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TOPOLOGICAL TRANSITIVITY ON THE TORUS

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ABSTRACT. T. Ding has shown that a topologically transitive flow on the torus given by a real analytic vector field is orbitally equivalent to a Kronecker flow on the torus, modified so as to have a finite number of fixed points, provided the original flow had only a finite number of fixed points. In this paper it is shown that the assumption that there are only finitely many fixed points is unnecessary.

Let T^2 be the torus represented in the usual way as the quotient of R^2 by the lattice of integral points. Suppose we are given real analytic differential equations

$$\frac{dx_i}{dt} = X_i(x_1, x_2) \quad (i = 1, 2)$$

with $X_i(x_1, x_2)$ periodic of period one in each of their variables. This yields a flow on T_2. Assume this flow is topologically transitive. In [1] it is shown that if there are only a finite number of stationary points for this flow there is a homeomorphism of T^2 onto T^2 sending the orbits of our flow onto the orbits of a flow given by

$$\frac{dx_i}{dt} = f(x_1, x_2) \quad \frac{dx_2}{dt} = \lambda f(x_1, x_2),$$

where λ is irrational and f is a smooth non-negative function vanishing at only a finite number of points. (Obviously the orbit structure of the latter flow depends only on λ and the set of points where $f(x_1, x_2)$ vanishes). We are going to show that the assumption that our original flow vanishes at only a finite number of points is unnecessary; that is that a real analytic topologically transitive flow on T^2 necessarily vanishes at only a finite number of points.

The points where our vector field vanishes is precisely the set of points where the real analytic function $X_1^2 + X_2^2$ vanishes. We will need the following result: if a real analytic function in a region of the plane has a zero that is not isolated, there is an open neighborhood of this point in the subspace K of points where the function vanishes that is homeomorphic to the set of rays in R^2 emanating from (0, 1) and passing through each of a finite set of points on the X_1-axis. Moreover, the number n of points in the set on the X_1-axis is even. We describe this situation by saying that our original point is a branching point of the set of zeros of our function with an even number of branches emanating from it. While this result is well known, I have been unable to find a proof in the literature. I am indebted to Brian Cole for the following demonstration.

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Recall first that if $f(Z_1, Z_2)$ is a complex analytic function vanishing at the origin in C^2 there exists a finite number of one to one analytic maps of the unit disc into C^2, each one sending 0 into $(0,0)$, such that the images of any two of these maps intersect only at $(0,0)$ and such that the union of these images constitutes an open neighborhood of $(0,0)$ in the subspace of C^2 consisting of those points at which $f(Z_1, Z_2)$ vanishes.

Given a real analytic function f in a neighborhood of the origin in R^2 such that $f(0,0) = 0$, extend to a complex valued function of (z_1, z_2) in a neighborhood of $(0,0)$ in C^2. Let $\big(g_1(t), h_1(t)\big), \ldots, \big(g_k(t), h_k(t)\big)$ be the one to one maps of the unit disc into C^2 alluded to above. Since $g_i(0) = 0$ for any i, we can find a complex analytic function $w_i(t)$ such that $w_i(0) = 0$, $w_i'(0) \neq 0$ and $g_i(t) = \big(w_i(t)\big)^n$ for some positive integer n. Switching our parameter from t to w_i for the i-th function from the unit disc to C^2 we can assume that our k functions whose images cover a neighborhood of $(0,0)$ in the zero set of f are of the form $\big(w^{n_1}, U_1(w)\big), \ldots, \big(w^{n_k} U_k(w)\big)$ where now the domain of our $U_i(w)$ is a disc about 0 of some radius r. The real zeros of f in the image of $\big(w^{n_i}, U_i(w)\big)$ come from those points at which both w^{n_i} and $U_i(w)$ are real. The points at which w^{n_i} is real are the union of a finite collection of line segments through the origin. If 0 is a limit point of the values of w which lie along one of these line segments and at which $U_i(w)$ is real, then $U_i(w)$ is real on this whole line segment. Thus the points in R^2 at which f is zero and which are in a suitable neighborhood of $(0,0)$ lie along the images of a finite collection of line segments. Each line segment gives rise to two branches through the origin at which f is zero; thus we get an even number of branches through $(0,0)$.

From the above result it follows that the set of points in T^2 where our vector field vanishes is a finite simplicial complex K of dimension at most one.

Suppose some component K_1 of this complex had more than one point. The discussion in [1] shows that if we have a dense orbit for a flow on T^2 and C is any simple closed curve on T^2, the orbit must intersect C. Since our original flow was topologically transitive, K_1 cannot contain a simple closed curve. This implies that K_1 is a dendrite, and any dendrite has at least two end points [4]. For a one dimensional simplicial complex, an end point is just a branching point with exactly one branch emanating from it. Since each point of K_1 would have to have an even number of branches emanating from it, K is zero dimensional and therefore finite. This concludes our proof. It is perhaps appropriate to mention the situation with respect to the final result in [1]. The author states that under his assumptions a topologically transitive flow on T^2 is metrically transitive. What he actually shows, however, is that for a topologically transitive flow any closed proper invariant set has measure zero. Of course this is far from enough to establish metric transitivity.

In discussing metric transitivity it is appropriate to consider those probability measures that are invariant under the flow. In our present situation it is apparent that any invariant measure can be represented as a sum of an invariant measure concentrated at the stationary points and one for which the measure of any single point is zero. Obviously it is only measures of the second kind that are of interest. In [3] Stepanoff showed that there are analytic flows on the torus that are orbitally equivalent to a topologically

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transitive straight line flow modified so as to have a unique stationary point for which no such invariant measure exists. In [2] Oxtoby gave a similar example where such an invariant probability measure does exist. Moreover he showed that in this case if such a probability measure exists it is unique and with respect to it the flow is metrically transitive. His proof extends trivially to the case where there are only finitely many stationary points. The result of the present paper shows that the same result holds in the general analytic topologically transitive case.

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