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INFINITELY MANY SOLUTIONS FOR NONLOCAL SYSTEMS INVOLVING FRACTIONAL LAPLACIAN UNDER NONCOMPACT SETTINGS

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Abstract

In this paper, we study a class of Brezis–Nirenberg problems for nonlocal systems, involving the fractional Laplacian $(-\Delta)^s$ operator, for $0 < s < 1$, posed on settings in which Sobolev trace embedding is noncompact. We prove the existence of infinitely many solutions in large dimension, namely when $N > 6s$, by employing critical point theory and concentration estimates.

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1. Introduction and main result

In this paper, we are concerned with the study of the existence of infinitely many solutions for the following elliptic system for a given $0 < s < 1$:

$$\begin{cases} (-\Delta)^s u = \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} + au + bv & \text{in } \Omega, \\ (-\Delta)^s v = \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v + bu + cv & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{S^s_Ω}$$

where $(-\Delta)^s$ is the fractional power of the positive $-\Delta$ Laplacian operator, $\Omega \subset \mathbb{R}^N$, $N > 2s$, is an open bounded domain and the constants α and β and the coefficients a, b and c satisfy the following assumptions, respectively:

($\mathcal{H}_1$) $\alpha > 1, \beta > 1,\ \alpha + \beta = 2^*_s := (2N/N - 2s)$;
($\mathcal{H}_2$) $a, c > 0, b \neq 0, ac - b^2 > 0$.

When $\alpha = \beta$, $a = c = \lambda > 0$, $b = 0$ and $u = v$, problem (S^s_Ω) reduces to the semilinear scalar fractional elliptic problem

$$\begin{cases}(-\Delta)^s u = |u|^{2^*_s-2}u + \lambda u & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega.\end{cases} \tag{P^s_Ω}$$

In [10], Devillanova and Solimini considered the following problem:

$$-\Delta u = |u|^{2^*-2}u + \lambda u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

(which corresponds to (P^s_Ω) for $s = 1$) and obtained infinitely many solutions for every $\lambda > 0$ provided $N \geq 7$ (that is, for $N > 6$). This work has been extended to the corresponding problem involving the p-Laplacian operator (with $p < N$)

$$-\Delta_p u = |u|^{p^*-2}u + \lambda |u|^{p-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

(where $\lambda > 0$ and $p^* := (pN/N - p)$) by Cao *et al.* They proved in [7] that the problem admits infinitely many solutions provided $N > p^2 + p$. Recently, Yan *et al.* approached problem (P^s_Ω) for $0 < s < 1$ and obtained in [23] the existence of infinitely many solutions for every $\lambda > 0$ provided $N > 6s$.

Systems similar to (S^s_Ω), but involving Laplacian or p-Laplacian operators, have been studied extensively in recent years; see [1, 9, 17, 19, 22] and the references therein. In particular, Liu studied the p-Laplacian systems with critical growth, that is, with $\alpha + \beta = p^*$, and obtained in [16] the existence of infinitely many solutions provided $N > p^2 + p$.

Motivated by the large use of fractional powers of the Laplacian in modelling diffusion processes (like Lévy stable diffusion processes, flame propagation, population dynamics and chemical reactions in liquids) and layer solutions (see [15, 20]), in this paper we extend the above-mentioned multiplicity results to system (S^s_Ω) in the corresponding critical case, that is, when $\alpha + \beta = 2^*_s$. We shall take advantage of the variational methods nowadays available thanks to the new formulation of the fractional Laplacian operator, given, for $s = 2^{-1}$, in [6] by Caffarelli and Silvestre, in [18] by Stinga and Torrea and in [8] by Capella *et al.* for $0 < s < 1$ (via harmonic extensions; see Definition 2.3 below). (This new formulation known as *Dirichlet to Neumann mapping* in the case $s = 2^{-1}$ of the fractional Laplacian $(-\Delta)^s$ allows us to transform nonlocal problems to local ones even if one has to deal with one more dimension.) We shall prove that, for any $0 < s < 1$, (S^s_Ω) has infinitely many solutions provided $N > 6s$.

We shall adapt some ideas and techniques used in [10] (see also [1, 7, 17, 23]). In particular, we shall introduce a sequence of perturbed (subcritical) systems (whose corresponding energy functional satisfies the Palais–Smale (P.S. for short) property at all energy levels) which 'approximates' system (S^s_Ω). Then suitable decay estimates on the set of the sequences consisting in solutions to the introduced subcritical approximating systems, and a (local) Pohozaev identity, will be used in order to establish a global compactness argument, which, thanks to min–max theorems

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employed on a genus homotopic class, will guarantee the existence of infinitely many critical values.

The paper is organized as follows. In Section 2, we review the fractional Laplacian, s-harmonic extension and the extended local problems posed on half-infinite cylinders. In Section 3, we establish a local Pohozaev identity, which allows the proof of the compactness result for sequences of solutions of the perturbed system and, finally, in Section 4, we show how this technique allows us to prove, through the application of classical min–max arguments combined with the topological genus, the existence of infinitely many solutions.

The main result is the following theorem.

THEOREM 1.1. *Under assumptions* ($\mathcal{H}_1$) *and* ($\mathcal{H}_2$), *problem* (S^s_Ω) *has infinitely many solutions provided* $N > 6s$.

2. Notation and preliminaries

In this paper we shall make use of the following notation. For any $0 < s < 1$, $(-\Delta)^s$ is the fractional power of the positive Laplacian operator $-\Delta$ acting on functions defined in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, $N > 2s$, and satisfying Dirichlet boundary conditions. In the paper we shall adopt the definition of $(-\Delta)^s$ given by the spectral decomposition of the $-\Delta$ operator (as in [4], which is based on [5]) according to which any normalized positive eigenfunction ψ (with related eigenvalue $\lambda = \|\nabla\psi\|^2_{L^2(\Omega)} = \int_\Omega -\Delta\psi\psi\,dx$) of $-\Delta$ operator is still an eigenfunction of the $(-\Delta)^s$ operator (with $\lambda^s = \|\nabla\psi\|^{2s}_{L^2(\Omega)}$ as related eigenvalue). (Note that then $\int_\Omega (-\Delta)^s\psi\psi\,dx = \lambda^s = (\int_\Omega -\Delta\psi\psi\,dx)^s$.) By taking into account that the sequence $(\psi_j)_{j\in\mathbb{N}}$ of (positive normalized) eigenfunctions of $-\Delta$ (with $(\lambda_j)_{j\geq 1}$ as corresponding sequence of eigenvalues) is an orthonormal basis of $L^2(\Omega)$, we can introduce the following subspace of $L^2(\Omega)$:

$$H(\Omega) := \Big\{ u = \sum_{j\geq 1} a_j\psi_j \in L^2(\Omega) : \sum_{j\geq 1} a_j^2\lambda_j^s < +\infty \Big\} \tag{2.1}$$

equipped with the norm

$$\|u\|_{H(\Omega)} = \Big(\sum_{j\geq 1} a_j^2\lambda_j^s\Big)^{1/2} \quad \text{for all } u = \sum_{j\geq 1} a_j\psi_j \in H(\Omega). \tag{2.2}$$

Denote by $H_0^s(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{H^s(\Omega)} = \|u\|_{L^2(\Omega)} + \Big(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{1/2}$$

and set

$$H_{00}^{1/2}(\Omega) := \Big\{ u \in H^{1/2} : \int_\Omega \frac{u^2(x)}{\mathrm{dist}(x,\partial\Omega)}\,dx < +\infty \Big\}.$$

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We have, see [5], that

$$H(\Omega) = \begin{cases} H_0^s(\Omega) & \text{if } 0 < s < 1, s \neq \frac{1}{2}, \\ H_{00}^{1/2}(\Omega) & \text{if } s = \frac{1}{2}. \end{cases}$$

We can define on $H(\Omega)$ the linear operator $(-\Delta)^s$ by setting

$$(-\Delta)^s u = \sum_{j\geq 1} a_j \lambda_j^s \psi_j \quad \text{for all } u = \sum_{j\geq 1} a_j \psi_j \in H(\Omega). \tag{2.3}$$

Moreover, since $(\psi_j)_{j\geq 1}$ is orthonormal in $L^2(\Omega)$, then $\|u\|_{H(\Omega)} = \|(-\Delta)^{s/2}u\|_{L^2(\Omega)}$, so that for all $u \in H(\Omega)$, $(-\Delta)^{s/2}u \in L^2(\Omega)$. Moreover, since for all $u, v \in H(\Omega)$,

$$\int_\Omega (-\Delta)^{s/2}u(-\Delta)^{s/2}v = \int_\Omega (-\Delta)^s uv$$

(that is, a certain kind of integration by parts formula holds), we can give the following definition on some elements of the set

$$E(\Omega) := H(\Omega) \times H(\Omega). \tag{2.4}$$

Definition 2.1. A pair of functions $(u, v) \in E(\Omega)$ is said to be a weak solution of system (S_Ω^s) if, for all $(\varphi_1, \varphi_2) \in E(\Omega)$,

$$\begin{aligned}\int_\Omega ((-\Delta)^{s/2}u(-\Delta)^{s/2}\varphi_1 + (-\Delta)^{s/2}v(-\Delta)^{s/2}\varphi_2)\, dx \\ - \int_\Omega (au\varphi_1 + b(v\varphi_1 + u\varphi_2) + cv\varphi_2)\, dx \\ - \frac{2\alpha}{\alpha+\beta}\int_\Omega |u|^{\alpha-2}u|v|^\beta \varphi_1\, dx - \frac{2\beta}{\alpha+\beta}\int_\Omega |u|^\alpha |v|^{\beta-2}v\varphi_2\, dx = 0.\end{aligned}$$

Remark 2.2. Note that condition $(\mathcal{H}_1)$ guarantees that the last two integrals in the above equality are finite. On the other hand, the condition $b \neq 0$ in $(\mathcal{H}_2)$ is used to exclude the semitrivial solutions of (S_Ω^s), that is, solutions of the form $(u, 0)$ and $(0, v)$ with u and v eigenfunctions of the $(-\Delta)^s$ operator.

Indeed, if $(\mathcal{H}_2)$ holds true, then the quadratic form $Q(x, y) := ax^2 + 2bxy + cy^2$ is positive definite and satisfies

$$\mu_1(x^2 + y^2) \leq Q(x, y) \leq \mu_2(x^2 + y^2) \quad \forall (x, y) \in \mathbb{R}^2,$$

where $\mu_1 < \mu_2$ are the eigenvalues of the coefficient matrix $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ of linear terms in (S_Ω^s).

Solutions to (S_Ω^s) will then be obtained as critical points of the following energy functional F defined by setting for all $(u, v) \in E_0^s(\Omega)$,

$$\begin{aligned}F(u, v) := \frac{1}{2}\int_\Omega (|(-\Delta)^{s/2}u|^2 + |(-\Delta)^{s/2}v|^2)\, dx \\ - \frac{1}{2}\int_\Omega (a|u|^2 + 2buv + c|v|^2)\, dx \\ - \frac{2}{\alpha+\beta}\int_\Omega |u|^\alpha |v|^\beta\, dx.\end{aligned} \tag{2.5}$$

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Notice that, in view of embedding theorems [11, Corollary 7.2], the functional F is well defined and critical points of F are solutions to (S^s_Ω).

We conclude this section by sketching the main ingredients of a recent technique developed in [6] (for $s = 2^{-1}$) and in [8] (for $0 < s < 1$) to deal with fractional powers of the Laplacian operator. In order to treat the nonlocal system (S^s_Ω), we shall follow the same approach in [6, 14] by introducing a *corresponding extension problem*, which allows us to investigate system (S^s_Ω) by studying a local system (which involves one more dimension) but still via classical variational methods.

First we need to introduce some notation to deal with the auxiliary dimension. We shall work on (subsets of) the upper half-space in $\mathbb{R}^{N+1}$, denoted by $\mathbb{R}^{N+1}_+$, and we shall split any element $z \in \mathbb{R}^{N+1}_+$ as a pair (x, y) with $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $y > 0$, so that

$$\mathbb{R}^{N+1}_+ := \mathbb{R}^N \times \mathbb{R}_+ = \{z = (x, y) = (x_1, \dots, x_N, y) \in \mathbb{R}^{N+1} \mid y > 0\}.$$

Given any (smooth bounded) domain $D \subset \mathbb{R}^N$, we shall denote by

$$C_D := D \times (0, +\infty) \subset \mathbb{R}^{N+1}_+$$

the half-cylinder with base D and by

$$\partial_L C_D := \partial D \times [0, +\infty) \quad \text{and} \quad \partial_B C_D := \overline{D} \times \{0\}$$

its lateral and base boundaries, respectively. Now we are ready to recall the following definition (given for instance in [4]).

DEFINITION 2.3. The s-harmonic extension $U = \mathcal{E}_s(u)$ of a function $u \in H(\Omega)$ to the (open) cylinder C_Ω is the (unique) weak solution to the problem

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla w(x, y)) = 0, & x \in \Omega,\ y > 0, \\ w(x, y) = 0, & x \in \partial\Omega,\ y > 0, \\ w(x, 0) = u(x), & x \in \Omega. \end{cases} \tag{2.6}$$

Notice that, for all $u \in H(\Omega)$, the related s-extension $\mathcal{E}_s(u)$ belongs to the space

$$H^1_{0,L}(C_\Omega) := \left\{ w \in H^1(C_\Omega) : w = 0 \text{ on } \partial_L C_\Omega,\ \int_{C_\Omega} y^{1-2s} |\nabla w|^2 \, dx \, dy < \infty \right\} \tag{2.7}$$

equipped with the homogeneous norm

$$\|w\|_{H^1_{0,L}(C_\Omega)} := \left(\int_{C_\Omega} y^{1-2s} |\nabla w|^2 \, dx \, dy \right)^{1/2}.$$

Since

$$\|u\|_{H(\Omega)} = \|\mathcal{E}_s(u)\|_{H^1_{0,L}(C_\Omega)} \quad \text{for all } u \in H(\Omega),$$

the extension operator $\mathcal{E}_s$ (which maps u into $\mathcal{E}_s(u)$) is an isometry between $H(\Omega)$ and $H^1_{0,L}(C_\Omega)$. *Vice versa*, for every function $w \in H^1_{0,L}(C_\Omega)$, set

$$\operatorname{tr}_N(w) = w(\cdot, 0),$$

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the related trace on the hyperspace $\{(x,y) \in \mathbb{R}^{N+1}_+ : y = 0\}$; we have that $\mathrm{tr}_N(w) \in H(\Omega)$. Moreover, by Definition 2.3,

$$\mathrm{tr}_N(\mathcal{E}_s(u)) = u \quad \forall u \in H(\Omega)$$

and, by using the divergence theorem and (2.7),

$$\int_{C_\Omega} \mathrm{div} w \, dx \, dy = \int_\Omega \mathrm{tr}_N(w) \, dx \quad \forall w \in H^1_{0,L}(\Omega). \tag{2.8}$$

We shall now express the fractional Laplacian operator $(-\Delta)^s$ of a function $u \in H(\Omega)$ through the trace of a suitable (directional) derivative (along the outward normal to $\partial_B(C_\Omega)$), denoted by $\boldsymbol{\partial}_s$, of its s-harmonic extension $\mathcal{E}_s(u)$. Indeed, for almost every $x \in \Omega$,

$$(-\Delta)^s u(x) = -k_s \lim_{y \to 0^+} y^{1-2s} \frac{\partial \mathcal{E}_s(u)}{\partial y}(x,y), \tag{2.9}$$

where $k_s = 2^{1-2s}\Gamma(1-s)/\Gamma(s)$ is a normalization constant and Γ denotes the Euler Gamma function. Formula (2.9), for $s = 2^{-1}$, justifies the name, given to the square root of the Laplacian, as the Dirichlet to Neumann map (since it maps u, that is, the Dirichlet data for $\mathcal{E}_s(u)$ on $\partial_B C_\Omega$ into the trace of its outer normal derivative $\partial\mathcal{E}_s(u)/\partial\nu$).

To shorten the notation, for any smooth bounded domain $D \subset \mathbb{R}^N$, we shall introduce the operator $\boldsymbol{\partial}_s$ defined by setting for all $w \in H^1_{0,L}(C_D)$,

$$\boldsymbol{\partial}_s w(x,0) := -k_s \lim_{y \to 0^+} y^{1-2s} \frac{\partial w}{\partial y}(x,y) \quad \forall x \in \Omega. \tag{2.10}$$

Then (2.9) states that, for all $u \in H(\Omega)$,

$$(-\Delta)^s u(x) = \boldsymbol{\partial}_s \mathcal{E}_s(u)(x,0) \quad \forall x \in \Omega. \tag{2.11}$$

In the remaining part of the paper we shall take the constant k_s in (2.9) equal to 1.

Now, by taking into account that $(-\Delta)^s u$ is expressed, through formula (2.11), in terms of its s-harmonic extension $\mathcal{E}_s(u)$, and since $\mathcal{E}_s(u)$ must solve (2.6), we can associate to system (S^s_Ω) the following (local) extended system:

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla w_1(x,y)) = 0, & (x,y) \in C_\Omega, \\ -\mathrm{div}(y^{1-2s}\nabla w_2(x,y)) = 0, & (x,y) \in C_\Omega, \\ w_1(x,y) = w_2(x,y) = 0, & (x,y) \in \partial_L C_\Omega, \\ \boldsymbol{\partial}_s w_1(x,0) = \dfrac{2\alpha}{\alpha+\beta}|w_1(x,0)|^{\alpha-2} w_1(x,0)|w_2(x,0)|^\beta & \\ \qquad + a w_1(x,0) + b w_2(x,0), & x \in \Omega, \\ \boldsymbol{\partial}_s w_2(x,0) = \dfrac{2\beta}{\alpha+\beta}|w_1(x,0)|^{\alpha}|w_2(x,0)|^{\beta-2} w_2(x,0) & \\ \qquad + b w_1(x,0) + c w_2(x,0), & x \in \Omega, \end{cases} \tag{$\mathrm{S}^s_{C_\Omega}$}$$

whose solutions (w_1, w_2) belong to the set

$$E^1_{0,L}(C_\Omega) := H^1_{0,L}(C_\Omega) \times H^1_{0,L}(C_\Omega)$$

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equipped with the norm

$$\|(w_1, w_2)\|_{E^1_{0,L}(C_\Omega)} = \|w_1\|_{H^1_{0,L}(C_\Omega)} + \|w_2\|_{H^1_{0,L}(C_\Omega)}.$$

By taking into account (2.10) and (2.8), we shall say that $(w_1, w_2) \in E^1_{0,L}(C_\Omega)$ is an *energy solution* to system ($S^s_{C_\Omega}$) if

$$\begin{aligned}
&\int_{C_\Omega} y^{1-2s}\nabla w_1 \nabla\varphi_1\,dx\,dy + \int_{C_\Omega} y^{1-2s}\nabla w_2\nabla\varphi_2\,dx\,dy\\
&\quad= \int_{C_\Omega} (aw_1\varphi_1 + b(w_2\varphi_1 + w_1\varphi_2) + cw_2\varphi_2)\,dx\,dy\\
&\qquad+\frac{2\alpha}{\alpha+\beta}\int_{C_\Omega} |w_1|^{\alpha-2}w_1|w_2|^{\beta}\varphi_1\,dx\,dy + \frac{2\beta}{\alpha+\beta}\int_{C_\Omega}|w_1|^{\alpha}|w_2|^{\beta-2}w_2\varphi_2\,dx\,dy
\end{aligned}$$

holds true for all $(\varphi_1, \varphi_2) \in E^1_{0,L}(C_\Omega)$.

The energy functional corresponding to system ($S^s_{C_\Omega}$) is then defined by setting for all $(w_1, w_2) \in E^1_{0,L}(C_\Omega)$,

$$\begin{aligned}
I(w_1, w_2) = \frac{1}{2}&\int_{C_\Omega} y^{1-2s}(|\nabla w_1|^2 + |\nabla w_2|^2)\,dx\,dy\\
&-\frac{1}{2}\int_{C_\Omega}(a|w_1|^2 + 2bw_1w_2 + c|w_2|^2)\,dx\,dy\\
&-\frac{2}{\alpha+\beta}\int_{C_\Omega}|w_1|^{\alpha}|w_2|^{\beta}\,dx\,dy.
\end{aligned}$$

REMARK 2.4. Formula (2.11) and Definition 2.3 allow us to see system ($S^s_{C_\Omega}$) as an 'extension' of the system (S^s_Ω). Indeed, if $(w_1, w_2) \in E^1_{0,L}(C_\Omega)$ satisfies ($S^s_{C_\Omega}$), then set $u = \mathrm{tr}_N(w_1)$ and $v = \mathrm{tr}_N(w_2)$; the pair (u, v) belongs to the space $E(\Omega)$ and is a solution to the original system (S^s_Ω). Moreover, the traces on Ω of critical points of I in $E^1_{0,L}(C_\Omega)$ are critical points in $E(\Omega)$ of the functional F defined by (2.5).

In order to shorten the notation, we shall consider the following quantities depending on the index: $i \in \{1, 2\}$

$$\gamma_i := \begin{cases}\alpha & \text{for } i = 1,\\ \beta & \text{for } i = 2,\end{cases}\qquad c_i := \begin{cases} a & \text{for } i = 1,\\ c & \text{for } i = 2.\end{cases}$$

In this way we can rewrite system ($S^s_{C_\Omega}$) as a pair of systems $\boldsymbol{S}_{1,2}$ and $\boldsymbol{S}_{2,1}$, where, for $i \neq j \in \{1, 2\}$,

$$\begin{cases}
-\mathrm{div}(y^{1-2s}\nabla w_i(x, y)) = 0, & (x, y) \in C_\Omega,\\
w_i(x, y) = 0, & (x, y) \in \partial_L\Omega,\\
\partial_s w_i(x, 0) = \dfrac{2\gamma_i}{\gamma_i+\gamma_j}|w_i(x, 0)|^{\gamma_i-2}w_i(x, 0)|w_j(x, 0)|^{\gamma_j} & \\
\qquad\qquad + c_iw_i(x, 0) + bw_j(x, 0), & i \neq j,\ x \in \Omega
\end{cases}\tag{$S_{i,j}$}$$

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and we shall also use the notation $\boldsymbol{S}^s_{i\neq j}(C_\Omega)$ to refer to system ($\mathrm{S}^s_{C_\Omega}$). Since we work under the critical growth assumption (see ($\mathcal{H}_1$)), the functional I does not satisfy the P.S. condition at every energy level, so we cannot apply the usual min–max theorems to obtain infinitely many solutions to ($\mathrm{S}^s_{C_\Omega}$) (and by Remark 2.4 to (S^s_Ω)). Following the original idea in [10] (employed also in [23]), we deal first with a suitable perturbed system. Given a suitably small positive number $\varepsilon > 0$, namely $0 < \varepsilon < \min\{\alpha - 1, \beta - 1, 2^*_s - 1\}$, and setting

$$\gamma_i^\varepsilon := \begin{cases} \alpha - \varepsilon & \text{for } i = 1, \\ \beta - \varepsilon & \text{for } i = 2, \end{cases} \tag{2.12}$$

we shall consider the perturbed system ($\mathrm{S}^{s,\varepsilon}_{C_\Omega}$) given by the pair of systems ($\boldsymbol{S}^\varepsilon_{1,2}$) and $\boldsymbol{S}^\varepsilon_{2,1}$, where, for $i \neq j \in \{1, 2\}$,

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla w_i(x,y)) = 0, & (x,y) \in C_\Omega, \\ w_i(x,y) = 0, & (x,y) \in \partial_L C_\Omega, \\ \partial_s w_i(x,0) = \dfrac{2\gamma_i^\varepsilon}{\gamma_i^\varepsilon + \gamma_j^\varepsilon}|w_i(x,0)|^{\gamma_i^\varepsilon - 2} w_i(x,0)|w_j(x,0)|^{\gamma_j^\varepsilon} + & \\ \qquad + c_i w_i(x,0) + b w_j(x,0), & i \neq j,\ x \in \Omega. \end{cases} \tag{$S^\varepsilon_{i,j}$}$$

The functional I^ε corresponding to system ($\mathrm{S}^{s,\varepsilon}_{C_\Omega}$) is defined by setting for all $(w_1, w_2) \in E^1_{0,L}(C_\Omega)$,

$$\begin{aligned} I^\varepsilon(w_1, w_2) = \frac{1}{2}\int_{C_\Omega} & y^{1-2s}(|\nabla w_1|^2 + |\nabla w_2|^2)\,dx\,dy \\ & - \frac{1}{2}\int_{C_\Omega} (a|w_1|^2 + 2bw_1w_2 + c|w_2|^2)\,dx\,dy + \\ & - \frac{2}{\alpha + \beta - 2\varepsilon}\int_{C_\Omega} |w_1|^{\alpha-\varepsilon}|w_2|^{\beta-\varepsilon}\,dx\,dy. \end{aligned}$$

Note that I^ε is even and satisfies the P.S. condition at all energy levels and so, by applying the symmetric mountain pass lemma (see [2, Corollary 2.9]), we deduce that system ($S^\varepsilon_{i,j}$) has infinitely many solutions for each given $\varepsilon > 0$. More precisely, it follows from [13, Theorem 6.1] that, fixing any small enough $\varepsilon > 0$ and any integer number k, there exist $c^\varepsilon_k > 0$ and $(w_1^{\varepsilon,k}, w_2^{\varepsilon,k})$ solutions to ($S^\varepsilon_{i,j}$) belonging to a suitable set A_k topologically dependent on k such that $I^\varepsilon(w_1^{\varepsilon,k}, w_2^{\varepsilon,k}) = c^\varepsilon_k$. In such a way one gets a sequence $(c^\varepsilon_k)_{k\in\mathbb{N}}$ of positive numbers and a sequence $((w_1^{\varepsilon,k}, w_2^{\varepsilon,k}))_{k\in\mathbb{N}}$ of solutions to ($S^\varepsilon_{i,j}$) such that $c^\varepsilon_k \to +\infty$ as $k \to +\infty$. The idea is then to fix an infinitesimal sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of real positive numbers and, for any $k \in \mathbb{N}$, to take the sequence $((w^k_{1,n}, w^k_{2,n}))_{n\in\mathbb{N}} := ((w_1^{\varepsilon_n,k}, w_2^{\varepsilon_n,k}))_{n\in\mathbb{N}}$ whose elements belong to the same set A_k. To obtain the existence of infinitely many solutions for (the approximated (extended) system) ($\mathrm{S}^s_{C_\Omega}$) (and by Remark 2.4 to system (S^s_Ω)), the first step is to investigate whether, for fixed $k \in \mathbb{N}$, $((w^k_{1,n}, w^k_{2,n})_{n\in\mathbb{N}}$ converges strongly in $E^1_{0,L}(C_\Omega)$ as $n \to +\infty$.

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3. Sequential compactness of the set of approximating balanced sequences in $E^1_{0,L}(C_\Omega)$

We shall adapt to this context the notion of a balanced sequence first introduced in [10].

Definition 3.1. Let $((w_{1,n}, w_{2,n}))_{n\in\mathbb{N}}$ be a sequence in $E^1_{0,L}(C_\Omega)$. We shall say that $((w_{1,n}, w_{2,n}))_{n\in\mathbb{N}}$ is a *balanced sequence* if for all $n \in \mathbb{N}$, $(w_{1,n}, w_{2,n})$ is a solution to $(S^\varepsilon_{i,j})$ for some $\varepsilon = \varepsilon_n > 0$. If, in addition, $\varepsilon_n \to 0$ as $n \to +\infty$, we shall say that $((w_{1,n}, w_{2,n}))_{n\in\mathbb{N}}$ is an *approximating balanced sequence*.

In this section, we shall establish the following compactness result.

Proposition 3.2. *Let $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold true and assume that $N > 6s$. Then every approximating balanced sequence which is bounded in $E^1_{0,L}(\Omega)$ is, modulo a subsequence, strongly convergent in $E^1_{0,L}(C_\Omega)$ to a solution to $(S^s_{C_\Omega})$.*

Before giving the proof of Proposition 3.2, we need some estimates for a solution (w_1, w_2) to $(S^\varepsilon_{i,j})$ for $\varepsilon > 0$. In correspondence to any smooth domain $D \subset \mathbb{R}^N$ such that $\Omega \subset\subset D$, we shall replace any solution (w_1, w_2) to $(S^s_{C_\Omega})$ by the related null extension to C_D obtained by taking null traces out of Ω, that is, by imposing $(w_1(x, 0), w_2(x, 0)) = (0, 0)$ in $D \setminus \Omega$.

We choose a constant $A > 0$ large enough so that for all $x, y \in \mathbb{R}$, and for all suitably small $\varepsilon > 0$, the following inequality system is verified.

$$\begin{cases} \left|\dfrac{2\alpha-\varepsilon}{\alpha+\beta-\varepsilon}|x|^{\alpha-2-\varepsilon}x|y|^{\beta-\varepsilon} + ax + by\right| \le 2(|x| + |y|)^{2^*_s-1} + A, \\ \left|\dfrac{2\beta-\varepsilon}{\alpha+\beta-\varepsilon}|x|^{\alpha-\varepsilon}|y|^{\beta-2-\varepsilon}y + bx + cy\right| \le 2(|x| + |y|)^{2^*_s-1} + A. \end{cases} \tag{3.1}$$

Let $w \in H^1_{0,L}(C_D)$, $w \ge 0$, be a solution to

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla w(x, y)) = 0, & x \in D,\ y > 0, \\ w(x, y) = 0, & x \in \partial D,\ y > 0, \\ \partial_s w(x, 0) = 2(|w(x, 0)| + |w(x, 0)|)^{2^*_s-1} + A, & x \in D. \end{cases} \tag{3.2}$$

Lemma 3.3. *Let $A > 0$ be as in (3.1) and w a solution of (3.2), where $D \subset\subset \mathbb{R}^N$. Then, if (w_1, w_2) is a solution to $(S^\varepsilon_{i,j})$ for some $\varepsilon > 0$,*

$$|w_i| \le w \quad \forall i \in \{1, 2\} \text{ and therefore } \tfrac{1}{2}(|w_1| + |w_2|) \le w. \tag{3.3}$$

Proof. By suitably choosing the domain D and the constant A, since (w_1, w_2) solves $(S^\varepsilon_{i,j})$, we find that, for $i = 1, 2$, $w \pm w_i$ satisfies the following system:

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla(w \pm w_i)(x, y)) = 0, & x \in D, y > 0, \\ w(x, y) \pm w_i(x, y) = 0, & x \in \partial D, y > 0, \\ \partial_s(w \pm w_i)(x, 0) \ge 0, & x \in D. \end{cases} \tag{3.4}$$

□

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The above lemma allows us to care about estimates on C_Ω to positive solutions w to (3.2).

Let

$$\begin{aligned} \mathcal{S}_{s,\alpha,\beta} &:= \inf_{(w_1,w_2)\in E^1_{0,L}(C_\Omega)} \frac{\int_{C_\Omega} y^{1-2s}(|\nabla w_1|^2 + |\nabla w_2|^2)\,dx\,dy}{\left(\int_\Omega |w_1(x,0)|^\alpha |w_2(x,0)|^\beta\,dx\right)^{2/2^*_s}} \\ &= \inf_{(w_1,w_2)\in E^1_{0,L}(C_\Omega)} \frac{\|(w_1,w_2)\|^2_{E^1_{0,L}(C_\Omega)}}{\|(\mathrm{tr}_N(w_1),\mathrm{tr}_N(w_2))\|^2_{L^{\alpha,\beta}_s(\Omega)}}, \end{aligned}$$

where

$$L^{\alpha,\beta}_s(\Omega) := \left\{ f = (f_1,f_2) : \|(f_1,f_2)\|^2_{L^{\alpha,\beta}_s(\Omega)} := \int_\Omega |f_1(x)|^\alpha |f_2(x)|^\beta d < +\infty \right\} \subset E^s_0(\Omega).$$

Then (see [14, Lemma 2.5])

$$\mathcal{S}_{s,\alpha,\beta} = \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*_s} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*_s} \right] \mathcal{S}(s,N),$$

where $\mathcal{S}(s,N)$ is the related (best) Sobolev constant defined by

$$\begin{aligned} \mathcal{S}(s,N) &:= \inf_{w\in H^1_{0,L}(C_\Omega)\setminus\{0\}} \frac{\int_{C_\Omega} y^{1-2s}|\nabla w(x,y)|^2\,dx\,dy}{(\int_\Omega |w(x,0)|^{2^*_s}\,dx)^{2/2^*_s}} \\ &= \inf_{w\in H^1_{0,L}(C_\Omega)\setminus\{0\}} \frac{\|w\|^2_{H^1_{0,L}(C_\Omega)}}{\|\mathrm{tr}_N(w)\|^2_{L^{2^*_s}(\Omega)}}, \end{aligned} \tag{3.5}$$

which is achieved if and only if $\Omega = \mathbb{R}^N$ by any function $w_\varepsilon = \mathcal{E}_s(u_\varepsilon)$ which is the s-harmonic extension of (translations of)

$$u_\varepsilon(x) := \frac{\varepsilon^{(N-2s)/2}}{(\varepsilon^2 + |x|^2)^{(N-2s)/2}}, \quad \varepsilon > 0, \quad x \in \mathbb{R}^N;$$

see for instance [3] and [21] for the case $s = 2^{-1}$.

Now we introduce the 'system on the whole space'

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla \tilde{w}_1(x,y)) = 0, & (x,y) \in \mathbb{R}^{N+1}_+, \\ -\mathrm{div}(y^{1-2s}\nabla \tilde{w}_2(x,y)) = 0, & (x,y) \in \mathbb{R}^{N+1}_+, \\ \partial_s \tilde{w}_1(x,0) = \dfrac{2\alpha}{\alpha+\beta}\tilde{w}_1^{\alpha-1}(x,0)\tilde{w}_2^{\beta}(x,0), & x \in \mathbb{R}^N, \\ \partial_s \tilde{w}_2(x,0) = \dfrac{2\beta}{\alpha+\beta}\tilde{w}_1^{\alpha}(x,0)\tilde{w}_2^{\beta-1}(x,0), & x \in \mathbb{R}^N. \\ \tilde{w}_1(x,0) > 0,\ \tilde{w}_2(x,0) > 0, & x \in \mathbb{R}^N. \end{cases} \tag{3.6}$$

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Set

$$W_1 = \left(\frac{2\alpha}{\alpha+\beta}\right)^{1/\alpha+\beta-2}\left(\frac{\beta}{\alpha}\right)^{\beta/2(\alpha+\beta-2)}\tilde{w}_1 \quad \text{and} \quad W_2 = \left(\frac{2\beta}{\alpha+\beta}\right)^{1/\alpha+\beta-2}\left(\frac{\alpha}{\beta}\right)^{\alpha/2(\alpha+\beta-2)}\tilde{w}_2;$$

then system (3.6) can be rewritten as

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla W_1(x,y)) = 0, & (x,y) \in \mathbb{R}^{N+1}_+, \\ -\mathrm{div}(y^{1-2s}\nabla W_2(x,y)) = 0, & (x,y) \in \mathbb{R}^{N+1}_+, \\ \boldsymbol{\partial}_s W_1(x,0) = W_1^{\alpha-1}(x,0)W_2^{\beta}(x,0), & x \in \mathbb{R}^N, \\ \boldsymbol{\partial}_s W_2(x,0) = W_1^{\alpha}(x,0)W_2^{\beta-1}(x,0), & x \in \mathbb{R}^N. \\ W_1(x,0) > 0,\ W_2(x,0) > 0, & x \in \mathbb{R}^N. \end{cases} \tag{3.7}$$

The following lemma guarantees that solutions to system (3.7) have the same trace on $\mathbb{R}^N$.

Lemma 3.4. *Let (W_1, W_2) be a solution to (3.7). Then $\mathrm{tr}_N(W_1) = \mathrm{tr}_N(W_2)$.*

Proof. Let (W_1, W_2) be a solution of (3.7). Then

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla(W_1 - W_2)) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \boldsymbol{\partial}_s(W_1 - W_2)(x,0) = W_1^{\alpha-1}(x,0)W_2^{\beta-1}(x,0)(W_2 - W_1) & \text{on } \mathbb{R}^N \times \{0\}, \\ W_1 > 0, \quad W_2 > 0 & \text{in } \mathbb{R}^N \times \{0\}. \end{cases} \tag{3.8}$$

By multiplying (3.8) by $(W_1 - W_2)(\cdot, 0)$ and integrating by parts,

$$\begin{aligned} &\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla_x(W_1 - W_2)(x,0)|^2\,dx\,dy \\ &\quad = -\int_{\mathbb{R}^N} W_1^{\alpha-1}(x,0)W_2^{\beta-1}(x,0)(W_1 - W_2)^2(x,0)\,dx \le 0, \end{aligned}$$

which implies that

$$W_1(\cdot,0) - W_2(\cdot,0) = \text{const.} \tag{3.9}$$

□

Notice that by the last requirement in (3.8), $W_1(\cdot,0) > 0$, $W_2(\cdot,0) > 0$ in $\mathbb{R}^N$. Therefore, from (3.7), it follows that $W_1^{\alpha-1}(\cdot,0)W_2^{\beta}(\cdot,0) = W_1^{\alpha}(\cdot,0)W_2^{\beta-1}(\cdot,0) = \boldsymbol{\partial}_s(W_1 - W_2)(\cdot,0)$ in $\mathbb{R}^N$. Hence, by (3.9), $W_1(\cdot,0) = W_2(\cdot,0)$, that is, $\mathrm{tr}_N(W_1) = \mathrm{tr}_N(W_2)$.

From Proposition A.1 in the Appendix, we have the global compactness result on (S^s_Ω). Let $(w_{1,n}, w_{2,n})_{n\in\mathbb{N}}$ be an approximating balanced sequence (see Definition 3.1) bounded in $E^s_0(\mathcal{C}_\Omega)$. Then, modulo subsequences, there exist k sequences of mutually diverging scalings (σ_n^i, x_n^i) (determined by the *scaling factor* σ_n^i and the *concentration point* x_n^i) such that as $n \to +\infty$,

$$\begin{cases} w_{1,n} = w_{1,0} + A_0^{-1}\displaystyle\sum_{i=1}^{k}(\sigma_n^i)^{N/2_s^*}W(\sigma_n^i(x - x_n^i), 0) + r_n^1, \\ w_{2,n} = w_{2,0} + B_0^{-1}\displaystyle\sum_{i=1}^{k}(\sigma_n^i)^{N/2_s^*}W(\sigma_n^i(x - x_n^i), 0) + r_n^2, \end{cases} \tag{3.10}$$

where $(w_{1,0}, w_{2,0}) \in E_0^s(C_\Omega)$ is a weak solution to system ($S^s_{C_\Omega}$), $\|(r_n^1, r_n^2)\|_{E_0^s(C_\Omega)} \to 0$ as $n \to +\infty$, W achieves the constant $\mathcal{S}(s, N)$, which is given in (3.5), and

$$A_0 = \bigg(\bigg(\frac{2\beta}{\alpha+\beta}\bigg)^2\bigg(\frac{\beta}{\alpha}\bigg)^{\beta-2}\bigg)^{1/2(\alpha+\beta-2)}, \quad B_0 = \bigg(\bigg(\frac{2\alpha}{\alpha+\beta}\bigg)^2\bigg(\frac{\alpha}{\beta}\bigg)^{\alpha-2}\bigg)^{1/2(\alpha+\beta-2)}.$$

The terms that constitute the sum $\sum_{i=1}^k (\sigma_n^i)^{N/2_s^*} W(\sigma_n^i(\cdot - x_n^i), 0)$ which appear in (3.10) are called bubbles or profiles of the pair $(w_{1,n}, w_{2,n})$.

As in [7, 10, 23], we shall introduce the following facts which are essentials to prove the strong convergence of $(u_n, v_n)_{n\in\mathbb{N}}$ in $E_0^s(C_\Omega)$. Referring to the profile decomposition (3.10) of the pair $(w_{1,n}, w_{2,n})$, we shall denote by σ_n one of the slowest concentration scaling factors σ_n^i (that is, σ_n is the lowest order infinity among the ones appearing in the bubbles) and we shall set x_n as the corresponding concentration point x_n^i.

Note that, since the number of the bubbles in $(w_{1,n}, w_{2,n})$ is finite and independent of n, we can always choose a constant $\overline{C} > 0$, independent of n, such that the region

$$\mathcal{A}_n^1 := (B_{(\overline{C}+5)\sigma_n^{-1/2}}(x_n, 0) \setminus B_{\overline{C}\sigma_n^{-1/2}}(x_n, 0)) \cap C_\Omega$$

does not contain any concentration point of $(w_{1,n}, w_{2,n})$ for every $n \in \mathbb{N}$. We set two thinner subsets as follows:

$$\mathcal{A}_n^2 := (B_{(\overline{C}+4)\sigma_n^{-1/2}}(x_n, 0) \setminus B_{(\overline{C}+1)\sigma_n^{-1/2}}(x_n, 0)) \cap C_\Omega$$

and

$$\mathcal{A}_n^3 := (B_{(\overline{C}+3)\sigma_n^{-1/2}}(x_n, 0) \setminus B_{(\overline{C}+2)\sigma_n^{-1/2}}(x_n, 0)) \cap C_\Omega.$$

Then, by applying inequality (3.3) and by using [23, Propositions 4.1 and 4.2], we get the following integral estimates.

Lemma 3.5. *Let Let $(w_{1,n}, w_{2,n})_{n\in\mathbb{N}}$ be an approximating balanced sequence. Then there exists a constant $C > 0$, independent of n, such that for all $p \geq 1$,*

$$\int_{\mathcal{A}_n^2} y^{1-2s}(|w_{1,n}|^p + |w_{2,n}|^p)\,dx\,dy \leq C\sigma_n^{-N+2_s^*/2_s^*}$$

and

$$\int_{\mathrm{tr}_N(\mathcal{A}_n^2)} (|w_{1,n}|^p + |w_{2,n}|^p)\,dx \leq C\sigma_n^{-N/2}.$$

Moreover,

$$\int_{\mathcal{A}_n^3} y^{1-2s}(|\nabla w_{1,n}|^2 + |\nabla w_{2,n}|^2)\,dx\,dy \leq C\sigma_n^{-N/2_s^*}.$$

Note that with $X = (x, y) \in \mathbb{R}_+^{N+1}$, we have the following local Pohozaev identity. Let

$$F(u_n, v_n, x, x_0, \nu) := \bigg[\frac{2}{\alpha_n+\beta_n}|u_n|^{\alpha_n}|v_n|^{\beta_n} + \frac{1}{2}(au_n^2 + 2bu_nv_n + cv_n^2)\bigg]((x - x_0)\cdot\nu)$$

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and

$$G(u_n, v_n, x, y, x_0, \nu) := y^{1-2s}\Big[\frac{N-2s}{2}\Big(u_n\frac{\partial u_n}{\partial \nu} + v_n\frac{\partial v_n}{\partial \nu}\Big) - \frac{1}{2}(|\nabla u_n|^2 + |\nabla v_n|^2)((X-z_0)\cdot\nu) + (\nabla u_n\cdot(X-z_0))\frac{\partial u_n}{\partial \nu} + (\nabla v_n\cdot(X-z_0))\frac{\partial v_n}{\partial \nu}\Big],$$

where $X = (x, y) \in \mathbb{R}^{N+1}_+$ and x_0 is a point in $\mathbb{R}^N$.

Lemma 3.6. *Let $(u_n, v_n)_{n\geq1}$ be a solution of $(S^{\varepsilon}_{i,j})$ and $\alpha_n = \alpha - \varepsilon_n$, $\beta_n = \beta - \varepsilon_n$, $\varepsilon = \varepsilon_n \to 0$ as $n \to +\infty$ and B_n be any bounded set in C_Ω. Then we have the local Pohozaev identity on $B_n \subset C_\Omega$ associated to equations of $(S^{\varepsilon}_{i,j})$:*

$$\frac{4\varepsilon_n N}{2^*_s(2^*_s - 2\varepsilon_n)}\int_{B_n\cap\{y=0\}}|u_n|^{\alpha_n}|v_n|^{\beta_n}\,dx + s\int_{B_n\cap\{y=0\}}(au_n^2 + 2bu_nv_n + cv_n^2)\,dx = \int_{\partial(B_n\cap\{y=0\})}F(u_n, v_n, x, x_0, \nu)\,d\sigma + \int_{\partial B_n\cap\{y>0\}}G(u_n, v_n, x, y, x_0, \nu)\,d\sigma, \tag{3.11}$$

where ν is the outward unit normal to ∂B_n and $z_0 = (x_0, 0)$ with x_0 a point in $\mathbb{R}^N$.

Proof. By employing the classical strategy of testing the equation against the test functions $(\nabla u_n\cdot(x-x_0))$ and $(\nabla v_n\cdot(x-x_0))$ and then using the divergence theorem,

$$\frac{N-2s}{2}\int_{\partial B_n} y^{1-2s}\Big(u_n\frac{\partial u_n}{\partial \nu} + v_n\frac{\partial v_n}{\partial \nu}\Big)d\sigma = \frac{1}{2}\int_{\partial B_n} y^{1-2s}(|\nabla u_n|^2 + |\nabla v_n|^2)((X-z_0)\cdot\nu)\,d\sigma - \int_{\partial B_n} y^{1-2s}\Big((\nabla u_n\cdot(X-z_0))\frac{\partial u_n}{\partial \nu} + (\nabla v_n\cdot(X-z_0))\frac{\partial v_n}{\partial \nu}\Big)d\sigma,$$

where ν is the outward unit normal to ∂B_n and $z_0 = (x_0, 0)$ with x_0 a point in $\mathbb{R}^N$.

Since (u_n, v_n) is a solution of $(S^{\varepsilon}_{i,j})$, we have the following equations:

$$\begin{cases} u_n(x,0)\boldsymbol{\partial}_s u_n(x,0) = \dfrac{2\alpha_n}{\alpha_n+\beta_n}|u_n(x,0)|^{\alpha_n}|v_n(x,0)|^{\beta_n} + au_n^2(x,0) + bv_n(x,0), \\ v_n(x,0)\boldsymbol{\partial}_s v_n(x,0) = \dfrac{2\beta_n}{\alpha_n+\beta_n}|u_n(x,0)|^{\alpha_n}|v_n(x,0)|^{\beta_n} + bu_n^2(x,0) + cv_n(x,0). \end{cases} \tag{3.12}$$

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We obtain from (3.12) that

$$\begin{aligned}
&\frac{N-2s}{2}\int_{B_n\cap\{y=0\}}(2|u_n|^{\alpha_n}|v_n|^{\beta_n}+au_n^2+2bu_nv_n+cv_n^2)\,dx\\
&\quad+\frac{N-2s}{2}\int_{\partial B_n\cap\{y>0\}}y^{1-2s}\Big(u_n\frac{\partial u_n}{\partial\nu}+v_n\frac{\partial v_n}{\partial\nu}\Big)\,d\sigma\\
&=\frac{1}{2}\int_{B_n\cap\{y=0\}}y^{1-2s}(|\nabla u_n|^2+|\nabla v_n|^2)((x-x_0)\cdot\nu)\,dx\\
&\quad+\frac{1}{2}\int_{\partial B_n\cap\{y>0\}}y^{1-2s}(|\nabla u_n|^2+|\nabla v_n|^2)(X\cdot\nu)\,d\sigma\\
&\quad-\int_{B_n\cap\{y=0\}}y^{1-2s}\Big((\nabla u_n\cdot(x-x_0))\frac{\partial u_n}{\partial\nu}+(\nabla v_n\cdot(x-x_0))\frac{\partial v_n}{\partial\nu}\Big)\,dx\\
&\quad-\int_{\partial B_n\cap\{y>0\}}y^{1-2s}\Big((\nabla u_n\cdot(X-z_0))\frac{\partial u_n}{\partial\nu}+(\nabla v_n\cdot(X-z_0))\frac{\partial v_n}{\partial\nu}\Big)\,d\sigma. \qquad (3.13)
\end{aligned}$$

Noting that the product scalar $((x-x_0)\cdot\nu)=0$, on $B_n\cap\{y=0\}$,

$$\frac{1}{2}\int_{B_n\cap\{y=0\}}y^{1-2s}(|\nabla u_n|^2+|\nabla v_n|^2)((x-x_0)\cdot\nu)\,dx=0. \qquad (3.14)$$

Moreover,

$$\begin{aligned}
&\int_{B_n\cap\{y=0\}}y^{1-2s}\Big((\nabla u_n\cdot(x-x_0))\frac{\partial u_n}{\partial\nu}+(\nabla v_n\cdot(x-x_0))\frac{\partial v_n}{\partial\nu}\Big)\,dx\\
&=\int_{B_n\cap\{y=0\}}(\nabla u_n\cdot(x-x_0))\Big(\frac{2\alpha_n}{\alpha_n+\beta_n}|u_n|^{\alpha_n-2}u_n|v_n|^{\beta_n}+au_n+bv_n\Big)\,dx\\
&\quad+\int_{B_n\cap\{y=0\}}(\nabla v_n\cdot(x-x_0))\Big(\frac{2\beta_n}{\alpha_n+\beta_n}|u_n|^{\alpha_n}|v_n|^{\beta_n-2}v_n+a_2u_n+a_3v_n\Big)\,dx\\
&=\int_{B_n\cap\{y=0\}}\nabla\Big(\frac{2}{\alpha_n+\beta_n}|u_n|^{\alpha_n}|v_n|^{\beta_n}+\frac{1}{2}(au_n^2+2bu_nv_n+cv_n^2)\Big)\cdot(x-x_0)\,dx\\
&=-N\int_{B_n\cap\{y=0\}}\Big(\frac{2}{\alpha_n+\beta_n}|u_n|^{\alpha_n}|v_n|^{\beta_n}+\frac{1}{2}(au_n^2+2bu_nv_n+cv_n^2)\Big)\,dx\\
&\quad+\int_{\partial(B_n\cap\{y=0\})}\Big(\frac{2}{\alpha_n+\beta_n}|u_n|^{\alpha_n}|v_n|^{\beta_n}+\frac{1}{2}(au_n^2+2bu_nv_n+cv_n^2)\Big)((x-x_0)\cdot\nu)\,d\sigma.
\end{aligned}$$
$$(3.15)$$

Therefore, from (3.13), (3.14) and (3.15), we infer that (3.11) holds. □

PROOF OF PROPOSITION 2. Let $(u_n,v_n)_{n\geq1}$ be a bounded sequence in $E_0^s(C_\Omega)$ composed of solutions for $(S_{i,j}^{\varepsilon})$. Thus, in order to prove the $E_0^s(C_\Omega)$-strong convergence in (3.10), we just need to show that the bubbles $(\sigma_n^i)^{(N-2s)/2}U(\sigma_n^i(x-x_n^i),0)$ $(1\leq i\leq k)$ in (3.10) will not appear in the decomposition of u_n and v_n. Since the proof is similar to that of

[7, Lemma 6.1], here we only give a sketch of it. From Lemma 3.5, we have the local Pohozaev identity on $B_n := B_{t_n\sigma_n^{-1/2}}((x_n, 0)) \cap C_\Omega \subset \mathbb{R}^{N+1}$ for the solution concentrating sequence (u_n, v_n) of $(S^{\varepsilon}_{i,j})$ with $\varepsilon = \varepsilon_n \to 0$,

$$\begin{aligned}\frac{4\varepsilon_n N}{2_s^*(2_s^* - 2\varepsilon_n)} &\int_{B_n\cap\{y=0\}} |u_n|^{\alpha_n}|v_n|^{\beta_n}\,dx \\ &+ s\int_{B_n\cap\{y=0\}} (au_n^2 + 2bu_nv_n + cv_n^2)\,dx \\ = \int_{\partial(B_n\cap\{y=0\})} &F(u_n, v_n, x, x_0, \nu)\,d\sigma + \int_{\partial B_n\cap\{y>0\}} G(u_n, v_n, x, y, x_0, \nu)\,d\sigma, \qquad (3.16)\end{aligned}$$

where ν is the outward unit normal to ∂B_n and $z_0 = (x_0, 0)$, $x_0 \in \mathbb{R}^N$. We decompose

$$\partial B_n \cap \{y > 0\} = \partial_i B_n \cup \partial_e B_n,$$

where $\partial_i B_n := \partial B_n \cap C_\Omega$ and $\partial_e B_n := B_n \cap \partial_L C_\Omega$.

We consider two different cases:

(1) $B_{t_n\sigma_n^{-1/2}}((x_n, 0)) \cap \{0\} \cap (\mathbb{R}^{N+1}\backslash C_\Omega) \neq \emptyset$;
(2) $B_{t_n\sigma_n^{-1/2}}((x_n, 0)) \cap \{0\} \subset C_\Omega$.

In case 1, we take $x_0 \in \mathbb{R}^N\backslash\Omega$ with $|x_0 - x_n| \le 2t_n\sigma_n^{-1/2}$ and $\nu.(X - (x_0, 0)) \le 0$ in $\partial_e B_n$, where ν is the outward unit normal to $\partial_L C_\Omega$.

Since $(u_n, v_n) = (0, 0)$ on $\partial_L C_\Omega$,

$$\begin{aligned}-\frac{1}{2}&\int_{\partial B_n\cap\{y>0\}} y^{1-2s}(|\nabla u_n|^2 + |\nabla v_n|^2)((X - z_0)\cdot\nu)\,d\sigma \\ &+ \int_{\partial B_n\cap\{y>0\}} y^{1-2s}\Big((\nabla u_n\cdot X - z_0)\frac{\partial u_n}{\partial\nu} + (\nabla v_n\cdot X - z_0)\frac{\partial v_n}{\partial\nu}\Big)\,d\sigma \\ &= \frac{1}{2}\int_{\partial B_n\cap\{y>0\}} y^{1-2s}(|\nabla u_n|^2 + |\nabla v_n|^2)((X - z_0)\cdot\nu)\,d\sigma \le 0.\end{aligned}$$

In case 2, $\partial_e B_n = \emptyset$, we take a point $x_0 = x_n$.

Since $(u_n, v_n) = (0, 0)$ on $\partial_L C_\Omega$ and, for n large enough, $(4\varepsilon_n N/2_s^*(2_s^* - 2\varepsilon_n)) > 0$,

$$\begin{aligned}s\int_{B_n\cap\{y=0\}} (au_n^2 + 2bu_nv_n + cv_n^2)\,dx &\le \int_{\partial_i(B_n\cap\{y=0\})} F(u_n, v_n, x, x_0, \nu)\,d\sigma \\ &+ \int_{\partial_i B_n} G(u_n, v_n, x, y, x_0, \nu)\,d\sigma. \qquad (3.17)\end{aligned}$$

Set $B'_n = B_{\sigma_n^{-1}}(x_n, 0) \cap C_\Omega$. Recalling that by (3.10), we have the following decomposition of (u_n, v_n):

$$(u_n, v_n) = (u_0, v_0) + (u_{n,1}, v_{n,1}) + (u_{n,2}, v_{n,2}),$$

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where $u_{n,1} = A_0^{-1} \sum_{j=1}^{k} (\sigma_n^i)^{(N-2s)/2} U(\sigma_n^i(x - x_n^i), 0)$ and

$$v_{n,1} = B_0^{-1} \sum_{j=1}^{k} (\sigma_n^i)^{(N-2s)/2} U(\sigma_n^i(x - x_n^i), 0)$$

with $\|(u_{n,2}, v_{n,2})\|_{E_0^s(C_\Omega)} \to 0$ as $n \to \infty$, then we deduce that for n large enough, $B_n' \subset B_n$.

$$\begin{aligned}
&\int_{B_n \cap \{y=0\}} (au_n^2 + 2bu_nv_n + cv_n^2)\,dx \\
&\quad \geq \int_{B_n' \cap \{y=0\}} (au_n^2 + 2bu_nv_n + cv_n^2)\,dx \\
&\quad \geq C \int_{B_n' \cap \{y=0\}} (u_n^2 + v_n^2)\,dx \\
&\quad \geq \frac{1}{2} \int_{B_n' \cap \{y=0\}} (|u_{n,1}|^2 + |v_{n,1}|^2)\,dx \\
&\qquad\quad - 2 \int_{B_n' \cap \{y=0\}} (|u_0|^2 + |v_0|^2 + |u_{n,2}|^2 + |v_{n,2}|^2)\,dx.
\end{aligned} \tag{3.18}$$

After a direct calculation,

$$\begin{aligned}
\int_{B_n' \cap \{y=0\}} |u_{n,1}|^2 + |v_{n,1}|^2\,dx &\geq C\sigma_n^{-2s}, \\
\int_{B_n' \cap \{y=0\}} |u_0|^2 + |v_0|^2\,dx &\leq C\sigma_n^{-N}, \\
\int_{B_n' \cap \{y=0\}} |u_{n,2}|^2 + |v_{n,2}|^2\,dx &\leq C(\|u_{n,2}\|^2_{L^{2_s^*}(\Omega)} + \|v_{n,2}\|^2_{L^{2_s^*}(\Omega)})\sigma_n^{-2s}.
\end{aligned} \tag{3.19}$$

Note that $\|(u_{n,2}, u_{n,2})\|_{E_0^s(C_\Omega)} \to 0$ as $n \to \infty$. Inserting (3.19) into (3.18), we get for n large enough,

$$\int_{B_n \cap \{y=0\}} (au_n^2 + 2bu_nv_n + cv_n^2)\,dx \geq C\sigma_n^{-2s}. \tag{3.20}$$

By the choice of $z_0 = (x_0, 0)$, as in [23], we only need to consider the right-hand side of (3.16) on $\partial_i B_n$. Using Lemma 3.4 and applying the Hölder inequality,

$$\begin{aligned}
&\int_{\partial_i(B_n \cap \{y=0\})} F(u_n, v_n, x, x_0, \nu)\,d\sigma + \int_{\partial_i B_n} G(u_n, v_n, x, x_0, \nu)\,d\sigma \\
&\leq C\sigma_n^{-1/2} \Big(\int_{\partial_i(B_n \cap \{y=0\})} |u_n|^{\alpha_n+\beta_n}\,d\sigma\Big)^{\alpha_n/\alpha_n+\beta_n} \Big(\int_{\partial_i(B_n \cap \{y=0\})} |v_n|^{\alpha_n+\beta_n}\,d\sigma\Big)^{\beta_n/\alpha_n+\beta_n} \\
&\qquad + \int_{\partial_i(B_n \cap \{y=0\})} (|\nabla u_n|^2 + |\nabla v_n|^2)|x - x_0|\,d\sigma \\
&\qquad + \Big(\int_{\partial_i(B_n \cap \{y=0\})} (|\nabla u_n|^2 + |\nabla v_n|^2)\,d\sigma\Big)^{1/2} \Big(\int_{\partial_i(B_n \cap \{y=0\})} (|u_n|^2 + |v_n|^2)\,d\sigma\Big)^{1/2} \\
&\leq C\sigma_n^{-(N-2s)/2}.
\end{aligned} \tag{3.21}$$

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Inserting (3.20) and (3.21) into (3.17),

$$\sigma_n^{-2s} \leq C\sigma_n^{-(N-2s)/2},$$

which is a contradiction for n large enough due to $N > 6s$. □

4. Proof of Theorem 1.1

For any positive integer k, define the Z_2-homotopy class $\mathcal{F}_k$ as follows:

$$\mathcal{F}_k := \{A : A \subset E_0^s(C_\Omega) \text{ is compact, } Z_2\text{-invariant, and } \gamma(A) \geq k\},$$

where the genus $\gamma(A)$ is the smallest integer m such that there exists an odd map $\phi \in C(A, \mathbb{R}^m \setminus \{0\})$. For $k = 1, 2, \ldots$, we define the min–max value (see [12, page 134] for example)

$$c_{k,\varepsilon} = \min_{A\in\mathcal{F}_k} \max_{(u,v)\in A} I^\varepsilon_{\lambda,\mu}(u,v).$$

It follows from [12, Corollary 7.12] that, for each small $\varepsilon > 0$, $c_{k,\varepsilon}$ is a critical value of $I^\varepsilon_{\lambda,\mu}$, since $I^\varepsilon_{\lambda,\mu}$ satisfies the P.S. condition. Thus, problem ($S^\varepsilon_{i,j}$) has a solution $(u_{k,\varepsilon}, v_{k,\varepsilon})$ such that $I^\varepsilon_{\lambda,\mu}(u_{k,\varepsilon}, v_{k,\varepsilon}) = c_{k,\varepsilon}$. Note that $I^\varepsilon_{\lambda,\mu}(tu_{k,\varepsilon}, tv_{k,\varepsilon}) \to -\infty$ uniformly with respect to $\varepsilon > 0$ as $t \to +\infty$; hence, $c_{k,\varepsilon}$ is uniformly bounded with respect to ε for each fixed k. By a direct calculation, we find that $\|(u_{k,\varepsilon}, v_{k,\varepsilon})\|_{E_0^s(C_\Omega)} \leq C$ uniformly with respect to ε for each fixed k. So now we can apply Proposition 3.2 and obtain a subsequence of $(u_{k,\varepsilon_n}, v_{k,\varepsilon_n})_{n\geq 1}$ such that, as $n \to +\infty$, $(u_{k,\varepsilon_n}, v_{k,\varepsilon_n}) \to (u_k, v_k)$ strongly in $E_0^s(C_\Omega)$ for some (u_k, v_k) and $c_{k,\varepsilon_n} \to c_k$. Then (u_k, v_k) is solution of (S^s_Ω) and $I(u_k, v_k) = c_k$. We are now ready to show that $I_{\lambda,\mu}$ has infinitely many critical point solutions. Note that c_k is nondecreasing in k. By an argument similar to the one used in the proof of Theorem 1.1 in [7], we distinguish two cases.

Case 1. Suppose that there are $1 < k_1 < \cdots k_i < \cdots$ satisfying

$$c_{k_1} < \cdots < c_{k_i} < \cdots .$$

In this case, we have infinitely many distinct critical points and, therefore, infinitely many solutions.

Case 2. We assume in this case that for some positive integer m, $c_k = c$ for all $k \geq m$. Suppose that for any $\delta > 0$, I has a critical point (u, v) with $I(u, v) \in (c - \delta, c + \delta)$ and $I(u, v) \neq c$. In this case, we are done. So from now on we assume that there exists a $\delta > 0$ such that I has no critical point (u, v) with $I(u, v) \in (c - \delta, c) \cup (c, c + \delta)$. In this case, using the deformation argument, we can prove that

$$\gamma(K_c) \geq 2,$$

where $K_c = \{(u, v) \in X : \ I(u, v) = c, \ I'(u, v) = 0\}$. As a consequence, I has infinitely many critical points. We can obtain infinitely many solutions for problem (S^s_Ω).

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Appendix

In this section, we give a global compactness result in the following proposition.

PROPOSITION A.1. *Suppose that* $(u_n, v_n)_{n\geq 1}$ *is a solution of* ($S^{\varepsilon}_{i,j}$) *with* $\varepsilon = \varepsilon_n \to 0$ *satisfying* $\|(u_n, v_n)\|_{E^s_0(C_\Omega)} \leq C$ *for* C *a constant independent of* n. *Then:*

- (u_n, v_n) *can be decomposed as*

$$\begin{cases} u_n = u_0 + A_0^{-1} \displaystyle\sum_{i=1}^{k} (\sigma_n^i)^{(N-2s)/2} U(\sigma_n^i (x - x_n^i), 0) + \omega_n^1, \\ v_n = v_0 + B_0^{-1} \displaystyle\sum_{i=1}^{k} (\sigma_n^i)^{(N-2s)/2} U(\sigma_n^i (x - x_n^i), 0) + \omega_n^2, \end{cases}$$

  where $(u_0, v_0) \in E^s_0(C_\Omega)$ *is a weak solution of problem* ($S^s_{C_\Omega}$), $\|(\omega_n^1, \omega_n^2)\|_{E^s_0(C_\Omega)} \to 0$ *as* $n \to +\infty$, U *achieves the constant* $\mathcal{S}(s, N)$, *which is given in* (3.5), *and*

$$a_0 = \bigg(\Big(\frac{2\beta}{\alpha+\beta} \Big)^2 \Big(\frac{\beta}{\alpha} \Big)^{\beta-2} \bigg)^{1/2(\alpha+\beta-2)}, \quad b_0 = \bigg(\Big(\frac{2\alpha}{\alpha+\beta} \Big)^2 \Big(\frac{\alpha}{\beta} \Big)^{\alpha-2} \bigg)^{1/2(\alpha+\beta-2)}.$$

  For $i = 1, \ldots, k$, $x_{n,i} \in \Omega$ *with* $\sigma_n^i d(x_{n,i}, \partial\Omega) \to +\infty$, $\sigma_n^i |x_{n,i}| \to +\infty$;
- *for* $i, j = 1, \ldots, k$, *if* $i \neq j$, *then, as* $n \to +\infty$,

$$\frac{\sigma_n^j}{\sigma_n^i} + \frac{\sigma_n^i}{\sigma_n^j} + \sigma_n^i \sigma_n^j |x_{n,i} - x_{n,j}|^2 \to +\infty.$$

PROOF. The proof follows without difficulty by modifying the proof of the concentration compactness result for (P^s_Ω) (see [6, 23]) and using Lemma 3.3. We omit the details for the sake of simplicity. □

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