chain condition on prime ideals and R is a UZD. The argument that we shall give below to show that Spec(R) is a finite set closely follows an argument of Heinzer and Lantz [**10**, Proposition 3.7]. By Lemma 5, R has only a finite number of maximal ideals say $M_1, \ldots, M_t$. Let, if possible, Spec(R) be an infinite set. Then Spec(R_{M_i}) is an infinite set for some $i \in \{1, \ldots, t\}$. Now R_{M_i} has Noetherian spectrum and so $M_iR_{M_i} = \sqrt{(y_1, \ldots, y_h)R_{M_i}}$ for some $y_j \in M_iR_{M_i}$ $(j = 1, \ldots, h)$ [**12**, Corollary 2.4]. It is then clear that Spec$(R_{M_i}[1/y_j])$ is an infinite set for some $j \in \{1, \ldots, h\}$. Since $R_{M_i}[1/(y_j)]$ is a UZD, it has only a finite number of maximal ideals say $N_1, \ldots N_s$. Note that each N_g $(g = 1, \ldots, s)$ is of the form $Q_gR_{M_i}[1/y_j]$ for some prime ideal $Q_gR_{M_i}$ of R_{M_i} such that $Q_gR_{M_i} \subset M_iR_{M_i}$. Notice that Spec$\left((R_{M_i}[1/y_j])_{N_g}\right)$ is an infinite set for some $g \in \{1, \ldots, s\}$. Further observe that $M_i \supset Q_g$. Now $(R_{M_i}[1/y_j])_{N_g} \simeq R_{Q_g}$ by [**2**, Proposition 11 (iii), p.70] and thus Spec(R_{Q_g}) is an infinite set and R_{Q_g} has Noetherian spectrum and is a UZD. Hence applying the above argument to the ring R_{Q_g} yields $H \in$ Spec(R) such that $Q_g \supset H$ and Spec(R_H) is infinite. So by repeating the above procedure we obtain a strictly descending sequence of prime ideals of R. This is in contradiction to the assumption that R has descending chain condition on prime ideals. Therefore Spec(R) is a finite set. ∎

The converse is obvious.

REMARK 11. (i) We mention an example to show that the hypothesis in Proposition 10 that R has Noetherian spectrum cannot be dropped. There exists a valuation ring V such that the set of prime ideals of V forms an infinite ascending chain $(0) \subset P_1 \subset$

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$P_2 \subset \cdots \subset M = \bigcup_{i=1}^{\infty} P_i$ [5, Example 5, p.578]. Thus $\operatorname{Spec}(V)$ is an infinite set but by Proposition 4 (ii), V is a UZD. Further, note that V has descending chain condition on prime ideals.

(ii) We now mention an example to show that the hypothesis in Proposition 10 that R has descending chain condition on prime ideals cannot be dropped.

Let F be a field and $\{X_i\}_{i=1}^{\infty}$ be a set of elements algebraically independent over F. Let $K = F(\{X_i\}_{i=1}^{\infty})$. Let G be the direct sum of countably many copies of $\mathbb{Z}$, the additive group of integers. We order G with reverse lexicographic ordering. Then there exists a valuation ring W on K with value group G by [7, Example 2.6]. It is easy to verify that the set of all prime ideals of W forms an infinite descending chain $M \supset P_1 \supset P_2 \supset \ldots$. But W is a UZD and W has Noetherian spectrum.

Next we consider the ascent and descent of UZD with respect to integral extension of rings.

PROPOSITION 12. *(i) Let $R \subset T$ be rings. Let T be integral over R. If T is a UZD then R is a UZD.*

(ii) Let B be a finite integral extension ring of a ring A. If B has finitely many minimal prime ideals and if A is a UZD then B is a UZD.

PROOF: (i) Let $\{P_\alpha\}_{\alpha\in\Lambda}$ be any family of prime ideals of R. Now for each P_α, there exists $Q_\alpha \in \operatorname{Spec}(T)$ such that $Q_\alpha \cap R = P_\alpha$ by [1, Theorem 5.10]. Since T is UZD, by Proposition 1, $\bigcup_{\alpha\in\Lambda} Q_\alpha = \bigcup_{i=1}^{s} H_i$ for some $H_i \in \operatorname{Spec}(T)$ $(i = 1, \ldots, s)$. Now it follows that $\bigcup_{\alpha\in\Lambda} P_\alpha = \bigcup_{\alpha\in\Lambda} (Q_\alpha \cap R) = \bigcup_{i=1}^{s} (H_i \cap R)$. Hence R is a UZD.

(ii) By hypothesis B has only a finite number of minimal prime ideals, say $Q_1, \ldots, Q_t$. Notice that each B/Q_i $(i = 1, \ldots, t)$ is a finite integral extension of $A/(Q_i \cap A)$ and $A/(Q_i \cap A)$ is a UZD (for $i = 1, \ldots, t$). We prove that B/Q_i is a UZD for each $i \in \{1, \ldots, t\}$. Then it will follow that B is a UZD. Hence it suffices to prove (ii) in the case in which B is an integral domain. Let K denote the quotient field of B. Let X be an indeterminate over K. Consider $V = K[[X]] = K + M$ where $M = XK[[X]]$. Let $B_1 = B + M$; $A_1 = A + M$. Since B is a finite integral extension of A, it follows that B_1 is a finite integral extension of A_1. As A is a UZD, A_1 is a Z.D. ring by [13, Remark 2.1]. Hence B_1 is a Z.D. ring by [9, Theorem 2.9]. Again by [13, Remark 2.1], B is a UZD. This completes the proof of (ii). ∎

REMARK 13. We mention an example to show that Proposition 12 (ii) does not extend to infinite integral extensions. Gilmer and Huckaba in [8, Example p.211] have constructed for a fixed prime p an infinite algebraic extension L of the field of rationals $\mathbb{Q}$ such that the integral closure $\overline{Z_p}$ of Z_p in L has an infinite number of maximal ideals.

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Since Z_p is a 1-dimensional quasilocal domain it is clear that Z_p is a UZD. As $\overline{Z_p}$ has an infinite number of maximal ideals, $\overline{Z_p}$ is not a UZD.

We conclude this note with the following Proposition which determines when every overring of an integral domain is a UZD.

Proposition 14. *Let R be an integral domain with quotient field K. Then each overring of R is a UZD if and only if the integral closure of R in K is a Prüfer domain with only finitely many maximal ideals.*

Proof: $(\Rightarrow)$ Let $\overline{R}$ denote the integral closure of R in K. Let $\mathrm{Q} \in \mathrm{Spec}\,(\overline{R})$. Let $\alpha \in K$, $\alpha \neq 0$. Let X be an indeterminate over $\overline{R}_{\mathrm{Q}}$. Let g denote the $\overline{R}_{\mathrm{Q}}$ homomorphism from $\overline{R}_{\mathrm{Q}}[X]$ to $\overline{R}_{\mathrm{Q}}[\alpha]$ determined by $g(X) = \alpha$. Now $\overline{R}_{\mathrm{Q}}[\alpha]$ is a UZD and hence it has only a finite number of maximal ideals. We assert that $\ker g \not\subseteq \mathrm{Q}\overline{R}_{\mathrm{Q}}[X]$. For if $\ker g \subseteq \mathrm{Q}\overline{R}_{\mathrm{Q}}[X]$ then $(\overline{R}_{\mathrm{Q}}[X])/(\mathrm{Q}\overline{R}_{\mathrm{Q}}[X]) \simeq (\overline{R}_{\mathrm{Q}})/(\mathrm{Q}\overline{R}_{\mathrm{Q}})[X]$ becomes a homomorphic image of $\overline{R}_{\mathrm{Q}}[\alpha]$ which would force $(\overline{R}_{\mathrm{Q}})/(\mathrm{Q}\overline{R}_{\mathrm{Q}})[X]$ to have only a finite number of maximal ideals, a contradiction. Hence $\ker g \not\subseteq \mathrm{Q}\overline{R}_{\mathrm{Q}}[X]$. So by [**6**, Lemma 19.14] either α or α^{-1} is in $\overline{R}_{\mathrm{Q}}$. Thus $\overline{R}_{\mathrm{Q}}$ is a valuation ring for each $\mathrm{Q} \in \mathrm{Spec}\,(\overline{R})$. Hence $\overline{R}$ is a Prüfer domain. Since $\overline{R}$ is a UZD, $\overline{R}$ has only a finite number of maximal ideals.

$(\Leftarrow)$ Let A be any overring of R. Let $\overline{A}$ denote the integral closure of A in K. Then $\overline{A}$ is a Prüfer domain with only a finite number of maximal ideals by [**6**, Theorem 26.1 (a) and Exercise 14, p. 331]. So $\overline{A}$ is a UZD by Proposition 4(ii). Now Proposition 12 (i) implies that A is a UZD. $\square$

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