THE RECURSIVE EQUIVALENCE TYPE OF A DECOMPOSITION OF AN ω-GROUP: THE *RET* OF A DECOMPOSITION

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1. Introduction

Let ε, Λ, Λ^*, Λ_R stand for the set of non-negative integers, isols, isolic integers and regressive isols respectively, and let $P(\tau)$ be the ω-group of Gödel numbers of permutations of the set $\tau \subset \varepsilon$ which move only finitely many elements of τ. The concept of an ω-group was studied by Hassett [5]. He proved in P12 of [5] that for an isolated set τ, the decomposition of $P(\tau)$ into conjugacy sets is a gc-decomposition if and only if τ is regressive. For the finite symmetric group on n elements, S_n, it is known that the order of the conjugacy class is $p(n)$, where $p(n)$ is the partition function. The author shows in this paper, using a result of Barback [1], that if $p_\Lambda(T)$ is Nerode's canonical extension of $p(n)$ to Λ and Req $(\tau) = T$, then $p_\Lambda(T) = \text{Req } C_\tau$, where C_τ is the decomposition of $P(\tau)$ into conjugacy sets. The reader is assumed to be familiar with the contents of [2], [4] and [5].

2. Basic concepts

NOTATIONS. (i) A function of n, say $a(n)$, may also be written a_n.

(ii) $v(n) = \{0, 1, 2, \cdots, n-1\}$.

(iii) In this paper we will denote the group of all finite permutations of a set σ, i.e. those permutations which move only finitely many elements of σ, by $\mathscr{P}(\sigma)$. We denote the set of elements in $P(\sigma)$ by σ^*, that is,

$$\sigma^* = \{f^* \in \varepsilon \,|\, f \in \mathscr{P}(\sigma)\}.$$

(iv) For a recursive function $f(x)$, we denote Nerode's canonical extension from Λ into Λ^* by f_Λ. We know by [1] that f_Λ maps Λ_R into itself if and only if f is eventually increasing.

[1] The results presented in this paper were taken from the author's doctoral dissertation written at Rutgers University under the direction of Professor J. C. E. Dekker.

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We need the following theorems, [3, Prop. 1] and [1, p. 36] respectively,

(1) *The function f has a one-to-one partial recursive extension if and only if f and f^{-1} have partial recursive extensions and f is one-to-one.*

(2) *Let $f(n)$ be an increasing recursive function and define $e(n)$ by $e(0) = f(0)$ and $e(n+1) = f(n+1) \neq f(n)$. Then $f_\Lambda(T) = \Sigma_{T+1} e_n$, for $T \in \Lambda_R$.* –

DEFINITION. A *partition* of the positive integer n is an expression of the form $n_1 + \cdots + n_k$, where $n_1, \cdots, n_k$ denote positive integers (not necessarily distinct) with n as sum. Two partitions of n are *equal*, if they only differ in the order of their terms; we may therefore assume that $1 \leqq n_1 \leqq n_2 \leqq \cdots \leqq n_k \leqq n$.

DEFINITION. The *partition function* is the function $p(n)$ from ε into ε such that $p(0) = 1$ and for $n \geqq 1$, $p(n)$ is the number of distinct partitions of n.

Several properties of $p(n)$ are discussed in Chapter 10 of [7]. It is readily seen that $p(n)$ is an increasing recursive function which is strictly increasing for $n \geqq 1$. Hence p_Λ is a function from Λ_R into itself. It is well-known from group theory that for $n \geqq 1$, $p(n)$ is the number of conjugacy sets of the symmetric group S_n. Assume that τ is a non-empty finite set of cardinality n. Then $P(\tau) \cong S_n$, hence

$$p(n) = \text{the number of conjugacy sets of } P(\tau).$$

Let τ be a non-empty isolated set and C_τ the decomposition of the isolic group $P(\tau)$ into conjugacy sets. By [5, P12] the *md*-class C_τ is a *gc*-class if and only if τ is regressive. Obviously, if τ is regressive, $RET(C_\tau)$ only depends on $\text{Req}(\tau)$.

NOTATION. If $T \in \Lambda_R$ and $T \neq 0$, $C_T = \text{Req}(C_\tau)$, for $\tau \in T$.

We observed that $C_T = p(n)$, in case $T = n > 0$. It is therefore a reasonable conjecture that $C_T = p_\Lambda(T)$, for $T \in \Lambda_R$, $T \neq 0$.

3. Main Result

THEOREM. *If T is a non-zero regressive isol, $C_T = p_\Lambda(T)$.*

PROOF. The result holds if T is finite, hence we assume that $T \in \Lambda_R - \varepsilon$ Let $\sigma \in T + 1$, t_n a regressive function ranging over σ and $\tau = \sigma - (t_0)$. Then $\tau \in T$. For $x \in P(\tau)$ we denote the conjugacy set of x by $C(x)$. In the proof of [5, P12] a function $h^*(x)$ from τ^* into itself is defined as follows: $h^*(1) = 1$; given any number $x = f^* \in P(\tau)$ with $x \neq 1$, that is, $f \neq i$, we can compute the cycle structure of f, say

$$(n(1), \cdots, n(k)), \text{ where } 2 \leqq n(1) \leqq \cdots \leqq n(k).$$

Let $n = n(1) + \cdots + n(k)$. Taking into account that $\tau = (t_1, t_2, \cdots)$, we define

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$$h_f = (t_1, \cdots, t_{n(1)})(t_{n(1)+1}, \cdots, t_{n(1)+n(2)}) \cdots (t_{n-n(k)+1}, \cdots, t_n),\ h^*(x) = (h_f)^*.$$

According to the proof, the function $h^*(x)$ is a gc-function of the md-class $C_\tau = \{C(x) \mid x \in \tau^*\}$. We conclude that

$$C_T = RET(C_\tau) = \text{Req } h^*(\tau^*). \tag{3}$$

Let $e(0) = p(0)$, $e(n+1) = p(n+1) - p(n)$. Thus by (2) we have that $p_\Lambda(T) = \Sigma_{T+1} e_n$, that is,

$$p_\Lambda(T) = \text{Req} \bigcup_{n=0}^{\infty} j[t_n, v(e_n)]. \tag{4}$$

We define $\beta = \bigcup_{n=0}^{\infty} j[t_n, v(e_n)]$, $\gamma = h^*(\tau^*)$.
In view of (3) and (4) the proof will be complete if we can show that $\beta \simeq \gamma$. Let us call a partition $(n_1, \cdots, n_k)$ of n,

of the *first* type, if $1 = n_1 \leqq n_2 \leqq \cdots \leqq n_k$,

of the *second* type, if $2 \leqq n_1 \leqq n_2 \leqq \ldots \leqq n_k$.

Let for $n \geqq 1$,

$p_1(n)$ = the number of partitions of n of the first type,

$p_2(n)$ = the number of partitions of n of the second type,

then $p(n) = p_1(n) + p_2(n)$, for $n \geqq 1$. We claim

$$p_2(n) = e(n), \qquad \text{for } n \geqq 1. \tag{5}$$

This is trivial for $n = 1$, since $p_2(1) = 0$, $e(1) = 0$. Now consider the numbers $\geqq 2$, that is, the numbers of the form $n+1$, with $n \geqq 1$. If $(1, n_2, \cdots, n_k)$ is a first type partition of $n+1$, then $(n_2, \cdots, n_k)$ is a partition of n. Moreover, all partitions of n can be obtained by dropping the initial "1+" from a first type partition of $n+1$. Hence $p_1(n+1) = p(n)$ and

$$e(n+1) = p(n+1) - p(n) = p(n+1) - p_1(n+1) = p_2(n+1).$$

For $n \geqq 2$ we have $e_n > 0$, that is, n has second type partitions. If $(n_1, \cdots, n_k)$ is a second type partition we denote it by $[n_1, \cdots, n_k]$ and define its Gödel number (G-number) by

$$[n_1, \cdots, n_k]^\# = p_1^{n_1} \cdots p_k^{n_k},$$

where p_i is the ith odd prime. For each $n \geqq 2$ we denote the class of second type partitions of n by T_n and order this class according to increasing G-numbers, say

$$T_n = (s_{n,0}, s_{n,1}, \cdots, s_{n,e(n)-1}).$$

We define the function $b_n(i)$ for $n, i \in \varepsilon$ by

$$b_n(i) = (s_{n,i})^{\#}, \quad \text{for } n \geqq 2,\ i < e_n,$$
$$b_n(i) = 0, \qquad \text{otherwise.}$$

Then $b_n(i)$ is a recursive function. Let $q(x)$ be a regressing function of t_n and

$$q^*(x) = (\mu y)[q^{y+1}(x) = q^y(x)], \text{ for } x \in \delta q.$$

Suppose a number

$$x = j(t_n, i) \in \beta = \bigcup_{n=0}^{\infty} j[t_n, \nu(e_n)]$$

is given. Then the numbers

$$t_n = k(x),\ n = q^*k(x),\ i = l(x),$$

can be computed. Clearly $i < e_n$, because $x \in \beta$. Since $e(0) = p(0) = 1$, $e(1) = 0$, $e(m) \geqq 1$, for $m \geqq 2$, we see that the number $n = q^*k(x)$ must be 0 or $\geqq 2$. If $n = 0$, we have $x = j(t_0, 0)$ and define $g(x) = 1$. Now assume $n \geqq 2$. Then the number $b_n(i)$ is of the form

$$b_n(i) = [n_1, \cdots, n_k]^{\#}, \text{ where } [n_1, \cdots, n_k] \in T_n.$$

In this case we define

$$g(x) = ((t_1, \cdots, t_{n(1)}) \cdots (t_{n-n(k)+1}, \cdots, t_n))^*.$$

Since the function t_n is regressive and the numbers t_n, n, i can be computed from x, so can the number $g(x)$. We claim

(6) $\qquad g$ *is one-to-one on* β,

(7) $\qquad g(\beta) = \gamma.$

Re (6). The function g maps $j(t_0, 0)$ onto 1 and any number $x \in \beta$ which is different from $j(t_0, 0)$ onto a member of γ which is different from 1.

Now assume that

$$x = j(t_n, i) \in \beta,\ x' = j(t_m, i') \in \beta,$$

where $n, m \geqq 2$ and $x \neq x'$. It follows that $n \neq m$, or $n = m$ and $i \neq i'$. If $n \neq m$, the classes T_n and T_m are disjoint, and hence $g(x)$ and $g(x')$ are G-numbers of finite permutations in $\mathscr{P}(\tau)$ whose cycle classes are distinct. On the other hand, if $n = m$, $i \neq i'$, we see that $T_n = T_m$ and $g(x)$ and $g(x')$ are G-numbers of finite permutations in $\mathscr{P}(\tau)$ whose cycle structures are distinct members of T_n. In each case $g(x) \neq g(x')$.

Re (7). The function g maps β into γ. Let $y = f^* \in \gamma$. If $y = 1$, $y = gj(t_0, 0)$ where $j(t_0, 0) \in \beta$. Now assume $y \neq 1$, i.e. $f \neq i$, say

$$f = (t_1, \cdots, t_{m(1)}) \cdots (t_{m-m(l)+1}, \cdots, t_m),$$
$$2 \leqq m(1) \leqq \cdots \leqq m(l),\ m = m(1) + \cdots + m(l).$$

Define i and x by

$$b_m(i) = [m_1, \cdots, m_l]^{\#},\ x = j(t_m, i),$$

then $i < e_m$, $x \in \beta$ and $g(x) = y$.

We have proved that g maps β one-to-one onto γ. Note that in the proof of (7) we have suggested how to compute g^{-1}. It can be shown that g and g^{-1} have partial recursive extensions. Hence by (1) we have $\beta \simeq \gamma$ and we are done.

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