PASCAL OVALS IN PROJECTIVE PLANES

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1. Introduction. A *projective plane* consists of a set of *points* and a set of *lines*, together with a relation of *incidence* between points and lines, such that

(i) any two distinct points P, Q are incident with a unique line PQ,

(ii) any two distinct lines p, q are incident with a unique point $p \cap q$,

(iii) there exist four points, no three of which are incident with the same line.

We shall use the usual geometrical terminology.

An *oval* $\mathscr{C}$ in a projective plane π is a set of points of π such that:

(i) no three points of $\mathscr{C}$ are collinear; this means that a line of π is either a *secant* of $\mathscr{C}$, containing two points of $\mathscr{C}$, or a *tangent* of $\mathscr{C}$, containing just one point of $\mathscr{C}$, or a *non-secant* of $\mathscr{C}$, containing no point of $\mathscr{C}$;

(ii) through any point $P \in \mathscr{C}$ there passes just one tangent of $\mathscr{C}$; this tangent is called the *tangent at* P *to* $\mathscr{C}$; P is the *point of contact* of the tangent; it follows from (i) that every other line through P is a secant of $\mathscr{C}$.

A hexagon $ABCDEF$ in π is a *Pascal hexagon* if the three points of intersection of opposite sides of the hexagon are collinear. The oval $\mathscr{C}$ is a *Pascal oval* if every hexagon inscribed in $\mathscr{C}$ is a Pascal hexagon. (This definition will be made more precise in § 2.)

In 1966, Buekenhout (**2**) proved that if a projective plane π contains a Pascal oval $\mathscr{C}$, then π is a Pappian plane (i.e., π satisfies the axiom of Pappus and can therefore be coordinatized by a field). In his proof he makes use of a theorem of Tits (**5**) on transitive permutation groups. More recently, Artzy (**1**) has given a simpler proof of Buekenhout's result, using coordinates on $\mathscr{C}$, but in his proof he imposes extra conditions on $\mathscr{C}$. These conditions have the effect of ensuring that the characteristic of the coordinatizing field is not 2.

My aims in this paper are (a) to show that, with slight modifications, Artzy's proof is valid without his extra conditions, (b) to emphasize from the start the distinction between planes of characteristic 2 and other planes, and (c) to prove as much as possible about $\mathscr{C}$ by synthetic methods, as a step towards a possible synthetic proof of the main theorem in the future, and as a means of providing alternative proofs of some of Artzy's lemmas. Included in (c) are alternative (and, I think, simpler) proofs of some of Buekenhout's lemmas.

2. Pascal ovals. If $P \in \mathscr{C}$, we shall denote the tangent at P by PP; since we shall be dealing with only one oval, this notation is unambiguous.

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A *hexagon* $ABCDEF$ inscribed in $\mathscr{C}$ is a cyclically ordered set of six points of $\mathscr{C}$, not necessarily distinct. The *sides* of this hexagon are AB, BC, CD, DE, EF, and FA. (If $A = B$, then the side AB is AA, the tangent at A, etc.) A hexagon inscribed in $\mathscr{C}$ is *degenerate* if two non-adjacent vertices coincide. Thus, for instance, if A, B, C, D, E, and F are distinct points of $\mathscr{C}$, then $ABADEF$ is degenerate but $ABCDEF$ and $AACDDF$ are not.

If a hexagon $ABCDEF$ inscribed in $\mathscr{C}$ is non-degenerate, then its six sides are distinct, and the three points of intersection of its pairs of opposite sides, *viz.* $P = AB \cap DE, Q = BC \cap EF, R = CD \cap FA$, are distinct. If P, Q, and R are collinear, then the hexagon is a *Pascal hexagon*, with *Pascal line* PQR. The oval $\mathscr{C}$ is a *Pascal oval* if every non-degenerate hexagon inscribed in $\mathscr{C}$ is a Pascal hexagon.

It is not difficult to show that, if every hexagon with *six distinct* vertices inscribed in $\mathscr{C}$ is a Pascal hexagon, then $\mathscr{C}$ is a Pascal oval.

In order to deal with degenerate hexagons as well, let us say that $ABCDEF$ is a Pascal hexagon with Pascal line l if AB, DE, and l have a common point, BC, EF, and l have a common point, and CD, FA, and l have a common point. Then it is easily verified that every degenerate hexagon has a Pascal line, which may not be unique.

From now on, $\mathscr{C}$ will denote a Pascal oval. We shall use the notation "$LM \cap XY \in l$" to mean that LM, XY, and l have a common point; thus we include in this notation the case where the lines LM and XY coincide. The following lemma will enable us to avoid the repeated discussion of degenerate hexagons in subsequent proofs.

2.1. *If $ABCDEF$ is a hexagon inscribed in $\mathscr{C}$, if l is a line not through any vertex of the hexagon, and if $AB \cap DE \in l$ and $BC \cap EF \in l$, then $CD \cap FA \in l$.*

Proof. If $ABCDEF$ is non-degenerate, then it is a Pascal hexagon, from which the result follows. If $ABCDEF$ is degenerate, we simply have to enumerate and test the various possible cases.

By considering the hexagon $ABCDBF$ and a line l through B but not through $CD \cap FA$, we see that 2.1 is false if l passes through a vertex.

If A and B are distinct points of $\mathscr{C}$, and if C lies on the secant AB, $C \neq A$, then "$AC \cap \mathscr{C}$" will denote the point B. Thus $AC \cap \mathscr{C}$ is the second point of intersection of the secant AC with $\mathscr{C}$. If AC is the tangent at A, then "$AC \cap \mathscr{C}$" will denote the point A. In particular, $AA \cap \mathscr{C} = A$.

The terms "tangent", "secant", "non-secant" will mean "tangent to $\mathscr{C}$", "secant of $\mathscr{C}$", "non-secant of $\mathscr{C}$".

In some of the subsequent proofs we shall indicate the positions of certain points in the figure rather than in the text. In the proof of 2.2, for instance, there is no need to say "let $X = AD \cap BC$". Some of the figures illustrate general cases rather than degenerate cases that may occur.

2.2 (Figure 1). *If PA and PB are distinct tangents through P, and if PCD is a secant, then $CC \cap DD \in AB$.*

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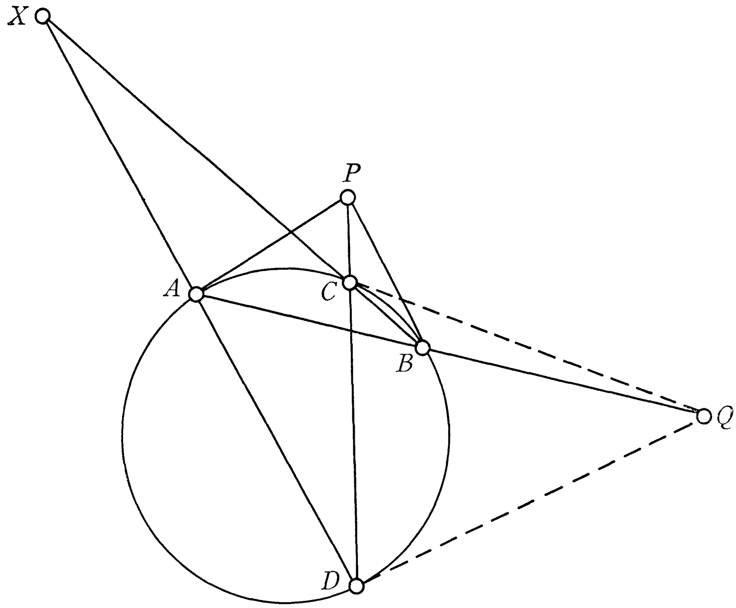

FIGURE 1

Proof. From hexagon $ABBCCD$, we have X, P, and $CC \cap AB$ collinear. From hexagon $AABCDD$, we have X, P, and $DD \cap AB$ collinear. Hence the result since X and P are distinct and $XP \neq AB$.

Cf. (**1**, Lemma 1); the present proof is similar but shorter.

2.3. *Either* (i) *all the tangents to* $\mathscr{C}$ *are concurrent, or* (ii) *through any point of any tangent other than the point of contact there passes just one more tangent.*

Proof. (i) Suppose that three tangents to $\mathscr{C}$, at A, B, and C say, meet at O. Let L be any other point of $\mathscr{C}$; suppose that OL is not a tangent, and let $LO \cap \mathscr{C} = M \neq L$. Then the tangents at L and M meet on BC, CA, and AB by 2.2, a contradiction since A, B and, C are not collinear. Hence OL is the tangent at L.

(ii) Suppose that no three tangents are concurrent. Let $P \neq A$ lie on the tangent at A. Let $C \in \mathscr{C}$, $C \neq A$. If PC is a tangent, there is nothing more to prove. If not, let $CP \cap \mathscr{C} = D \neq C$ (Figure 1). Let $CC \cap DD = Q$. Then QA is not a tangent. Let $AQ \cap \mathscr{C} = B \neq A$. Then $AA \cap BB \in CD$ (2.2), hence $AA \cap BB = P$ and PB is the second tangent through P.

If case (i) of 2.3 holds, the point of concurrence of the tangents is the *centre* of $\mathscr{C}$, which we shall always denote by O, and $\mathscr{C}$ is a *central oval*. In a finite plane it is easy to show that every line through the centre of a central oval is a tangent, but in an infinite plane some of these lines may be non-secants. Artzy's extra condition on $\mathscr{C}$ (see § 1) is that case (ii) of 2.3 holds, so that $\mathscr{C}$ is not a central oval.

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COROLLARY TO 2.3. *If $\mathscr{C}$ is not a central oval, and if $P \notin \mathscr{C}$, then through P there pass either two tangents or none.*

If $\mathscr{C}$ is not a central oval, a point P not on $\mathscr{C}$ is an *exterior* or *interior* point according as the number of tangents passing through P is two or none.

Even if $\mathscr{C}$ is not a Pascal oval, we can prove 2.3 if π is a *finite* plane (**4**, Theorems 3, 5). The next two lemmas are also true even if $\mathscr{C}$ is not a Pascal oval. By considering the tangent and secants through a point of $\mathscr{C}$, we can easily prove the following result.

2.4. *If π is a finite plane of order n, then $\mathscr{C}$ contains $n + 1$ points. If π is an infinite plane, then $\mathscr{C}$ contains an infinite number of points.*

If π is a finite plane, if P is not the centre of $\mathscr{C}$, and if $P \notin \mathscr{C}$, it follows from 2.3 that there passes at least one secant through P. Using a simple counting argument, we can now prove the following result.

2.5. *If π is a finite plane, if P is not the centre of $\mathscr{C}$, and if $P \notin \mathscr{C}$, then there exists a non-secant through P.*

We shall use this result in the proof of 4.2.

3. Axial mappings. Let P be a point not on $\mathscr{C}$. The mapping $L \rightarrow LP \cap \mathscr{C}$ (where $L \in \mathscr{C}$) is a one-to-one mapping of $\mathscr{C}$ onto itself, called the *involution* (*on* $\mathscr{C}$) with *centre* P, denoted by $\mathbf{P}$. If P is not the centre of $\mathscr{C}$, then $\mathbf{P}$ has period 2. If P is the centre of $\mathscr{C}$, then $\mathbf{P}$ is the identity mapping on $\mathscr{C}$, but we shall still call $\mathbf{P}$ an involution in this case. (*Note.* If $\mathscr{C}$ is a non-central oval, then the identity mapping on $\mathscr{C}$ is *not* an involution.)

Let l be a line. If α is a one-to-one mapping of $\mathscr{C}$ onto itself, *not the identity*, such that $L(M\alpha) \cap M(L\alpha) \in l$ for all $L, M \in \mathscr{C}$, then α is an *axial mapping* (*on* $\mathscr{C}$) with axis l.

3.1. *Let α be an axial mapping with axis l. Then*
(i) *any fixed point of α lies on l,*
(ii) *there exists $F \in \mathscr{C}$, $F \notin l$, such that $F\alpha \notin l$,*
(iii) *the points of intersection of l with $\mathscr{C}$ (if any) are fixed points of α.*

Proof. (i) Let A be a fixed point of α, L a non-fixed point. Then

$$A = A(L\alpha) \cap L(A\alpha) \in l.$$

(ii) If l is a non-secant, then we can choose any point of $\mathscr{C}$ for F. If l is a secant, let F be any point of $\mathscr{C}$, $F \notin l$. (Such a point exists since $\mathscr{C}$ contains at least three points.) Suppose that $F\alpha = A \in l$. Then $A\alpha \neq A$ since α is one-to-one. Hence $A(F\alpha) \cap F(A\alpha) = AA \cap F(A\alpha) \notin l$, a contradiction. This argument fails if l is a tangent, since we then have $AA = l$. If l is the tangent at A, there exist two distinct points $F, F' \in \mathscr{C}$, $F, F' \neq A$. Since $F\alpha \neq F'\alpha$, we may suppose without loss of generality that $F\alpha \neq A$, so that $F\alpha \notin l$.

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(iii) Let F be defined as in (ii), and let A be a point of intersection of l with $\mathscr{C}$. If $A\alpha \neq A$, then $A(F\alpha) \cap F(A\alpha) \notin l$, a contradiction.

COROLLARY. *The fixed points of* α *are precisely the points of intersection of* l *and* $\mathscr{C}$, *hence* α *has at most two fixed points.*

3.2. *Let* Q *and* R *be distinct points not on* $\mathscr{C}$. *Then* $\mathbf{QR}$ *is an axial mapping with axis* QR.

Proof. Let $L, M \in \mathscr{C}$. If $L \in QR$ (so that $L\mathbf{QR} = L$) or if $M \in QR$, then trivially $L(M\mathbf{QR}) \cap M(L\mathbf{QR}) \in l$.

Otherwise (Figure 2), applying 2.1 to hexagon $L(L\mathbf{Q})(L\mathbf{QR})M(M\mathbf{Q})(M\mathbf{QR})$, we have $L(M\mathbf{QR}) \cap M(L\mathbf{QR}) \in QR$.

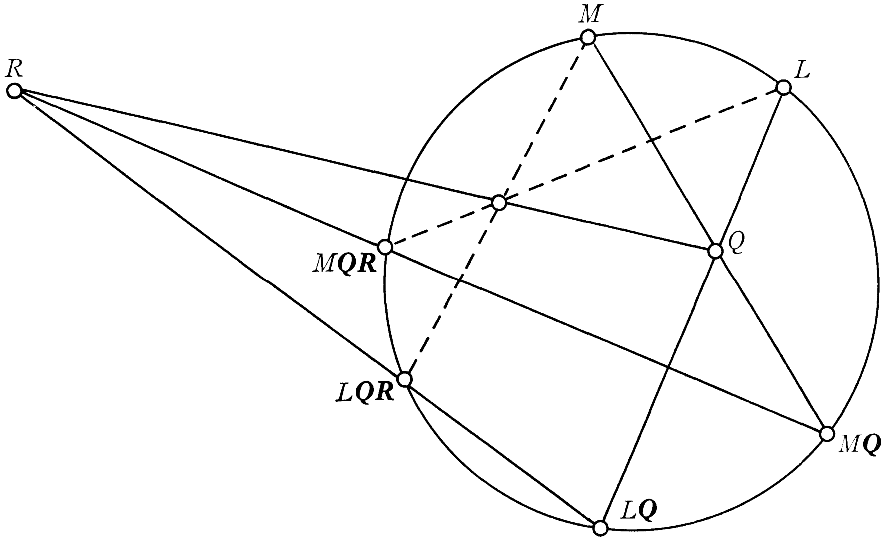

FIGURE 2

3.3. *If* α *is an axial mapping with axis* l, *and if* $P \in l$, $P \notin \mathscr{C}$, *then there exists a unique* $S \in l$, $S \notin \mathscr{C}$, *such that* $\alpha = \mathbf{PS}$.

Proof (Figure 3). Let F be defined as in 3.1(ii). Let $FP \cap \mathscr{C} = G$; then $G \notin l$. Let $G(F\alpha) \cap l = S$; then $S \notin \mathscr{C}$.

Let $L \in \mathscr{C}$. If $L \in l$, then $L\alpha = L$ (3.1(iii)) $= L\mathbf{PS}$. If $L \notin l$, let $L\mathbf{P} = M$. Then $F(L\alpha) \cap L(F\alpha) \in l$ and $FG \cap LM \in l$, thus from hexagon $FG(F\alpha)LM(L\alpha)$ we have $G(F\alpha) \cap M(L\alpha) \in l$; therefore $S \in M(L\alpha)$ and $M\mathbf{S} = L\alpha$. Hence $L\alpha = M\mathbf{S} = L\mathbf{PS}$.

Hence $\alpha = \mathbf{PS}$. The uniqueness of S is trivial.

3.4. *Given a line* l, *and distinct points* $F, H \in \mathscr{C}$, *not on* l, *there exists a unique axial mapping* α *with axis* l *such that* $F\alpha = H$.

Proof (Figure 3). Let $P \in l$, $P \notin \mathscr{C}$. Let $F\mathbf{P} = G$ ($G \notin l$), $GH \cap l = S$ ($S \notin \mathscr{C}$). Then $S \neq P$, hence $\mathbf{PS}$ is an axial mapping with the required properties (3.2). Any other axial mapping with the required properties can be expressed as $\mathbf{PS}'$ for some $S' \in l$ (3.3). Since we require $F\mathbf{PS}' = H$, we see that $S' = GH \cap l = S$, thus the mapping is unique.

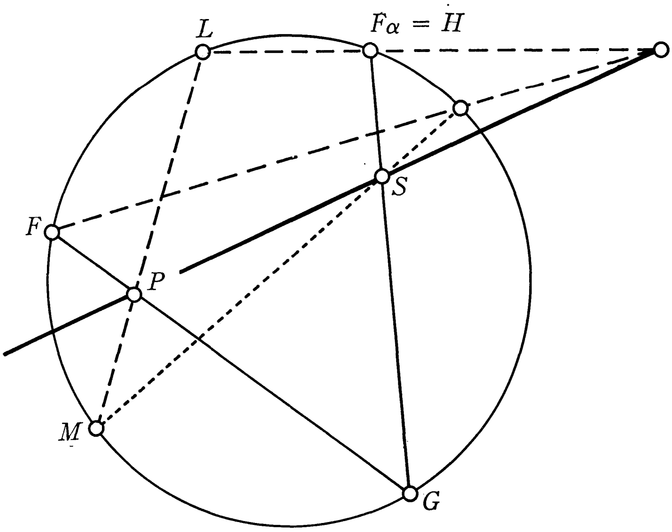

FIGURE 3

3.5. *If* A, B, C *and* A', B', C' *are triples of distinct points of* $\mathscr{C}$, *then there exists a unique axial mapping* α *such that* $A\alpha = A'$, $B\alpha = B'$, $C\alpha = C'$, *unless* $A = A'$, $B = B'$, $C = C'$.

Proof (Figure 4). Suppose without loss of generality that $C \neq C'$. Then $AC' \cap CA'$ and $BC' \cap CB'$ are well-defined and distinct. Let l be their join. Neither C nor C' lies on l. If the required mapping exists, then l must be its axis and it must map C onto C'. By 3.4, there exists a unique axial mapping α with these two properties; clearly $A\alpha = A'$, $B\alpha = B'$ from our definition of l.

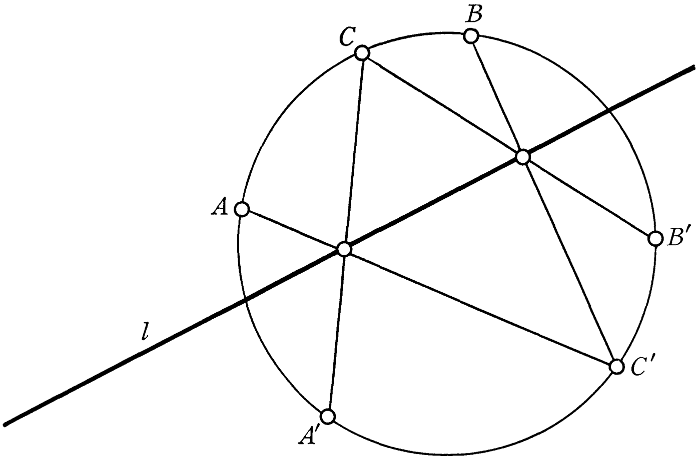

FIGURE 4

4. Involutions.

4.1 (cf. **2**, *théorème* 3, *lemme* 4.1). *If P, Q, R are collinear points not on $\mathscr{C}$, then there exists $S \notin \mathscr{C}$, collinear with P, Q, R, such that* $\mathbf{PQR} = \mathbf{S}$.

Proof. This is trivial if $Q = R$. If $Q \neq R$, then $\mathbf{QR}$ is an axial mapping with axis QR (3.2), and $P \in QR$. Hence there exists $S \in QR$ such that $\mathbf{QR} = \mathbf{PS}$ (3.3). Hence $\mathbf{PQR} = \mathbf{PPS} = \mathbf{S}$.

4.2. *If P, Q, R, $S \notin \mathscr{C}$ and if $P \neq Q \neq R \neq S$, then either* $\mathbf{PQRS} = 1$ *or* $\mathbf{PQRS}$ *has at most two fixed points*. (If $P = Q$ or $Q = R$ or $R = S$, the conclusion follows immediately from 3.2 and 3.1, Corollary, but we do not require the lemma in these special cases.)

Proof. If $\mathbf{PQRS}$ has more than two fixed points, let A, B, and C be three fixed points, and let $(A, B, C)\mathbf{PQ} = (A', B', C')$. Then $(A', B', C')\mathbf{RS} = (A, B, C)$, thus $(A, B, C)\mathbf{SR} = (A', B', C')$. Now $\mathbf{PQ}$ and $\mathbf{SR}$ are axial mappings, therefore we cannot have $A = A'$, $B = B'$, $C = C'$ (3.1, Corollary). Hence $\mathbf{PQ} = \mathbf{SR}$ (3.5), thus $\mathbf{PQRS} = 1$.

4.3 (cf. **2**, *lemme* 4.4). *If $P, Q, R, S \notin \mathscr{C}$, then there exist U, $V \notin \mathscr{C}$ such that* $\mathbf{PQRS} = \mathbf{UV}$.

Proof. If $P = Q$ or $Q = R$ or $R = S$, the proof is trivial. If $P \neq Q \neq R \neq S$, the basic idea of the proof is as follows: if the lines PQ and RS have a common point $T \notin \mathscr{C}$, then $\mathbf{PQRS} = (\mathbf{PQT})(\mathbf{TRS}) = \mathbf{UV}$ for suitable U, V (4.1). The proof below deals also with the possibility that PQ and RS may be distinct lines meeting on $\mathscr{C}$.

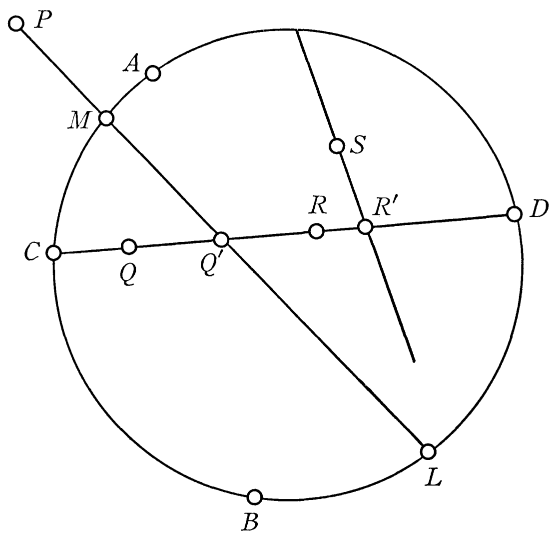

FIGURE 5

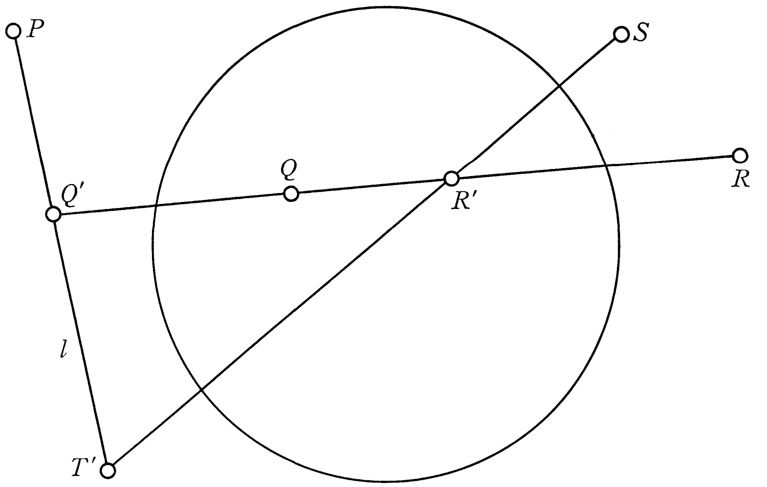

FIGURE 6

If **PQRS** = 1, then **PQRS** = **UU** for any $U \notin \mathscr{C}$. Otherwise, **PQRS** has at most two fixed points A, B, say (4.2) and QR meets $\mathscr{C}$ in at most two points C, D, say (Figure 5). *If there exist more than eight points on* $\mathscr{C}$ then there exists a secant or tangent PLM, distinct from PA, PB, PC, PD. Let $PLM \cap QR = Q'$. Then $Q' \notin \mathscr{C}$ and L, M are not fixed points of **PQRS**. There exists $R' \in QR$ such that **QR** = **Q′R′** (3.2, 3.3). If $PQ' \cap R'S \in \mathscr{C}$, then $PQ' \cap R'S = L$ or M. Hence either L**PQRS** = L**PQ′R′S** = L**R′S** = L, or M**PQRS** = M, a contradiction. Hence $PQ' \cap R'S = T'$ (say) $\notin \mathscr{C}$. Hence **PQRS** = **PQ′R′S** = (**PQ′T′**)(**T′R′S**) = **UV** for suitable U, V (4.1).

We deal with the case where there are not more than eight points on $\mathscr{C}$ by giving a proof that is valid for all finite planes. Suppose that π is finite. If P is not the centre of $\mathscr{C}$, then there exists a non-sectant l through P (2.5). Let $l \cap QR = Q'$ (Figure 6). Then $Q' \notin \mathscr{C}$. There exists $R' \in QR$, $R' \notin \mathscr{C}$, such that **QR** = **Q′R′** (3.2, 3.3). Let $T' = PQ' \cap R'S$. Then $T' \in l$ hence $T' \notin \mathscr{C}$. We now complete the proof as in the previous paragraph. If P is the centre of $\mathscr{C}$ but S is not, we can give a similar proof using a non-secant through S. If P and S are both at the centre of $\mathscr{C}$, then **PQRS** = **QR**.

COROLLARY (cf. **2**, *lemme* 4.5). *The axial mappings on* $\mathscr{C}$, *together with the identity mapping, form a strictly triply transitive group G.*

This follows from 4.3, 3.2, 3.3, 3.5.

4.4. *Every involution* (*except the identity in the case of a central oval*) *is an axial mapping.*

Proof. (a) Let $\mathscr{C}$ be a central oval, and P a point not on $\mathscr{C}$ distinct from the centre O. Then **P** = **PO**, which is an axial mapping by 3.2; the axis is PO.

(b) Let $\mathscr{C}$ be a non-central conic, and P an exterior point. Let PA and PB be the tangents from P, and let PCC' and PDD' be distinct secants through P

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(supposing that there exists more than one secant through P) (Figure 7). Let $p = AB$. The points W and X are distinct and $W, X \in p$ (2.2), thus $WX = p$. From hexagon $CCD'C'C'D$, we have W, Z, Y collinear. From hexagon $CD'D'C'DD$, we have X, Z, Y collinear. But $Z \neq Y$, thus $W, X \in ZY$. Hence $Z, Y \in WX = p$.

Since $\mathbf{P}$ maps C, D onto C', D' and A, B onto A, B, it follows that $\mathbf{P}$ is an axial mapping with axis p.

If there exists only one secant through P, the result follows directly from 2.2.

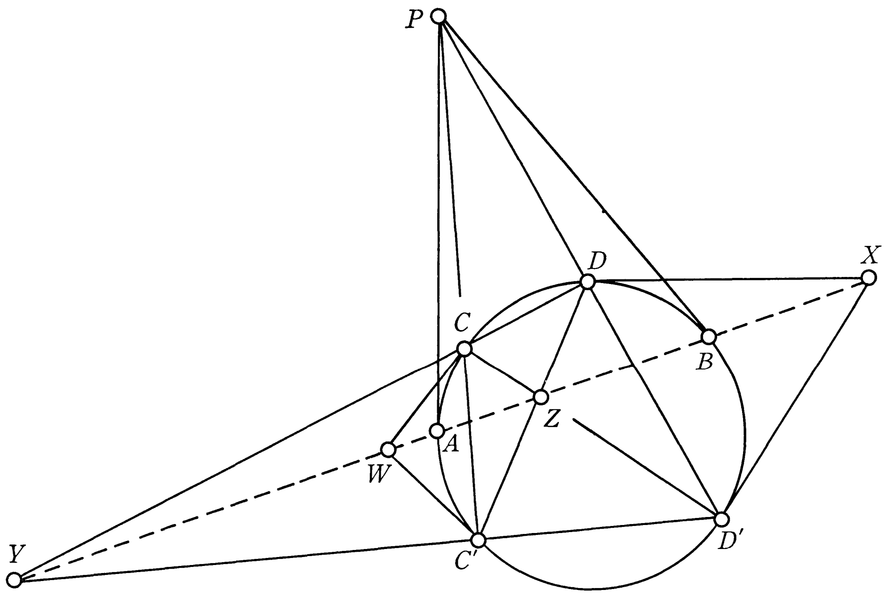

FIGURE 7

(c) Let $\mathscr{C}$ be a non-central oval, and P an interior point (Figure 8). Let PAB be a secant through P, and let the tangents at A, B meet at Q. By (*b*), $\mathbf{Q}$ is an axial mapping with axis AB. Hence there exists $R \in AB$ such that $\mathbf{Q} = \mathbf{PR}$ (3.3). Hence $\mathbf{P} = \mathbf{QR}$, thus $\mathbf{P}$ is an axial mapping (3.2).

COROLLARY. *The strictly triply transitive group G of axial mappings* (4.3, Corollary) *contains all the involutions and is generated by them.*

We shall now call the elements of G *projectivities* (*on* $\mathscr{C}$). Every non-identical projectivity is an axial mapping.

These results are sufficient for the proof of Buekenhout's theorem using coordinates in § 6, but it is interesting to see how we can now develop the theory of poles and polars for non-central ovals.

5. Poles and polars. We shall suppose in this section that $\mathscr{C}$ is not a central oval. If $P \notin \mathscr{C}$, we define the *polar* of P to be the axis of $\mathbf{P}$, denoted by p. If $L \in \mathscr{C}$, we define the *polar* of L to be the tangent at L, denoted by l.

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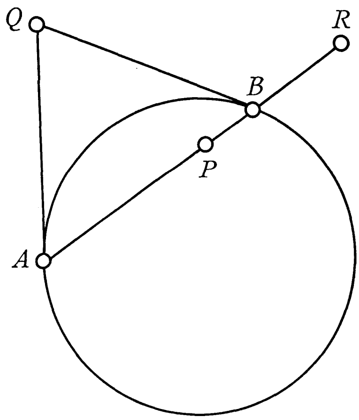

FIGURE 8

5.1. *If P lies on the polar of Q, then Q lies on the polar of P.*

Proof. (a) If $Q \in \mathscr{C}$, the result is trivial.

(b) If $Q \notin \mathscr{C}$ but $P \in \mathscr{C}$, then P is a fixed point of $\mathbf{Q}$ (3.1(iii)), thus PQ is the tangent at P, from which the result follows.

(c) If $Q \notin \mathscr{C}$, $P \notin \mathscr{C}$, then P lies on the axis of $\mathbf{Q}$, thus there exists R on the axis of Q, $R \notin \mathscr{C}$, such that $\mathbf{Q} = \mathbf{PR}$ (3.3). Hence $\mathbf{P} = \mathbf{QR}$, thus QR is the axis of $\mathbf{P}$ (3.2).

5.2. *If $P \notin \mathscr{C}$, then$P \notin p$.*

Proof. If $P \in p$, the axis of $\mathbf{P}$, then there exists $Q \in p$ such that $\mathbf{P} = \mathbf{PQ}$ (3.3). Hence $\mathbf{Q} = 1$, a contradiction.

5.3. *The polar of P is a secant, tangent or non-secant according as P is an exterior point, a point of $\mathscr{C}$ or an interior point.*

Proof. It follows from the proof of 4.4 that the polar of an exterior point is a secant. Let P be an interior point, and suppose that p meets $\mathscr{C}$ at Q. Then $Q \in p$ and thus $P \in q$, the tangent at Q. Hence P is not an interior point.

5.4. *Distinct points have distinct polars.*

Proof. Since the tangents at distinct points of $\mathscr{C}$ are distinct, we have only to exclude the possibility that two points $P, Q \notin \mathscr{C}$ have the same polar p. Suppose this is the case. Then $P, Q \notin p$ (5.2), thus the lines PQ and p are distinct. Let $R = PQ \cap p$, and let S be another point of p. Then R lies on the polars of P, Q, thus $r = PQ$. Similarly $s = PQ$. Hence R lies on the polars of R and S, hence $r = RS = p$, a contradiction.

5.5. *Given a line p, there exists a unique point P such that p is the polar of P.*

Proof. Let $Q, R \in p$, $Q \neq R$. Then $q \neq r$ (5.4). Let $P = q \cap r$. Then p is the polar of P (5.1). The uniqueness of P follows from 5.4.

We call P (in 5.5) the *pole* of p.

As an immediate consequence of the preceding lemmas, we have the following result.

5.6. *The mapping* $P \leftrightarrow p$ *is a one-to-one mapping of the sets of points and lines of* π *onto each other, of period* 2; *the mapping preserves incidence.*

Such a mapping is called a *polarity*, and the mapping in 5.6 is the *polarity determined by* $\mathscr{C}$. A polarity maps collinear points onto concurrent lines and vice-versa.

Let us call an oval as defined in § 1 a *point oval*, and its tangents *tangent lines*. Using the dual definition we can define a *line oval*, each line of which contains just one *tangent point* (i.e., a point lying on no other line of the line oval).

From 2.3 we easily deduce the following result.

5.7. *The tangents of* $\mathscr{C}$ *form a line oval* $\mathscr{C}^*$, *whose tangent points are the points of* $\mathscr{C}$.

We can call the dual of a Pascal point oval a *Pascal line oval.*

5.8. $\mathscr{C}^*$ *is a Pascal line oval.*

Proof. Let $abcdef$ be a non-degenerate hexagon of lines of $\mathscr{C}^*$ (i.e., tangents of $\mathscr{C}$) and let A be the pole of a with respect to $\mathscr{C}$, etc. We use 5.6 in the following argument. The vertices of $abcdef$ are the poles of the sides of $ABCDEF$. (If, say, $a = b$, then the vertex $a \cap b$ must be interpreted as A.) The three lines p, q, r joining opposite vertices of $abcdef$ are the polars of the three points of intersection P, Q, R of opposite sides of $ABCDEF$. But P, Q, R are collinear, thus p, q, r are concurrent.

In accordance with the usual nomenclature, we can therefore say that $\mathscr{C}$ is a *Brianchon oval.*

6. Coordinates. Our method of introducing coordinates is basically the same as Artzy's (**1**). Most of Artzy's proofs are valid for central conics also, but we give here alternative proofs of some of the results, based on the theorems of the previous sections. In order to understand the method more easily, consider first the hyperbola in a Euclidean plane whose equation is $xy = 1$ (Figure 9). The coordinate axes ZX, ZY are the asymptotes, where Z is the origin and X, Y are the points at infinity on the axes, i.e., ZX and ZY are the tangents to the hyperbola at X and Y. Any point $A \neq X$, Y on the hyperbola has coordinates of the form (a^{-1}, a), where $a \neq 0$. We can label A by means of its *parameter* a in brackets: $A = (a)$. To X, Y we give the labels (0), (∞). It is easily verified that the lines $(a)(b)$ and $(0)(a + b)$ meet on ZY, and that the lines $(a)(b)$ and $(1)(ab)$ meet on XY (Figure 9).

We therefore choose three distinct points X, Y, $I \in \mathscr{C}$ and let

$$XX \cap YY = Z$$

(Figure 10). To each point of $\mathscr{C}$ we assign a parameter. These parameters are just symbols associated with the points; distinct points must have distinct

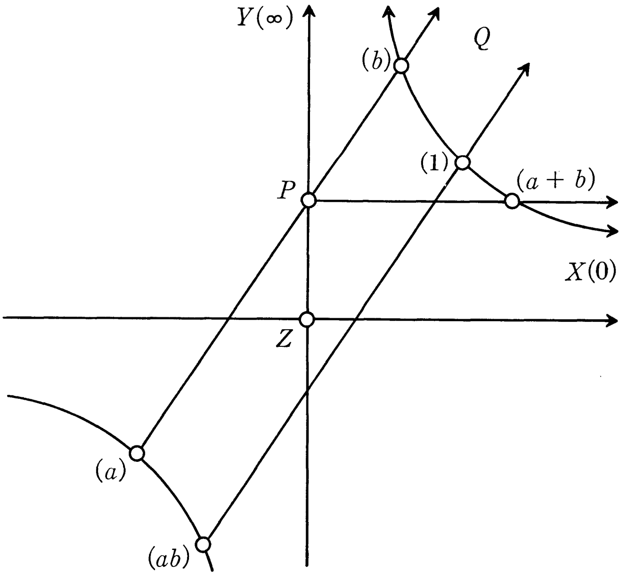

FIGURE 9

parameters. To X, Y, I we assign the special parameters 0, ∞, 1. If A has parameter a, we write $A = (a)$. Let $\mathscr{F}$ denote the set of all parameters except ∞.

If $a, b \in \mathscr{F}$, let $(a)(b) \cap ZY = P$, and let $(a)(b) \cap XY = Q$. We define $a + b$ to be the parameter of the point $(0)P \cap \mathscr{C}$, and we define $a \cdot b$, usually written as ab, to be the parameter of the point $(1)Q \cap \mathscr{C}$. Then $a + b$ and ab both belong to $\mathscr{F}$ (Figure 10).

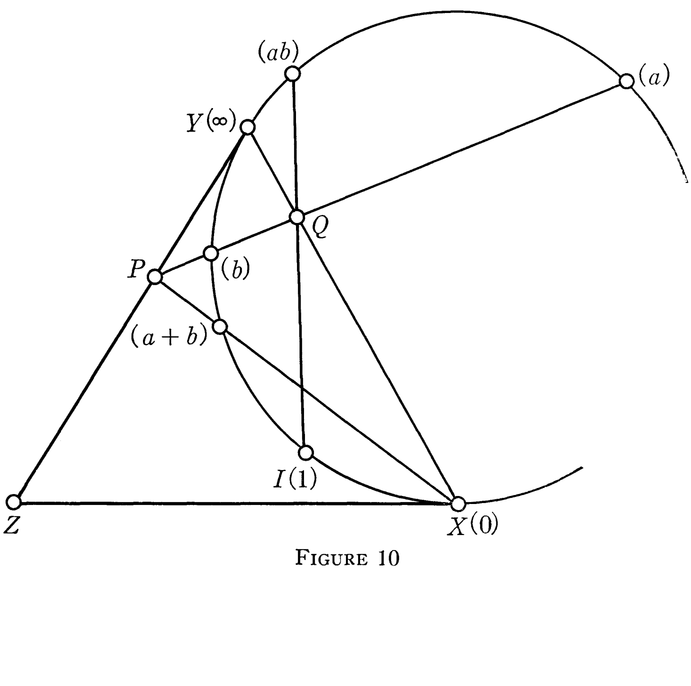

FIGURE 10

6.1. *For all* $a, b \in \mathscr{F}$, (i) $a + b = b + a$, (ii) $a + 0 = 0 + a = a$, (iii) *there exists a unique* $-a \in \mathscr{F}$ *such that* $a + (-a) = -a + a = 0$, (iv) $ab = ba$, (v) $a1 = 1a = a$, (vi) $a0 = 0a = 0$, (vii) *if* $a \neq 0$, *then there exists a unique* $a^{-1} \in \mathscr{F}$ *such that* $aa^{-1} = a^{-1}a = 1$, (viii) *if* $\mathscr{C}$ *is a central oval, then* Z *is the centre, and* $-a = a$; *if* $\mathscr{C}$ *is not a central oval, then* $-a \neq a$ *unless* $a = 0$.

These results follow immediately from the definitions of addition and multiplication.

6.2 (**1**, Lemma 2). *For all* $a, b, c \in \mathscr{F}$, $(a + b) + c = a + (b + c)$ *and* $(ab)c = a(bc)$.

This result follows immediately from 2.1.

It is convenient and reasonable to extend the definitions of addition and multiplication to the larger set $\mathscr{F} \cup \{\infty\}$, as follows:

$$a + \infty = \infty + a = \infty \quad \text{if } a \neq \infty,$$
$$a\infty = \infty a = \infty \quad \text{if } a \neq 0, \qquad 0^{-1} = \infty, \quad \infty^{-1} = 0.$$

Note that $\infty + \infty$, 0∞ and $\infty 0$ are not defined.

6.3. (i) *The mapping* $(x) \to (-x)$ *is the involution on* $\mathscr{C}$ *with centre* Z.

(ii) *The mapping* $(x) \to (x^{-1})$ *is the involution with centre* $II \cap XY$.

(iii) *The mapping* $(x) \to (ax)$, *where* $a \neq 0, \infty$, *is either the projectivity with axis* XY *that maps* (1) *onto* (a), *or the identity mapping if* $a = 1$.

(iv) *The mapping* $(x) \to (a + x)$, *where* $a \neq \infty$, *is either the projectivity with axis* ZY *that maps* (0) *onto* (a), *or the identity mapping if* $a = 0$.

These results follow immediately from the definitions of addition and multiplication, and from the lemmas in §§ 3, 4.

6.4 (cf. **1**, Lemma 3). *For all* $a, c \in \mathscr{F}$, $a(-c) = -(ac)$.

Proof. If $c = 0$, the result is trivial. If $c \neq 0$, the mappings $(x) \to (x(-c))$ and $(x) \to (-(xc))$, being products of projectivities on $\mathscr{C}$ (6.3(i)(iii)), are themselves projectivities (4.3, Corollary). These two projectivities both map (0), (1), (∞) onto (0), $(-c)$, (∞), and hence the projectivities are equal (by 3.5 if $-c \neq 1$; if $-c = 1$, then both projectivities are the identity mapping by 3.1, Corollary). Hence, putting $x = a$, we have $a(-c) = -(ac)$.

6.5. *For all* $a, b, c \in \mathscr{F}$, $a(b + c) = ab + ac$.

Proof (cf. **1**, Lemma 5). If $a = 0$ or $c = 0$, the result is trivial. If $a \neq 0$, $c \neq 0$, then the mappings $(x) \to (a(x + c))$ and $(x) \to (ax + ac)$, being products of projectivities on $\mathscr{C}$ (6.3(iii)(iv)), are themselves projectivities (4.3, Corollary). These two projectivities both map (0), $(-c)$, (∞) onto (ac), (0), (∞) (6.4), and hence the projectivities are equal (3.5). Hence, putting $x = b$, we have $a(b + c) = ab + ac$.

From 6.1, 6.2, 6.5 we deduce the following result.

6.6. $(\mathscr{F}; +, \cdot)$ *is a field.*

We now define coordinates for points of π not on XY by analogy with the Euclidean situation for the hyperbola $xy = 1$. If $P \notin XY$, let $YP \cap \mathscr{C} = (a^{-1})$ and let $XP \cap \mathscr{C} = (b)$. Then $a, b \neq \infty$. We define the coordinates of P to be (a, b). Then distinct points have distinct coordinates and every ordered pair (a, b), where $a, b \in \mathscr{F}$, is the pair of coordinates of some point $P \notin XY$. The point (a), where $a \neq 0$, has coordinates (a^{-1}, a). Any point on ZX has coordinates of the form $(a, 0)$; any point on ZY has coordinates of the form $(0, b)$.

6.7. *In the plane* π,
(i) *any line through* Y, *other than* XY, *has an equation of the form* $x = a$,
(ii) *any line not through* Y *has an equation of the form* $y = mx + b$.

Proof. (i) Any point on the line $(a^{-1})Y$ has, by definition, a as its first coordinate, thus this line has equation $x = a$.

(ii) Any point on the line $(b)X$ has, by definition, b as its second coordinate, so this line has equation $y = b$, i.e., $y = 0x + b$.

Let l be a line not through X or Y, meeting YZ at $(0, b)$ and meeting the line $x = 1$ at $(1, m + b)$, where $m \neq 0$. We shall show that l has the equation $y = mx + b$.

If $x \in \mathscr{F}$, let $(x, x\theta)$ be the point of l whose first coordinate is x (Figure 11); then θ is a one-to-one mapping of $\mathscr{F}$ onto itself, and $0\theta = b$, $1\theta = m + b$. The

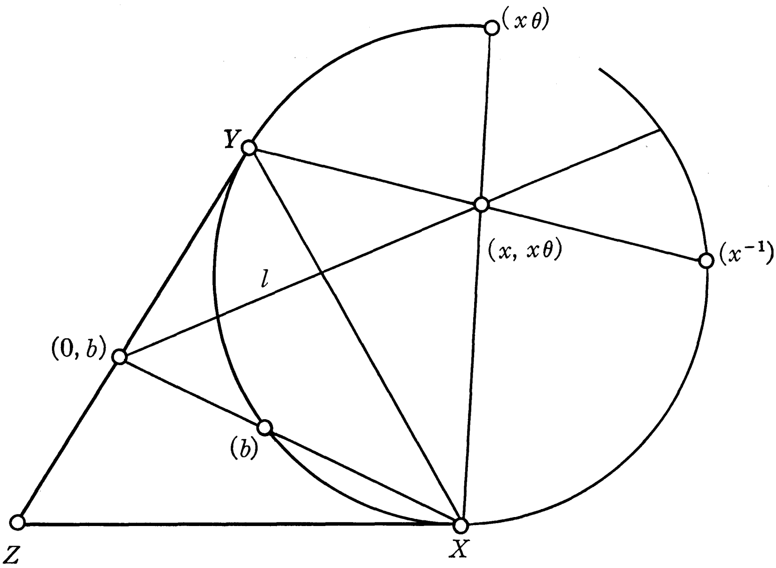

FIGURE 11

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mapping α: $(x^{-1}) \rightarrow (x\theta)$ is the projectivity on $\mathscr{C}$ with axis l that maps X onto Y, if we define $\infty\theta = \infty$. Hence the mapping β: $(x) \rightarrow (x\theta)$, being the product of $(x) \rightarrow (x^{-1})$ and α, is a projectivity on $\mathscr{C}$ (6.3(ii), 4.3, Corollary), and $(0)\beta = (b)$, $(1)\beta = (m + b)$, $(\infty)\beta = (\infty)$. But the mapping γ: $(x) \rightarrow (mx+b)$ is a projectivity (6.3(iii)(iv), 4.3, Corollary), and $(0)\gamma = (b)$, $(1)\gamma = (m + b)$, $(\infty)\gamma = (\infty)$. Hence $\beta = \gamma$ (by 3.5 unless $b = 0$ and $m = 1$; if $b = 0$ and $m = 1$, then $\beta = \gamma = 1$ by 3.1, Corollary). Hence $x\theta = mx + b$, thus the equation of l is $y = mx + b$.

We call m the *gradient* of l. Now $\mathscr{F}$ is a field, thus the simultaneous equations $y = m_1x + b_1$, $y = m_2x + b_2$ $(m_1 \neq m_2)$ have a solution in $\mathscr{F}$, but the simultaneous equations $y = mx + b_1$, $y = mx + b_2$ $(b_1 \neq b_2)$ have no solution in $\mathscr{F}$. Hence two lines meet on XY if and only if they have the same gradient.

Since π, or rather the affine plane obtained from π by removing XY and the points on it, can be coordinatized by the field $\mathscr{F}$ in the manner described above, and since the coordinates of the points of $\mathscr{C}$ satisfy the equation $xy = 1$, we have now proved the following result.

6.8. *π is a Pappian plane, and $\mathscr{C}$ is a conic in π.*

For remarks on the history of this method of assigning coordinates to points of a conic (in a Pappian plane) see (**3**, Chapter 11).

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