$$\alpha = n^{-\frac{1}{2}} \sum a_i , \qquad \beta = n^{-\frac{1}{2}} \sum b_i .$$

Then

$$\sum a_i^2 \sum b_i^2 - (\sum a_i b_i)^2 \geq \alpha^2 \sum b_i^2 - 2\alpha\beta \sum a_i b_i + \beta^2 \sum a_i^2$$

with the sign of equality if and only if the three row vectors $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$, $e = n^{-\frac{1}{2}} (1, \ldots, 1)$ are linearly dependent.

H. Schwerdtfeger

P 20. Given a plane square lattice of side length one and a positive integer n. Form all the sets S of n lattice points. Let $L(S)$ denote the length of the boundary of the convex closure of S. Estimate $\min_S L(S)$ in terms of n. (cf. H.D. Block, Proc. Amer. Math. Soc. 8 (1957), 860-862.)

P. Scherk

P 21. It is possible to metrize an affine plane (with preservation of its natural topology) in such a way that on every affine line the metric is Euclidean, but that the whole plane does not become Euclidean, viz. by a Minkowski metric. Similarly, is it possible to metrize an affine space A^n $(n > 2)$ in such a way that in every affine plane the metric is Minkowskian, but that the whole space is not Minkowskian?

H. Helfenstein

SOLUTIONS

P 1. Let $f(x)$ be a Lebesgue integrable function on some interval $a - \varepsilon \leq x \leq b + \varepsilon$, $\varepsilon > 0$, and let $F_h(x) = h^{-1} \int_x^{x+h} f(t)dt$. An important theorem in the theory of Lebesgue integration states that $\lim_{h \to 0} F_h(x) = f(x)$ for almost all x. Show that we also have $\lim_{h \to 0} \int_a^b | F_h(x) - f(x) | dx = 0$.

W.A.J. Luxemburg

Solution by the proposer. We restrict ourselves to the case $h > 0$ as the case $h < 0$ is similar.

$$\int_a^b |F_h(x) - f(x)|\, dx = \int_a^b \left| h^{-1} \int_x^{x+h} f(t)dt - f(x)\right| dx$$

$$= \int_a^b \left| h^{-1} \int_o^h (f(t+x)-f(x))dt\right| dx \leq h^{-1} \int_a^b \int_o^h |f(t+x)-f(t)|\, dt\, dx$$

$$= h^{-1} \int_o^h \left(\int_a^b |f(t+x)-f(t)|\, dx\right) dt .$$

(This last integral exists if h is sufficiently small, as f is integrable over $(a-\varepsilon, b+\varepsilon)$.) If we put $\omega_1(\delta) =$ $\sup \left(\int_a^b |f(x+t)-f(x)|\, dx,\ 0 < t \leq \delta \right)$ then we know that $\lim_{\delta \to 0} \omega_1(\delta) = 0$. (This can be proved as follows; given $\varepsilon > 0$ we can find a continuous function $\varphi(x)$, which is continuous for all x and such that $\int_{a-\varepsilon}^{b+\varepsilon} |f(x) - \varphi(x)|\, dx < \varepsilon/3$, as $\varphi(x)$ is uniformly continuous on $(a-\varepsilon, b+\varepsilon)$ we can find a $\delta(\varepsilon) > 0$ such that $|\varphi(x+t) - \varphi(x)| < \varepsilon/3(b-a)$ for all $|t| < \delta$. If $0 < t < \min(\varepsilon, \delta)$ then we have $\int_a^b |f(x+t)-f(x)|\, dx \leq \int_a^b |f(x+t) - \varphi(x+t)|\, dx +$ $\int_a^b |\varphi(x+t) - \varphi(x)|\, dx + \int_a^b |\varphi(x) - f(x)|\, dx \leq \varepsilon/3 + \varepsilon/3 +$ $\int_a^b |\varphi(x+t)-\varphi(x)|\, dx \leq \varepsilon$ if $0 < t < \delta$.) But as $\int_a^b |F_h(x) - f(x)|\, dx \leq \omega_1(h)$ we have $\lim_{h \to 0} \int_a^b |F_h(x) - f(x)|\, dx = 0$.

P 9. Let f(x) be a polynomial with integer coefficients. Let a_o be an integer and $a_{n+1} = f(a_n)$, $n = 0, 1, 2, \ldots$. Prove that if $a_o = a_k$ $(k > 0)$ then $a_n = a_{n+2}$ for all $n \geq 0$.

J. Lambek and L. Moser

Solution by the proposers. Clearly $(a_n - a_{n-1}) \mid (f(a_n) - f(a_{n-1}))$, that is $(a_n - a_{n-1}) \mid (a_{n+1} - a_n)$ for all $n > 0$. Let $b_n = a_{n+1} - a_n$. Then $b_n \mid b_{n+1}$ $(n \geq 0)$ and so $b_n^2 \leq b_{n+1}^2$ $(n \geq 0)$. If $a_o = a_k$ then $a_1 = f(a_o) = f(a_k) = a_{k+1}$ and in general $a_n = a_{n+k}$ $(n \geq 0)$. Hence also $b_n = b_{n+k}$ $(n \geq 0)$. Now $b_{n+1}^2 \leq b_{n+2}^2 \leq \ldots \leq b_{n+k}^2 = b_n^2$. Thus $b_n^2 \leq b_{n+1}^2 \leq b_n^2$ and $b_n^2 = b_{n+1}^2$ $(n \geq 0)$. Consider two cases:

Case 1. $b_n = b_{n+1}$ for all n. In this case the $a_o, a_1, \ldots$ is an arithmetic progression and since $a_o = a_k$ all the a's must be equal so that $a_n = a_{n+2}$.

Case 2. For some m, $b_m = -b_{m+1}$. This implies $a_m = a_{m+2}$. i.e. If $a_m = c$ and $f(c) = d$ then $f(d) = c$. Hence beyond a certain stage the a's are alternately c and d. However since $a_o = a_k = a_{2k} = \ldots$, a_o must be either c or d and the proof is complete.