Compositio Mathematica **110:** 263–276, 1998.

The second pluri-genus of surface singularities

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Received: 3 June 1996; accepted in final form 21 October 1996

Abstract. This paper studies the second pluri-genus of surface singularities. We give a formula for this invariant of a Gorenstein singularity, and several inequalities relating the invariant with the Milnor number, Tjurina number and the modality of a hypersurface singularity.

Mathematics Subject Classifications: 1991 Primary 32S10; Secondary 14J17, 32S30, 32S45.

Key words: surface singularity, pluri-genus, deformation, Milnor number, Tjurina number, modality.

Introduction

Let (X, x) be a normal surface singularity over $\mathbb{C}$ and $f : (M, E) \to (X, x)$ the minimal good resolution of the singularity (X, x), i.e., the smallest resolution for which an exceptional divisor E consists of non-singular curves intersecting transversally, with no three through one point. The geometric genus of the singularity (X, x) is defined by $p_g(X, x) = \dim_{\mathbb{C}} H^1(\mathcal{O}_M)$. Watanabe [24] introduced pluri-genera $\{\delta_m(X, x)\}_{m\in\mathbb{N}}$ (for $n(\geqslant 2)$-dimensional normal isolated singularities) which carry more precise information of the singularity. It is well-known that, for a normal surface singularity $(X, x), \delta_m(X, x) = 0$ for any $m \in \mathbb{N}$ if and only if (X, x) is a log-terminal singularity (quotient singularity), and $\delta_m(X, x) \leqslant 1$ for any $m \in \mathbb{N}$ if and only if (X, x) is a log-canonical singularity (see [8]).

In this paper we study the second pluri-genus of certain normal surface singularities, so ‘a singularity’ always means a normal surface singularity over $\mathbb{C}$.

In the first section, we summarize notations, definitions and basic facts which will be used in this paper.

In the second section, we will show that $\delta_2(X, x)$ is determined by $p_g(X, x)$ and the weighted dual graph of (X, x), and $\delta_2(X, x) \geqslant \dim_{\mathbb{C}} H^1(\Theta_E)$ holds for certain singularities.

In the last section, we consider relations among the invariants δ_2, p_g, μ, τ and the modality.

1. Preliminaries

A. *Basic facts on singularities*

(1.1) Let (X, x) be a surface singularity and $f\colon (M, E) \to (X, x)$ a minimal good resolution of the singularity (X, x). It is well-known that there is a unique minimal good resolution. Let $E = \bigcup_{i=1}^{k} E_i$ be the decomposition of the exceptional set E into irreducible components. A cycle D is an integral combination of the E_i, i.e., $D = \sum_{i=1}^{k} d_i E_i$ with $d_i \in \mathbb{Z}$. There is a natural partial ordering between cycles defined by comparing the coefficients. A cycle D is said to be positive if $D \geqslant 0$ and $D \neq 0$. For any two positive cycles V and W, there is an exact sequence

$$0 \to \mathcal{O}_W \bigotimes_{\mathcal{O}_M} \mathcal{O}_M(-V) \to \mathcal{O}_{V+W} \to \mathcal{O}_V \to 0. \tag{1.1.1}$$

The weighted dual graph of (X, x) is the information of the genera of the E_i and the intersection matrix $(E_i \cdot E_j)$, or the graph such that each vertex of which represents a component of E weighted by its intersection number and each edge corresponds to an intersection point of the components (cf. [9]). A component E_i of E is called a central curve if which has positive genus or intersects more than two other components. The weighted dual graph of a singularity is said to be star-shaped, if it is a tree as the graph where at most one vertex is the central curve. The connected components of a star-shaped graph minus the central curve are called the branches.

NOTATION 1.2. Let Y be a normal variety over $\mathbb{C}$, $\mathcal{M}$ a sheaf of $\mathcal{O}_Y$-modules, D a divisor on Y and F a closed subset of Y. We use the following notation

$$\mathcal{M}(D) = \mathcal{M} \bigotimes_{\mathcal{O}_Y} \mathcal{O}_Y(D),$$

$$H^i(\mathcal{M}) = H^i(Y, \mathcal{M}), \qquad H^i_F(\mathcal{M}) = H^i_F(Y, \mathcal{M}),$$

$$h^i(\mathcal{M}) = \dim_{\mathbb{C}} H^i(\mathcal{M}), \qquad h^i_F(\mathcal{M}) = \dim_{\mathbb{C}} H^i_F(\mathcal{M}).$$

We denote by K the canonical divisor on M.

DEFINITION 1.3. (cf. [24]) We define the pluri-genera $\{\delta_m(X, x)\}_{m \in \mathbb{N}}$ as follows

$$\delta_m(X, x) = \dim_{\mathbb{C}} \frac{H^0(\mathcal{O}_{M-E}(mK))}{H^0(\mathcal{O}_M(mK + (m-1)E))}.$$

Note that $\delta_1(X, x) = p_g(X, x)$.

(1.4) We take the following characterization of Du Bois singularity as its definition.

PROPOSITION 1.5. (Steenbrink [17, (3.6)]). *A normal surface singularity* (X, x) *is a Du Bois singularity if and only if the natural map* $H^1(\mathcal{O}_M) \to H^1(\mathcal{O}_E)$ *is an isomorphism.*

THEOREM 1.6. (Steenbrink [17, (3.7), (3.8)]. cf. [6, Th. 2.3]). (1) *If* (X, x) *is a rational singularity, then* (X, x) *is Du Bois.*

(2) *Let* (X, x) *be a Gorenstein singularity. Then* (X, x) *is a Du Bois singularity if and only if it is a rational double point, a simple elliptic or a cusp singularity.*

THEOREM 1.7. (Ishii [7, Theorem 2.3]). *Every resolution of a Du Bois singularity is a good resolution, where a good resolution means a resolution of the singularity for which the exceptional divisor is of normal crossings.*

(1.8) Ishii [7] noted that there exist Du Bois singularities with arbitrarily large geometric genus.

B. *Deformations*

(1.9) We use the notation above. We denote by D_X the functor (on artin rings) of deformations of a singularity (X, x). In [20], Wahl introduced the equisingular functor ES_M of deformations of (M, E) to which all E_i lift, and which blow down to deformations of (X, x). It is well-known that a deformations of M blows down if and only if $h^1(\mathcal{O}_M)$ does not jump (cf. [20, (4.3)]). Hence equisingular deformations preserve the geometric genera and the weighted dual graphs of singularities.

In [10, 11, 12], Laufer studied deformations of M in the analytic category. For a Gorenstein singularity (X, x), an equisingular deformation of (M, E) induces a topologically constant deformation of (X, x), and the converse holds, too (see [12, V, VI]).

(1.10) Let $\Omega^1_M\langle E\rangle$ be the sheaf of 1-forms with logarithmic poles along E, and $\mathcal{S}$ its dual. Then there are exact sequences (cf. [22])

$$0 \to \Omega^1_M \to \Omega^1_M\langle E\rangle \to \bigoplus_{i=1}^{k} \mathcal{O}_{E_i} \to 0; \tag{1.10.1}$$

$$0 \to \mathcal{S} \to \Theta_M \to \bigoplus_{i=1}^{k} \mathcal{O}_{E_i}(E_i) \to 0; \tag{1.10.2}$$

$$0 \to \Theta_M(-E) \to \mathcal{S} \to \Theta_E \to 0. \tag{1.10.3}$$

By (1.10.2), we have the following exact sequence

$$0 \to H^1(\mathcal{S}) \to H^1(\Theta_M) \to H^1\left(\bigoplus_{i=1}^{k} \mathcal{O}_{E_i}(E_i)\right) \to 0.$$

https://doi.org/10.1023/A:1000265619270 Published online by Cambridge University Press

There is a versal deformation $\pi: \overline{M} \to (Q, o)$ of (M, E) with tangent space $T_{Q,o} \cong H^1(\Theta_M)$, and a submanifold (P, o) with tangent space $T_{P,o} \cong H^1(\mathcal{S})$ such that all of the E_i lift to above P (cf. [10, 11]).

THEOREM 1.11. (Wahl [20]). (1) *ES_M is smooth and the natural map $ES_M \to D_X$ is injective.*

(2) *If any deformation of (M, E) to which all E_i lift blows down to a deformation of (X, x), then $T(ES_M) = H^1(\mathcal{S})$, where $T(ES_M)$ denotes the tangent space of ES_M. If $p_g(X, x) \leqslant 1$, then this condition is satisfied.*

(1.12) A function $h \in \mathbb{C}\{z_1, \dots, z_n\} = B$ is called a quasi-homogeneous polynomial of degree d with weights $(\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, if $h(t^{\alpha_1} z_1, \dots, t^{\alpha_n} z_n) = t^d h(z_1, \dots, z_n)$ for any $t \in \mathbb{C}$. We assume that $GCD(\alpha_1, \dots, \alpha_n) = 1$. A singularity (X, x) is said to be quasi-homogeneous if (X, x) is defined by quasi-homogeneous polynomials with common weights. Let (X, x) be a quasi-homogeneous singularity defined by an ideal $I \subset B$. Let us recall that the tangent space T_X^1 of D_X is given by the exact sequence

$$\mathrm{Hom}_A(\Omega_B^1 \bigotimes A, A) \to \mathrm{Hom}_A(I/I^2, A) \to T_X^1 \to 0,$$

where $A = B/I$. Since $\mathrm{Hom}_A(I/I^2, A)$ is graded, so is T_X^1: we write $T_X^1 = \bigoplus_{i \in \mathbb{Z}} T_X^1(i)$.

THEOREM 1.13. (Pinkham [16, 4.6]). $T(ES_M) = \bigoplus_{i \geqslant 0} T_X^1(i)$.

2. The second pluri-genus

(2.1) We use the same notation as in the first section. Let $f\ (M, E) \to (X, x)$ be a minimal good resolution, except in Lemma 2.3.

The following theorem will be proved in (3.2).

THEOREM 2.2. *Let (X, x) be a Gorenstein singularity which is not a rational double point. Then*

$$H^1(\mathcal{O}_M(2K + E)) = 0.$$

LEMMA 2.3. *Let $f\colon (M, E) \to (X, x)$ be a minimal resolution of the singularity (X, x), i.e., $K \cdot E_i \geqslant 0$ for all i. If (X, x) is not a rational double point, then*

$$H^1(\mathcal{O}_M(2K + E)) = 0.$$

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Proof of Lemma 2.3. There is an exact sequence

$$0 \to \mathcal{O}_M(2K) \to \mathcal{O}_M(2K+E) \to \mathcal{O}_E(2K+E) \to 0.$$

Since K is nef, $H^1(\mathcal{O}_M(2K)) = 0$, and hence $H^1(\mathcal{O}_M(2K+E)) \cong H^1(\mathcal{O}_E(2K+E))$. By duality, $h^1(\mathcal{O}_E(2K+E)) = h^0(\mathcal{O}_E(-K))$. We will show that $H^0(\mathcal{O}_E(-K)) = 0$. By assumption, (X,x) is not a rational double point. Hence we may assume that $K \cdot E_1 > 0$. Let $\{Z_i\}_{i=0,1,\ldots,k}$ be a computation sequence for E: $Z_0 = 0, Z_1 = E_1 = E_{i_1}, \ldots, Z_j = Z_{j-1} + E_{i_j}, \ldots, Z_k = Z_{k-1} + E_{i_k} = E$, where $Z_{j-1} \cdot E_{i_j} > 0$ for $j = 2, \ldots, k$. Since $(-K - Z_{j-1}) \cdot E_{i_j} < 0$ for $j = 1, \ldots, k$, it follows that $H^0(\mathcal{O}_{E_{i_j}}(-K - Z_{j-1})) = 0$ for $j = 1, \ldots, k$. From the following exact sequences (cf. (1.1.1))

$$0 \to \mathcal{O}_{E_{i_j}}(-K - Z_{j-1}) \to \mathcal{O}_{Z_j}(-K) \to \mathcal{O}_{Z_{j-1}}(-K) \to 0,$$

we have inductively that $H^0(\mathcal{O}_{Z_j}(-K)) = 0$ for $j = 1, \ldots, k$. We have thus proved the lemma. □

THEOREM 2.4. *Let (X,x) be a Du Bois singularity which is not a rational double point. Then*

$$H^1(\mathcal{O}_M(2K+E)) = 0.$$

Proof. Let $g\colon (M_1, F_1) \to (X,x)$ be the minimal resolution. Then the exceptional divisor F_1 is of normal crossings and $H^1(\mathcal{O}_{M_1}(2K_{M_1} + F_1)) = 0$ by Theorem 1.7 and Lemma 2.3. Let $\pi : (M_2, F_2) \to (M_1, F_1)$ be the blow-up of a double point w of F_1, and $C = \pi^{-1}(w)$. We have $\mathcal{O}_{M_2}(2K_{M_2} + F_2) = \pi^*\mathcal{O}_{M_1}(2K_{M_1} + F_1) \otimes \mathcal{O}_{M_2}(C)$. By the projection formula,

$$R^i\pi_*\mathcal{O}_{M_2}(2K_{M_2} + F_2) \cong R^i\pi_*\mathcal{O}_{M_2}(C) \bigotimes \mathcal{O}_{M_1}(2K_{M_1} + F_1).$$

From the following spectral sequence

$$E_2^{p,q} = H^p(R^q\pi_*\mathcal{O}_{M_2}(2K_{M_2} + F_2)) \Rightarrow H^{p+q}(\mathcal{O}_{M_2}(2K_{M_2} + F_2)),$$

we have an exact sequence

$$\begin{aligned} 0 &\to H^1(\pi_*\mathcal{O}_{M_2}(C) \bigotimes \mathcal{O}_{M_1}(2K_{M_1} + F_1)) \\ &\to H^1(\mathcal{O}_{M_2}(2K_{M_2} + F_2)) \\ &\to H^0(R^1\pi_*\mathcal{O}_{M_2}(C) \bigotimes \mathcal{O}_{M_1}(2K_{M_1} + F_1)). \end{aligned}$$

From the exact sequence

$$0 \to \mathcal{O}_{M_2} \to \mathcal{O}_{M_2}(C) \to \mathcal{O}_C(C) \to 0,$$

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we get $\pi_*\mathcal{O}_{M_2}(C) \cong \mathcal{O}_{M_1}$ and $R^1\pi_*\mathcal{O}_{M_2}(C) = 0$. Hence $H^1(\mathcal{O}_{M_2}(2K_{M_2} + F_2)) \cong H^1(\mathcal{O}_{M_1}(2K_{M_1} + F_1)) = 0$.

Since M is obtained by resolving the double points of the irreducible components of the exceptional set, applying the argument above, if it is needed, we have $H^1(\mathcal{O}_M(2K + E)) = 0$. □

COROLLARY 2.5. *Let* (X, x) *be a Gorenstein or a Du Bois singularity. Then*

$$\delta_2(X, x) = h^1_E(\mathcal{O}_M(2K + E)) = h^1(\mathcal{O}_M(-K - E)).$$

Proof. By duality, $h^1_E(\mathcal{O}_M(2K + E)) = h^1(\mathcal{O}_M(-K - E))$. If (X, x) is a rational double point, then $h^1(\mathcal{O}_M(-K - E)) = h^1(\mathcal{O}_M(-E)) = 0$ (since (X, x) is a Du Bois singularity), and $\delta_2(X, x) = 0$ (cf. Introduction). If (X, x) is not a rational double point, using the theorems above, we have an exact sequence

$$0 \to H^0(\mathcal{O}_M(2K + E)) \to H^0(\mathcal{O}_{M-E}(2K))$$

$$\to H^1_E(\mathcal{O}_M(2K + E)) \to 0.$$

By definition, $\delta_2(X, x) = h^1_E(\mathcal{O}_M(2K + E))$. □

COROLLARY 2.6. *If* (X, x) *is a Gorenstein singularity with* $p_g(X, x) \geqslant 1$, *then*

$$\delta_2(X, x) = p_g(X, x) - \tfrac{1}{2}(2K + E) \cdot (K + E).$$

Proof. Using $h^1(\mathcal{O}_M(2K + E)) = 0$, the theorem of Riemann–Roch (e.g., [15]) implies the result. □

COROLLARY 2.7. [25]. *Let* (X, x) *be a hypersurface singularity with* $p_g(X, x) = 1$. *Then* $\delta_2(X, x) \leqslant 4$.

Proof. If (X, x) is a Du Bois singularity (see Theorem 1.6), we may assume $K = -E$. Then $\delta_2(X, x) = 1$ by Corollary 2.6.

We assume that (X, x) is not a Du Bois singularity. Then $H^1(\mathcal{O}_E) = 0$ by Proposition 1.5. Hence $-E \cdot (K + E)/2 = \chi(\mathcal{O}_E) = 1$. Then we have that $\delta_2(X, x) = 2 - K \cdot (K + E)$ by Corollary 2.6.

If $f : (M, E) \to (X, x)$ is not minimal, then by [9, Prop. 3.5], we have the star-shaped graph which consists of four rational curves, such that the self-intersection number of the central curve E_1 is -1. Then we obtain that $K = -2E_1 - E_2 - E_3 - E_4$ and $K \cdot (K + E) = 1$. Hence $\delta_2(X, x) = 1$.

If $f : (M, E) \to (X, x)$ is minimal, then by [9, Th. 3.4, Th. 3.13], we get $K \cdot K \geqslant -3$. Since $K \cdot E > 0$, we have $\delta_2(X, x) = 2 - K \cdot K - K \cdot E \leqslant 4$. □

Remark 2.8. In exactly the same way as above, we can prove the following: *If* (X, x) *is a complete intersection singularity with* $p_g(X, x) = 1$, *then* $\delta_2(X, x) \leqslant 5$.

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COROLLARY 2.9. *Let (X, x) be a Gorenstein or a Du Bois singularity. Then*

$$\delta_2(X, x) \geqslant h^1(\Theta_E).$$

Proof. For a locally free sheaf $\mathcal{F}$ of rank 2 on M, $\mathcal{F} \cong \mathrm{Hom}_{\mathcal{O}_M}(\mathcal{F}, \mathcal{O}_M) \otimes_{\mathcal{O}_M} \bigwedge^2 \mathcal{F}$. Hence we get isomorphisms

$$\Theta_M(-E) \cong \Omega^1_M(-K-E) \quad \text{and} \quad \mathcal{S} \cong \Omega^1_M\langle E\rangle(-K-E).$$

Then the exact sequences (1.10.1) and (1.10.3) give

$$h^1(\Theta_E) \cong h^1\left(\bigoplus_{i=1}^{k} \mathcal{O}_{E_i}(-K-E)\right). \tag{2.9.1}$$

From the following exact sequence (cf. [17, (1.5)])

$$0 \to \mathcal{O}_E \to \bigoplus_{i=1}^{k} \mathcal{O}_{E_i} \to \bigoplus_{i<j} \mathcal{O}_{E_i \cap E_j} \to 0,$$

we have a surjective map

$$H^1(\mathcal{O}_E(-K-E)) \to H^1\left(\bigoplus_{i=1}^{k} \mathcal{O}_{E_i}(-K-E)\right).$$

By Corollary 2.5 and (2.9.1), we get

$$\delta_2(X, x) \geqslant h^1(\mathcal{O}_E(-K-E)) \geqslant h^1(\Theta_E).$$ □

3. Complete intersections

(3.1) We use the same notation as in the first section. Let (X, x) be a Gorenstein singularity with contractible X. Let Z be a cycle such that $\mathcal{O}_M(K) \cong \mathcal{O}_M(-Z)$. If (X, x) is not a rational double point, then $Z \geqslant E$.

Let $\mathcal{C}$ be the sheaf on M defined by an exact sequence

$$0 \to \mathcal{C} \to \mathbb{C}_M \to \mathbb{C}_E \to 0.$$

If $Z \geqslant E$, then the exterior differentiation gives an exact sequence (cf. [22, (1.5), (1.6)])

$$0 \to \mathcal{C} \to \mathcal{O}_M(-Z) \xrightarrow{d} \Omega^1_M\langle E\rangle(-Z) \xrightarrow{d} \Omega^2_M(-Z+E) \to 0. \tag{3.1.1}$$

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As X is contractible, $H^i(\mathcal{C}) = 0$ for all i. Hence $H^i(\mathcal{O}_M(-Z)) \cong H^i(d\mathcal{O}_M(-Z))$ for all i. In particular, $H^i(d\mathcal{O}_M(-Z)) \cong H^i(\mathcal{O}_M(K)) = 0$ for $i \geqslant 1$.

(3.2) *Proof of Theorem* 2.2. From (3.1.1), we have an exact sequence

$$\begin{aligned} H^1(\Omega^1_M\langle E\rangle(-Z)) &\to H^1(\Omega^2_M(-Z+E)) \\ &\to H^2(d\mathcal{O}_M(-Z)) = 0. \end{aligned}$$

By Wahl's vanishing theorem [19], $H^1(\Omega^1_M\langle E\rangle(-Z)) = 0$. Hence

$$H^1(\mathcal{O}_M(2K+E)) \cong H^1(\Omega^2_M(-Z+E)) = 0. \qquad \square$$

(3.3) In the rest of this section, we always assume that (X, x) is a complete intersection singularity which is not a rational double point. Let $\mu(X, x)$ and $\tau(X, x)$ denote Milnor number and Tjurina number of (X, x), respectively. We need the following results of Greuel [4, 5] (cf. [14]).

PROPOSITION 3.4. (1) $\mu(X, x) = h^1_{\{x\}}(d\Omega^1_X)$, *and* $\tau(X, x) = h^1_{\{x\}}(\Omega^1_X)$ [5, *p*. 168].

(2) $H^q_{\{x\}}(\Omega^p_X) = 0$ *for* $p + q \leqslant 1$ [5, *Prop*. 2.3].

(3) *The following sequences are exact* [4, *Satz* 4.4]*:*

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \to d\mathcal{O}_X \to 0;$$

$$0 \to d\mathcal{O}_X \to \Omega^1_X \to d\Omega^1_X \to 0.$$

(4) $H^0_{\{x\}}(d\Omega^1_X) = 0$ [4, *Lemma* 4.5].

(3.5) From (3.1.1), we have an exact sequence

$$\begin{aligned} 0 &\to H^1_E(d\mathcal{O}_M(-Z)) \to H^1_E(\Omega^1_M\langle E\rangle(K)) \\ &\to H^1_E(\mathcal{O}_M(2K+E)) \to H^2_E(d\mathcal{O}_M(-Z)) \\ &\to H^2_E(\Omega^1_M\langle E\rangle(K)). \end{aligned}$$

By Corollary 2.5, $h^1_E(\mathcal{O}_M(2K+E)) = \delta_2(X, x)$, and by the duality, $h^1_E(\Omega^1_M\langle E\rangle(K)) = h^1(\mathcal{S})$. If we set

$$\rho = \dim_{\mathbb{C}} \ker\left(H^2_E(d\mathcal{O}_M(-Z)) \to H^2_E(\Omega^1_M\langle E\rangle(K))\right),$$

we have

$$\delta_2(X, x) = h^1(\mathcal{S}) + \rho - h^1_E(d\mathcal{O}_M(-Z)).$$

We note that $h^1_E(d\mathcal{O}_M(-Z)) \leqslant h^1(\mathcal{S})$.

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LEMMA 3.6. $h^1_E(d\mathcal{O}_M(-Z)) = h^1_{\{x\}}(d\mathcal{O}_X) + p_g(X,x) - 1$.

Proof. From the following exact sequence

$$0 \to H^0(d\mathcal{O}_M(-Z)) \to H^0(d\mathcal{O}_{M-E}) \to H^1_E(d\mathcal{O}_M(-Z)) \to 0,$$

and isomorphisms

$$H^0(d\mathcal{O}_M(-Z)) \cong H^0(\mathcal{O}_M(K)) \cong H^0(f_*\mathcal{O}_M(K)),$$

we see that

$$H^1_E(d\mathcal{O}_M(-Z)) \cong \frac{H^0(d\mathcal{O}_{X-\{x\}})}{H^0(f_*\mathcal{O}_M(K))}.$$

Using (2) and (3) of Proposition 3.4, we obtain

$$H^1_{\{x\}}(d\mathcal{O}_X) \cong \frac{H^0(d\mathcal{O}_{X-\{x\}})}{H^0(d\mathcal{O}_X)}.$$

Let $\mathcal{M}$ be an ideal sheaf of $\mathcal{O}_X$ which defines the singular point x. Since X is contractible

$$H^0(\mathcal{M}) \cong H^0(d\mathcal{M}) \cong H^0(d\mathcal{O}_X).$$

As (X,x) is a Gorenstein singularity with $p_g(X,x) \geqslant 1$, we have $f_*\mathcal{O}_M(K) \subset \mathcal{M}$. It is well-known that

$$p_g(X,x) = \dim_{\mathbb{C}} \frac{H^0(\mathcal{O}_X)}{H^0(f_*\mathcal{O}_M(K))}$$

for a Gorenstein singularity (X,x). Now the result follows from

$$h^1_E(d\mathcal{O}_M(-Z)) - h^1_{\{x\}}(d\mathcal{O}_X) = \dim_{\mathbb{C}} \frac{H^0(\mathcal{M})}{H^0(f_*\mathcal{O}_M(K))} = p_g(X,x) - 1. \qquad \square$$

LEMMA 3.7. $\rho = \mu(X,x) - \tau(X,x) + h^1_{\{x\}}(d\mathcal{O}_X)$.

Proof. Since $H^1(d\mathcal{O}_M(-Z)) = H^2(d\mathcal{O}_M(-Z)) = 0$, we have

$$\begin{aligned} H^2_E(d\mathcal{O}_M(-Z)) &\cong H^1(d\mathcal{O}_{M-E}) \cong H^1(d\mathcal{O}_{X-\{x\}}) \\ &\cong H^2_{\{x\}}(d\mathcal{O}_X). \end{aligned}$$

Similarly, we get

$$H^2_E(\Omega^1_M\langle E\rangle(K)) \cong H^2_{\{x\}}(\Omega^1_X).$$

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Then

$$\rho = \dim_{\mathbb{C}} \ker \left(H^2_{\{x\}}(d\mathcal{O}_X) \to H^2_{\{x\}}(\Omega^1_X) \right).$$

From Proposition 3.4, we have an exact sequence

$$0 \to H^1_{\{x\}}(d\mathcal{O}_X) \to H^1_{\{x\}}(\Omega^1_X) \to H^1_{\{x\}}(d\Omega^1_X)$$
$$\to H^2_{\{x\}}(d\mathcal{O}_X) \to H^2_{\{x\}}(\Omega^1_X),$$

and hence $\rho = \mu(X,x) - \tau(X,x) + h^1_{\{x\}}(d\mathcal{O}_X)$. □

THEOREM 3.8. $\delta_2(X,x) = h^1(\mathcal{S}) + \mu(X,x) - \tau(X,x) - p_g(X,x) + 1$.

Proof. The theorem is immediately obtained from (3.5), Lemma 3.6 and Lemma 3.7. □

COROLLARY 3.9. *Let* $\pi: \overline{X} \to T$ *be a deformation of* (X,x) *which is obtained from an equisingular deformation of* (M,E). *We set* $X_t = \pi^{-1}(t)$ *for* $t \in T$. *Then*

$$\tau(X_t) \geqslant \mu(X,x) - \delta_2(X,x) \quad \textit{for any } \; t \in T. \tag{3.9.1}$$

In particular, if $p_g(X,x) = 1$, *then* $\tau(X_t) \geqslant \mu(X,x) - 5$.

Proof. We note that X_t is a complete intersection isolated singularity for any $t \in T$. From (3.5) and Lemma 3.6, $h^1(\mathcal{S}) \geqslant p_g - 1$. By Theorem 3.8, we have that $\delta_2(X_t) \geqslant \mu(X_t) - \tau(X_t)$. By Corollary 2.6, δ_2 is determined by p_g and the weighted dual graph of the singularity, and so is μ by [17, (2.26)]. The property of the equisingular deformations implies that

$$\delta_2(X_t) = \delta_2(X,x) \quad \text{and} \quad \mu(X_t) = \mu(X,x).$$

Then we get (3.9.1). If $p_g(X,x) = 1$, then $\delta_2(X,x) \leqslant 5$ by Remark 2.8. We have thus proved the corollary. □

(3.10) Let $h \in \mathbb{C}\{z_0, z_1, z_2\} = \mathcal{O}_{\mathbb{C}^3,o}$ define an isolated singularity (X,o) at the origine. Let J_h be an ideal of $\mathcal{O}_{\mathbb{C}^3,o}$ generated by $\partial h/\partial z_0, \partial h/\partial z_1$ and $\partial h/\partial z_2$. $Q_h = \mathcal{O}_{\mathbb{C}^3,o}/J_h$ is called Jacobian algebra. It is well known that

$$\mu(X,o) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^3,o}}{J_h} \quad \text{and} \quad \tau(X,o) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^3,o}}{(J_h, h)},$$

and that $\mu(X,o) = \tau(X,o)$ if and only if h is quasi-homogeneous (after a change of coordinates). If h is a quasi-homogeneous polynomial of degree d, then Q_h is graded with $Q_h = \bigoplus_{i \geqslant 0} Q_h(i)$, and there are the natural isomorphisms $T^1_X(i-d) \cong Q_h(i)$.

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We set $\mu = \mu(X, o)$. Let $\varphi_1, \ldots, \varphi_\mu$ be $\mathbb{C}$-basis of $\mathcal{O}_{\mathbb{C}^3,o}/J_h$. Then we define a function $H(z,t) \in \mathbb{C}\{z_0, z_1, z_2, t_1, \ldots, t_\mu\} = \mathcal{O}_{\mathbb{C}^3 \times \mathbb{C}^\mu, o}$ as following

$$H(z,t) = h + \sum_{i=1}^{\mu} t_i \varphi_i,$$

and we set

$$Y(X,o) = \{(t_0) \in (\mathbb{C}^\mu, o) | \mu(H(z,t_0)) = \mu\},$$

where $\mu(H(z,t_0))$ denotes Milnor number of the singularity defined by $H(z,t_0)$. Then $Y(X,o)$ is an analytic subset of $(\mathbb{C}^\mu, o)$.

DEFINITION 3.11. The modality $m(X,o)$ of the singularity (X,o) is the dimension of $Y(X,o)$ (cf. [2]). If (X,o) is defined by a quasi-homogeneous polynomial h of degree d, then the inner modality $m_0(X,o)$ of the singularity (X,o) is defined as the dimension of the vector space $\bigoplus_{i \geqslant d} Q_h(i)$ (cf. [26]). Note that $m_0(X,o) \leqslant m(X,o)$.

COROLLARY 3.12. *Let (X,o) be a hypersurface singularity with $p_g(X,o) = 1$ defined by $h \in \mathcal{O}_{\mathbb{C}^3,o}$. Then $\delta_2(X,o) \leqslant m(X,o)$.*

If (X,o) is quasi-homogeneous, then $\delta_2(X,o) = m_0(X,o) \leqslant 4$.

Proof. Let $(\mathbb{C}^{\tau(X,o)}, o)$ be the versal deformation space of the singularity (X,o) and $p\colon (\mathbb{C}^{\mu(X,o)}, o) \to (\mathbb{C}^{\tau(X,o)}, o)$ be a projection corresponding to the natural map of the tangent spaces $\mathcal{O}_{\mathbb{C}^3,o}/J_h \to \mathcal{O}_{\mathbb{C}^3,o}/(J_h, h)$. There is a submanifold P of $(\mathbb{C}^{\tau(X,o)}, o)$ which represents ES_M (cf. B of Preliminaries). By the property of the equisingular deformations, $p^{-1}(P) \subset Y(X,o)$. By Theorem 1.11, we see that the dimension of $p^{-1}(P)$ is $h^1(\mathcal{S}) + \mu(X,o) - \tau(X,o)$. Hence

$$h^1(\mathcal{S}) + \mu(X,o) - \tau(X,o) \leqslant m(X,o).$$

From Theorem 3.8, we get $\delta_2(X,o) \leqslant m(X,o)$.

We assume that h is a quasi-homogeneous polynomial of degree d. Then Theorems 3.8, 1.11 and 1.13 and (3.10) imply that

$$\delta_2(X,o) = h^1(\mathcal{S}) = \dim_{\mathbb{C}} \bigoplus_{i \geqslant d} Q_h(i) = m_0(X,o).$$

By Corollary 2.7, $\delta_2(X,o) \leqslant 4$. We have thus proved the corollary. □

Remark 3.13. If the invariance of Milnor number implies the invariance of the topological type for two-dimensional hypersurface singularities (cf. [13]), then, in the proof above, we have $p^{-1}(P) = Y(X,o)$ (cf. (1.9)). In this case, $Y(X,o)$ is nonsingular, and $\delta_2(X,o) = m(X,o)$ holds.

It is known that for any quasi-homogeneous hypersurface singularity (X,o), an inequality $\delta_2(X,o) \geqslant m_0(X,o)$ holds (see [26]).

DEFINITION 3.14. A function $h \in \mathcal{O}_{\mathbb{C}^3,o}$ is said to be *semi-quasi-homogeneous* of degree d with weights $(\alpha_0, \alpha_1, \alpha_2)$ if it is of the form $h = h_0 + h_1$, where h_0 is a quasi-homogeneous polynomial of degree d with weights $(\alpha_0, \alpha_1, \alpha_2)$ which defines an isolated singularity and all of the monomials of h_1 have degree strictly greater than d (cf. [1, 12.1]).

COROLLARY 3.15. *Let (X, o) be a singularity defined by a semi-quasi-homogeneous function $h \in \mathcal{O}_{\mathbb{C}^3,o}$ with weights $(1, 1, 1)$. Then $\delta_2(X, o) \geqslant m(X, o)$.*

Proof. We write $h = h_0 + h_1$ as the definition above. Let (X_0, o) be a singularity defined by h_0. Then by [3], $m_0(X_0, o) = m(X_0, o)$. Hence we have that $\delta_2(X_0, o) \geqslant m(X_0, o)$ by [26]. On the other hand, (X, o) is a fibre in an equisingular deformation of (X_0, o) by [1, Th. 12.1] and Theorem 1.13. Since the modality is upper semi-continuous by [2], we have

$$\delta_2(X, o) = \delta_2(X_0, o) \geqslant m(X_0, o) \geqslant m(X, o). \qquad \square$$

(3.16) We assume that the weighted dual graph of (X, x) is a star-shaped graph. We set $E = E_0 \bigcup E^{(1)} \bigcup \cdots \bigcup E^{(\beta)}$, where E_0 is the central curve, and $E^{(i)}$ the branches. The curves of $E^{(i)}$ are denoted by $E_{i,j}$, $1 \leqslant j \leqslant r_i$, where $E_0 \cdot E_{i,1} = E_{i,j} \cdot E_{i,j+1} = 1$. We set $b_{i,j} = -E_{i,j} \cdot E_{i,j}$.

Let us introduce some result of [18]. Let F be a divisor on E_0 with $\mathcal{O}_{E_0}(-E_0) \cong \mathcal{O}_{E_0}(F)$, and P_i the intersection point $E_0 \bigcap E_{i,1}$ for $i = 1, \ldots, \beta$. We define a $\mathbb{Q}$-divisor D on E_0 as follows: $D = F - \sum_{i=1}^{\beta} q_i P_i$, where $q_i \in \mathbb{Q}$ is defined by

$$\frac{1}{q_i} = b_{i,1} - \cfrac{1}{b_{i,2} - \cfrac{1}{\cdots - \cfrac{1}{b_{i,r_i}}}} \quad \text{for } i = 1, \ldots, \beta.$$

Let $R = \bigoplus_{n \geqslant 0} H^0(\mathcal{O}_{E_0}(nD))T^n \subset \mathbb{C}(E_0)[T]$, where $\mathbb{C}(E_0)$ is the field of rational functions of E_0, and T an indeterminate. Then $\operatorname{Spec}(R)$ is a normal surface singularity, we denote by (Y, y), and the weighted dual graph of (Y, y) is the same as that of (X, x).

By contracting the branches $E^{(1)} \cup \cdots \cup E^{(\beta)}$, we get a normal surface M' with cyclic quotient singularities. Let $\Phi\colon (M', E') \to (X, x)$ be the morphism induced canonically, where E' is the image of E_0. We define a filtration on $\mathcal{O}_X$ by $F^n = \Phi_* \mathcal{O}_{M'}(-nE')$ for $n \in \mathbb{Z}$. Note that $F^n = \mathcal{O}_X$ for $n \leqslant 0$. Let $\mathcal{R} = \bigoplus_{n \in \mathbb{Z}} F^n T^n$, and let $G = \bigoplus_{n \geqslant 0} (F^n / F^{n+1}) T^n$. Then the natural map $\mathbb{C}[T^{-1}] \to \mathcal{R}$ defines a deformation of $\operatorname{Spec}(G)$ with general fibre isomorphic to (X, x), since $G \cong \mathcal{R}/T^{-1}\mathcal{R}$ and $\mathcal{O}_X \cong \mathcal{R}/(T^{-1} - a)\mathcal{R}$ for $a \in \mathbb{C} - \{0\}$ (cf. [18, (5.15)]).

By [18,(6.3)], we have that $p_g(Y, y) = p_g(X, x)$ if and only if $R = G$.

COROLLARY 3.17. *Let (X, o) be a hypersurface singularity with $p_g(X, o) = 1$ such that the weighted dual graph of it is a star-shaped graph. Then (X, o) is*

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defined by a semi-quasi-homogeneous function of which the quasi-homogeneous part defines a singularity (X_0, o) *with* $m_0(X_0, o) = \delta_2(X, o)$.

In particular, for such a singularity with $\delta_2(X, o) \leqslant 2$, *we have* $\delta_2(X, o) = m(X, o)$.

Proof. We use the notation of (3.16). The weighted dual graph of (X, o) determines the embedding dimension of (Y, y) and $p_g(Y, y)$ (cf. [9]): then (Y, y) is a quasi-homogeneous hypersurface singularity with $p_g(Y, y) = 1$. Let h_0 be a quasi-homogeneous function of degree d which defines (Y, y), i.e., $R \cong \mathbb{C}[z_0, z_1, z_2]/(h_0)$. By (3.16) and [23, (1.12), (3.4)], (X, o) is a fibre in a deformation of (Y, y) which is obtained from an equisingular deformation. Then there is a function $h_1 \in \mathcal{O}_{\mathbb{C}^3,o}$ of which the image of the natural map $\mathcal{O}_{\mathbb{C}^3,o} \to Q_{h_0}$ is in $\bigoplus_{i \geqslant d} Q_{h_0}(i)$ such that $h_0 + h_1$ defines (X, o) (cf. Th. 1.13, (3.10)). Since $R = G$, we may assume that all of monomials of h_1 have degree strictly greater than d. Hence $h_0 + h_1$ is a semi-quasi-homogeneous function. Let $(X_0, o) = (Y, y)$. We have $\delta_2(X, o) = \delta_2(X_0, o) = m_0(X_0, o)$ by Corollary 3.12.

Quasi-homogeneous hypersurface singularities with $p_g = 1$ and $m_0 \leqslant 4$ are listed in [26]. The lists of all the singularities for which $m \leqslant 2$ are given in [1, 15.1]. Then we see the last assertion. □

(3.18) In [25], we proved the equality of Corollary 2.6 for Gorenstein singularities with $p_g = 1$, and classified the weighted dual graphs of those with $\delta_2 \leqslant 2$. Then we have the following.

Let (X, x) be a Gorenstein singularity with $p_g(X, x) = 1$. Then $\delta_2(X, x) = 1$ if and only if (X, x) is a simple elliptic, a cusp or a singularity obtained (in the sense of [21, (5.2)]) from a unimodular singularity, and $\delta_2(X, x) = 2$ if and only if (X, x) is a singularity obtained (in the sense above) from a bimodular singularity.

Acknowledgements

The author would like to thank Professor M. Tomari who kindly adviced on the subject of (3.16). He also thanks Professor Kimio Watanabe for helpful advices and the referee for useful suggestions.

References

1. Arnold, V. I., Gusein-Zade, S. M. and Varchenko, A. N.: *Singularities of differentiable maps* Volume I Birkhäuser, Boston, 1985.
2. Gabriélov, A. M.: Bifurcations, Dynkin diagrams, and modality of isolated singularities, *Functional Anal. Appl.* 8 (1974) 94–98.
3. Gabriélov, A. M. and Kushnirenko, A. G.: Description of deformations with constant Milnor number for homogeneous functions, *Functional Anal. Appl.* 9 (1975), 329–331.
4. Greuel, G.-M.: Der Gauß-Manin-Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten, *Math. Ann.* 214 (1975), 235–266.
5. Greuel, G.-M.: Dualität in der lokalen Kohomologie isolierter Singularitäten, *Math. Ann.* 250 (1980) 157–173.
6. Ishii, S.: On isolated Gorenstein singularities, *Math. Ann.* 270 (1985), 541–554.

7. Ishii, S.: Du Bois singularities on a normal surface *Adv. Stud. Pure Math.* 8 (1986), 153–163.
8. Ishii, S.: The asymptotic behavior of plurigenera for a normal isolated singularity, *Math. Ann.* 286 (1990), 803–812.
9. Laufer, H.: On minimally elliptic singularities, *Amer. J. Math.* 99 (1977), 1257–1295.
10. Laufer, H.: Versal deformations for two-dimensional pseudoconvex manifolds, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* 7 (1980), 511–521.
11. Laufer, H.: Lifting cycles to deformations of two-dimensional pseudoconvex manifolds, *Trans. Amer. Math. Soc.* 266 (1981), 183–202.
12. Laufer, H.: Weak simultaneous resolution for deformations of Gorenstein surface singularities, *Pros. Symp. Pure Math.* 40, Part 2 (1983), 1–30.
13. Tráng, Lê Dũng and Ramanujan, C.: The invariance of Milnor's number implies the invariance of the topological type, *Amer. J. Math.* 98 (1976), 67–78.
14. Looijenga, E. and Steenbrink, J.: Milnor number and Tjurina number of complete intersections, *Math. Ann.* 271 (1985), 121–124.
15. Morales, M.: Calcul de quelques invariants des singularités de surface normale, *Enseign. Math*. 31 (1983), 191–203.
16. Pinkham, H.: Deformations of normal surface singularities with $\mathbb{C}^*$-action, *Math. Ann.* 232 (1978), 65–84.
17. Steenbrink, J.: Mixed Hodge structures associated with isolated singularities, *Proc. Symp. Pure Math*. 40, Part 2 (1983) 513–536.
18. Tomari, M. and Watanabe, Kei-ichi: Filtered rings, filtered blowing-ups and normal two-dimensional singularities with 'star-shaped' resolution, *Publ. RIMS, Kyoto Univ.* 25 (1989), 681–740.
19. Wahl, J.: Vanishing theorems for resolutions of surface singularities, *Invent. Math.* 31 (1975), 17–41.
20. Wahl, J.: Equisingular deformations of normal surface singularities, I, *Ann. Math.* 104 (1976), 325–365.
21. Wahl, J.: Simultaneous resolution and discriminantal loci, *Duke Math. J*. 46 (1979), 341–375.
22. Wahl, J.: A characterization of quasi-homogeneous Gorenstein surface singularities, *Compositio Math.* 55 (1985), 269–288.
23. Wahl, J.: Deformations of quasi-homogeneous surface singularities, *Math. Ann.* 280 (1988), 105–128.
24. Watanabe, K.: On plurigenera of normal isolated singularities. I, *Math. Ann.* 250 (1980), 65–94.
25. Watanabe, K. and Okuma, T.: *Characterization of unimodular singularities and bimodular singularities by the second plurigenus*, preprint.
26. Yoshinaga, E. and Watanabe, K.: On the geometric genus and the inner modality of quasihomogeneous isolated singularities, *Sci. Rep. Yokohama Nat. Univ.* Sect. I 25 (1978), 45–53.
27. Yoshinaga, E. and Suzuki, M.: Normal forms of non-degenerate quasihomogeneous functions with inner modality $\leqslant 4$, *Invent. Math.* 55 (1979), 185–206.