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COMPACTNESS PROPERTIES OF CRITICAL NONLINEARITIES AND NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract We prove the compactness of critical Sobolev embeddings with applications to nonlinear singular Schrödinger equations and provide a unified treatment in dimensions $N > 2$ and $N = 2$, based on a somewhat unexpectedly broad array of parallel properties between spaces $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and H_0^1 of the unit disc. These properties include Leray inequality for $N = 2$ as a counterpart of Hardy inequality for $N > 2$, pointwise estimates by ground states $r^{(2-N)/2}$ and $\sqrt{\log(1/r)}$ of the respective Hardy-type inequalities, as well as compactness of the limiting Sobolev embeddings once the Sobolev norm is appended by a potential term whose growth at singularities exceeds that of the corresponding Hardy-type potential.

Keywords: Trudinger–Moser inequality; Hardy inequality; critical nonlinearity; Schrödinger equation; singular potentials

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1. Introduction

The goal of the present work is to address questions of compactness for Schrödinger type equations in dimensions $N \geqslant 2$. It is well known that two-dimensional problems differ from higher-dimensional ones in several ways, such as critical rates of growth, energy spaces and corresponding embeddings into Lebesgue-type spaces.

The starting point of our study is the Schrödinger operator

$$L = -\Delta + V(x) \tag{1.1}$$

on a domain $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, where V is a continuous function away from the origin. The idea is to present a unified approach where the differences between dimensions $N = 2$ and $N > 2$ can be understood as a realization of common phenomena. For that, when $N = 2$ we shall pick $\Omega = B$ (the unit ball, rather than $\Omega = \mathbb{R}^2$) as the closest counterpart of $\Omega = \mathbb{R}^N$ when $N > 2$.

A reason for this apparent dichotomy lies in the fact that the kinetic energy

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x$$

does not define a bona fide functional space when $N = 2$. Namely, the completion of $C_0^\infty(\mathbb{R}^2)$ under the kinetic energy norm lacks a continuous embedding even into the space of distributions (note that the zero element in the completion space consists of the class of all constant functions). Indeed, the affinity between the unit ball in $\mathbb{R}^2$ and the whole space $\mathbb{R}^N$ for $N > 2$ becomes transparent when we consider the Hardy inequality,

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x \geqslant C_N \int_{\mathbb{R}^N} \frac{u^2}{r^2} \, \mathrm{d}x, \quad u \in C_0^\infty(\mathbb{R}^N), \tag{1.2}$$

where the optimal constant $C_N = (\frac{1}{2}(N-2))^2$ expresses the optimal coercivity of the energy space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ in terms of weighted L^2-norms (and the uncertainty principle, for that matter). The mentioned optimality means that no larger weight can replace C_N/r^2 (cf. [**6**]). On the other hand, when $N = 2$ and $C_N = 0$ (yielding incompatibility with the uncertainty principle), one can reestablish the following Hardy-type inequality on the unit ball $B = B_1(0) \subset \mathbb{R}^2$:

$$\int_B |\nabla u|^2 \, \mathrm{d}x \geqslant \frac{1}{4} \int_B \frac{u^2}{r^2(\log(1/r))^2} \, \mathrm{d}x, \quad u \in C_0^\infty(B) \tag{1.3}$$

(cf. [**5**]), where again the weight $1/4r^2(\log(1/r))^2$ cannot be improved.

Going back to the Schrödinger operator L in (1.1), the above discussion suggests the use of radial potentials V which exceed the optimal threshold const.$/r^2$ when $N > 2$ or const.$/r^2(\log(1/r))^2$ when $N = 2$, thus inducing energy spaces which are smaller than $\mathcal{D}^{1,2}$ for $N > 2$ or $H_0^1(B)$ for $N = 2$. So, we shall be considering the following hypotheses:

(VN) $V(r) + (\mu_H - \alpha)V_H(r) \geqslant 0$, $V(r)/V_H(r) \to +\infty$ when $r \to 0$ or $r \to \infty$, for $N > 2$,

(V2) $V(r) + (\mu_H - \alpha)V_H(r) \geqslant 0$, $V(r)/V_H(r) \to +\infty$ when $r \to 0$ or $r \to 1$, for $N = 2$,

where $\alpha > 0$,

$$V_H(r) := \begin{cases} \dfrac{1}{r^2} & \text{for } N > 2, \\ \dfrac{1}{r^2(\log(1/r))^2} & \text{for } N = 2, \end{cases}$$

and

$$\mu_H := \begin{cases} \left(\dfrac{N-2}{2}\right)^2 & \text{for } N > 2, \\ \frac{1}{4} & \text{for } N = 2. \end{cases}$$

Typical potentials satisfying the above conditions are

$$V(r) := \begin{cases} \dfrac{|\log r|}{r^2} & \text{for } N > 2, \\ \dfrac{\log\log(1/r)}{r^2(\log(1/r))^2} & \text{for } N = 2. \end{cases}$$

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In the case $N > 2$, related results have been considered by Su *et al.* in [**11**, **12**] by a different method with assumptions yielding compactness on embedding of weighted radial Sobolev spaces. Our focus in this paper is not on compactness for weighted Sobolev inequalities in general as in [**11**, **12**], but rather on appropriate, and more refined, conditions that yield weak continuity of the standard (i.e. with weight 1) critical nonlinearities. Our approach also extends to the case $N = 2$ and, as mentioned at the start of this section, aims at presenting a unified view of both cases $N > 2$ and $N = 2$.

From now on, the energy space E_V is defined as the subspace of radially symmetric functions in the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| := \left[\int_\Omega (|\nabla u|^2 + V(r)u^2)\,\mathrm{d}x\right]^{1/2},$$

where $\Omega = \mathbb{R}^N$ for $N > 2$ and $\Omega = B$ for $N = 2$. It follows from (VN) and (V2) that E_V is continuously embedded into E_0, which stands for the radial subspace of $\mathcal{D}^{1,2}(\mathbb{R}^N)$ when $N > 2$ and for the radial subspace of $H_0^1(B)$ when $N = 2$, endowed with the norm $\|\nabla u\|_2$.

We recall two standard inequalities characterizing integrability of functions in Sobolev spaces, namely, the limiting Sobolev inequality for $N > 2$,

$$\sup_{\|\nabla u\|_2 = 1} \Phi(u) < \infty, \quad \text{where } \Phi(u) = \int_{\mathbb{R}^N} |u|^{2^*}\,\mathrm{d}x,\ 2^* = \frac{2N}{N-2}, \tag{1.4}$$

and the Trudinger–Moser inequality for $N = 2$ (cf. [**7**, **14**]),

$$\sup_{\|\nabla u\|_2 = 1} \Phi(u) < \infty, \quad \text{where } \Phi(u) = \int_B \mathrm{e}^{4\pi u^2}\,\mathrm{d}x, \tag{1.5}$$

with 4π being the best such constant. We point out that, while for $N > 2$ the limiting Sobolev inequality defines a continuous embedding of E_0 into L^{2^*}, the Trudinger–Moser inequality defines a continuous embedding of E_0 into the Orlicz space $\exp L^2$ induced by the functional Φ (for the sake of comparison, we remind the reader that $\mathcal{D}^{1,2}(\mathbb{R}^2)$ lacks continuous embedding even into $\mathcal{D}'$). The radial subspace $\exp L^2_{\mathrm{rad}}$ of $\exp L^2$ can be endowed with an equivalent norm (for details, we refer the reader to [**3**]):

$$\|u\| = \sup_r \frac{|u(r)|}{\sqrt{\log(1/r)}}. \tag{1.6}$$

Neither the embedding $E_0 \hookrightarrow L^{2^*}$ for $N > 2$ nor the embedding $E_0 \hookrightarrow \exp L^2$ for $N = 2$ is compact. However, if instead of E_0 we consider E_V with a potential V satisfying (VN) and (V2), then the corresponding embeddings become compact. Our main results are Theorems 2.4, 2.6, 3.2 and 3.4. The last two results are applications to critical growth Schrödinger equations of the compact embedding theorems (Theorems 2.4 and 2.6). An illustrative consequence of the latter is the following theorem.

Theorem 1.1. *Let $\Phi\colon E_V \to \mathbb{R}$ be as in (1.4) (respectively, (1.5)). If V satisfies (VN) (respectively, (V2)), then the embedding of E_V into L^{2^*} (respectively, $\exp L^2$) is compact and the functional Φ defined by (1.4) (respectively, (1.5)) is weakly continuous.*

Remark 1.2. Conditions of Theorem 1.1 are sharp in the following sense.

- If V/V_H is bounded near one of the endpoints in (VN) and (V2), then the corresponding embedding is not compact. The lack of compactness is verified on sequences $g_{t_n}u$, where $u \in C_0^\infty$ is fixed,

$$g_t u(r) := \begin{cases} t^{(N-2)/2}u(tr) & \text{for } N > 2, \\ t^{1/2}u(r^{1/t}) & \text{for } N = 2, \end{cases} \qquad t > 0, \tag{1.7}$$

  with t_n going in a direction related to the endpoint in question (e.g. for $N = 2$, e^{-1/t_n} converges to 0 and 1, respectively).

- If we have equality in the first condition of (VN) and (V2) and $\alpha = 0$, the operator (1.1) is no longer positive definite. In this case the operator (1.1) for $N > 2$ corresponds to the Hardy inequality that has the generalized ground state $r^{(2-N)/2}$, and for $N = 2$ it corresponds to the Leray inequality that has the generalized ground state $\sqrt{\log(1/r)}$.

- The theorem is false if the restriction to the subspace of radial functions is omitted. In particular, if $N > 2$, $u \in C_0^\infty(\mathbb{R}^N)$ and $\sup_{r \geqslant 1} |V(r)| < \infty$, then $u_y := u(\cdot - y) \rightharpoonup 0$ in E_V as $|y| \to \infty$, but

$$\int_{\mathbb{R}^N} |u_y|^{2^*}\,\mathrm{d}x = \int_{\mathbb{R}^N} |u|^{2^*}\,\mathrm{d}x.$$

  For $N = 2$ a counter-example is given by a sequence concentrating (by means of logarithmic dilations (1.7)) at a point bounded away from both the origin and ∂B.

2. Critical nonlinearities in E_V

In this section we prove Theorem 1.1 and derive some of its consequences. The main technical component of the proof will be the operators (1.7), which are isometries on E_0 and form a multiplicative group, as can be shown by an easy calculation.

Due to the lack of compactness in the limiting embeddings, a sequence in E_0 will vanish in the target space only if its convergence to zero in E_0 is stronger than the weak convergence. Indeed, we have the following.

Proposition 2.1. *If $(u_k) \subset E_0$ is such that for every $t_k > 0$,*

$$g_{t_k}u_k \rightharpoonup 0 \quad \text{in } E_0, \tag{2.1}$$

then $u_k \to 0$ in L^{2^*} for $N > 2$ and $u_k \to 0$ in $\exp L^2_{\mathrm{rad}}$ for $N = 2$.

Proof. For $N = 2$, this result is [**1**, Lemma 3.3]. For $N > 2$, the result is a restriction of [**9**, Theorem 2] to the radial case, once one observes that radiality prevents the formation of rescaled profiles at any other point but the origin, and condition (2.1) prevents formation of profiles at the origin as well. For completeness, we include an independent proof in the appendix. □

In the terminology of [**13**], the above proposition states that the embeddings $E_0 \hookrightarrow L^{2^*}$ and $E_0 \hookrightarrow \exp L^2_{\mathrm{rad}}$ are *cocompact* with respect to the group of transformations (1.7). Another important embedding of E_0 that is cocompact with respect to (1.7) is given by the inequalities

$$\|r^{(N-2)/2}u\|_\infty \leqslant C(N)\|\nabla u\|_2, \quad N > 2, \tag{2.2}$$

and

$$\|(\log(1/r))^{-1/2}u\|_\infty \leqslant C(N)\|\nabla u\|_2, \quad N = 2 \tag{2.3}$$

(see [**10**] and [**7**], respectively). In what follows, the notation L^p will refer to the space $L^p(\Omega, W\,\mathrm{d}x)$, $p < \infty$, the Lebesgue space of Ω with measure $W\,\mathrm{d}x$, $W > 0$, with norm

$$\|u\|_{W,p} = \biggl(\int_\Omega |u|^p \,\mathrm{d}x\biggr)^{1/p}.$$

Somewhat inconsistently, we denote by $L^\infty(\Omega, W\,\mathrm{d}x)$ the space of Lebesgue measurable functions with the norm

$$\|uW\|_\infty = \sup_{x\in\Omega} |u(x)|W(x).$$

Then, we have the following.

Proposition 2.2. *If the sequence u_k is as in Proposition 2.1, then $\|r^{(N-2)/2}u_k\|_\infty \to 0$ for $N > 2$ and $\|(\log(1/r))^{-1/2}u_k\|_\infty \to 0$ for $N = 2$.*

Proof. Note first that the case $N = 2$ here is identical to the case $N = 2$ in Proposition 2.1, since the norm appearing in (2.3) has already been adopted in § 1 (see (1.6)) as the standard norm for $\exp L^2_{\mathrm{rad}}$. For $N > 2$ we have that

$$r_k^{(N-2)/2}u_k(r_k\,\cdot) \rightharpoonup 0$$

for every sequence $r_k > 0$. Let us use the trace value of a function on the unit sphere S^{N-1}, integrated over this sphere, as a continuous test functional for the weak convergence. This yields $r_k^{(N-2)/2}u_k(r_k) \to 0$, which in turn, since r_k is arbitrary, gives $\|r^{(N-2)/2}u_k\|_\infty \to 0$. □

We further note that, for $N > 2$, one has the inequality

$$\int_{\mathbb{R}^N} r^{\alpha_p}|u|^p \,\mathrm{d}x \leqslant C(N,p)\|\nabla u\|_2^p, \quad \text{where } \alpha_p = \tfrac{1}{2}(p-2)(N-2) - 2,\ p \in [2,\infty). \tag{2.4}$$

Indeed, for $p = 2$ this is the Hardy inequality and for $p > 2$ it is an elementary consequence of the Hardy inequality and the radial estimate (2.2). Observe that, for $p = 2^*$, we have $\alpha_p = 0$, which gives the limiting Sobolev inequality as a consequence of the Hardy inequality.

Similarly, for $N = 2$, as has been observed in [**1**], the following inequality follows from Leray inequality (1.3) and the radial estimate (2.3):

$$\int_B \frac{|u|^p}{r^2(\log(1/r))^{\beta_p}}\,\mathrm{d}x \leqslant C(N,p)\|\nabla u\|_2^p, \quad \text{where } \beta_p = \tfrac{1}{2}(p+2),\ p \in [2,\infty). \tag{2.5}$$

It is now clear that inequalities (2.4) and (2.5) also define cocompact embeddings; namely, we have the following.

Corollary 2.3. *If the sequence u_k is as in Proposition 2.1, then*

$$\int_{\mathbb{R}^N} r^{\alpha_p}|u_k|^p\,\mathrm{d}x \to 0, \quad p \in (2,\infty),\ N > 2, \tag{2.6}$$

and

$$\int_B \frac{|u_k|^p}{r^2(\log(1/r))^{\beta_p}}\,\mathrm{d}x \to 0, \quad p \in (2,\infty),\ N = 2. \tag{2.7}$$

Unlike embeddings of E_0 that are cocompact but not compact, the space E_V admits the following compact embeddings.

Theorem 2.4. *For every $p \in [2,\infty)$ the space E_V is compactly embedded into the space $L^p(\mathbb{R}^N, r^{\alpha_p}\,\mathrm{d}x)$ for $N > 2$ and into the space*

$$L^p\biggl(B, r^{-2}\biggl(\log\frac{1}{r}\biggr)^{-\beta_p}\mathrm{d}x\biggr) \quad \textit{for } N = 2.$$

Proof. Consider first the case $p = 2$. By (VN) we have, for $N > 2$,

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{u_k^2}{r^2}\,\mathrm{d}x &= \int_{R^{-1}\leqslant r\leqslant R} \frac{u_k^2}{r^2}\,\mathrm{d}x + \epsilon_R \int_{\{r<R^{-1}\}\cup\{r>R\}} V u_k^2\,\mathrm{d}x \\ &\leqslant \int_{R^{-1}\leqslant r\leqslant R} \frac{u_k^2}{r^2}\,\mathrm{d}x + \epsilon_R\|u_k\|_{E_V}^2,\end{aligned}$$

where $\epsilon_R \to 0$ as $R \to \infty$. If we assume that $u_k \rightharpoonup 0$ and use Sobolev embedding in bounded domains, it follows that

$$\limsup_{k\to\infty} \int_{\mathbb{R}^N} \frac{u_k^2}{r^2}\,\mathrm{d}x \leqslant \epsilon_R \limsup_{k\to\infty} \|u_k\|_{E_V}^2.$$

Since R is arbitrary, it follows that $u_k \to 0$ strongly in $L^2(\mathbb{R}^N, 1/r^2\,\mathrm{d}x)$. Similarly, when $N = 2$, we have, by (V2),

$$\begin{aligned}&\int_B \frac{u_k^2}{r^2(\log(1/r))^2}\,\mathrm{d}x \\ &\quad = \int_{R^{-1}\leqslant\log(1/r)\leqslant R} \frac{u_k^2}{r^2(\log(1/r))^2}\,\mathrm{d}x + \epsilon_R \int_{\{\log(1/r)<R^{-1}\}\cup\{\log(1/r)>R\}} V u_k^2\,\mathrm{d}x \\ &\quad \leqslant \int_{R^{-1}\leqslant\log(1/r)\leqslant R} \frac{u_k^2}{r^2(\log(1/r))^2}\,\mathrm{d}x + \epsilon_R\|u_k\|_{E_V}^2,\end{aligned} \tag{2.8}$$

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where $\epsilon_R \to 0$ as $R \to \infty$. If we assume that $u_k \rightharpoonup 0$ and use Sobolev embedding in bounded domains, it follows that

$$\limsup_{k\to\infty} \int_B \frac{u_k^2}{r^2(\log(1/r))^2}\,\mathrm{d}x \leqslant \epsilon_R \limsup_{k\to\infty} \|u_k\|_{E_V}^2,$$

and we conclude similarly that $u_k \to 0$ strongly in

$$L^2\biggl(B, \frac{1}{r^2(\log(1/r))^2}\,\mathrm{d}x\biggr),$$

since R is arbitrary.

The case $2 < p < \infty$ follows immediately from the case $p = 2$ and the retrospective pointwise estimates (2.2) for $N > 2$ and (2.3) for $N = 2$. $\square$

As a matter of fact, the case $p = \infty$ also holds true. Before stating the theorem we need the following elementary estimate, which follows from the fundamental theorem of calculus and the Cauchy–Schwarz inequality in a similar manner to (2.2) and (2.3).

Lemma 2.5. *Assume that $u \in C_0^1(B)$ is a radial function on the unit disc, let $\lambda \neq 1$ and let*

$$N_\lambda(u) = \biggl(\int_B |\nabla u|^2 \biggl(\log\frac{1}{r}\biggr)^{\lambda}\mathrm{d}x\biggr)^{1/2}. \tag{2.9}$$

Then

$$\sup_{r\in(0,1)} |u(r)|\biggl(\log\frac{1}{r}\biggr)^{(\lambda-1)/2} \leqslant \frac{1}{\sqrt{2\pi|\lambda-1|}} N_\lambda(u). \tag{2.10}$$

Theorem 2.6. *The space E_V is compactly embedded into $L^\infty(\mathbb{R}^N, r^{(N-2)/2}\,\mathrm{d}x)$ for $N > 2$ and into the space $L^\infty(B, (\log(1/r))^{-1/2}\,\mathrm{d}x)$ for $N = 2$.*

Proof. Let $u_k \rightharpoonup 0$ and consider the case $N > 2$.

Case 1. Assume first that all the u_k have support outside the annulus

$$A_R = \{x \in \mathbb{R} \mid 1/R \leqslant |x| \leqslant R\},$$

with $R > 0$. Since $r^2V(r) \to \infty$ as $r, r^{-1} \to \infty$, there exist $M_R > 0$, $\lim_{R\to\infty} M_R = +\infty$, such that $V(r) \geqslant M_R/r^2$ for all $r, r^{-1} \geqslant R$. Let $\lambda = \lambda_R > 0$ be any of the two roots of the equation

$$\lambda_R(\lambda_R + N - 2) = M_R. \tag{2.11}$$

As a matter of convenience we can decrease M_R, so that M_R still goes to infinity as $R \to \infty$ but λ_R is now an integer. It follows that $\varphi_R(r) = r^{\lambda_R}$ solves

$$-\Delta\varphi + \frac{M_R}{r^2}\varphi = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\},$$

and we obtain

$$\int_{A_R^c} \varphi_R^2 |\nabla(\varphi_R^{-1}u_k)|^2 r^{N-1}\,\mathrm{d}r = \int_{A_R^c} \biggl(|\nabla u_k|^2 + \frac{M_R}{r^2}u_k^2\biggr) r^{N-1}\,\mathrm{d}r \leqslant C,$$

where $C > 0$ is such that

$$\|u_k\|^2 \leqslant C$$

for all $k \in \mathbb{N}$. Therefore, we have

$$\int_{\mathbb{R}^N} |\nabla v|^2 r^{N_R - 1}\,\mathrm{d}r \leqslant C, \tag{2.12}$$

where $v = r^{-\lambda_R} u_k$ and $N_R = N + 2\lambda_R$. It follows from the pointwise estimate (2.2) in $\mathcal{D}^{1,2}(\mathbb{R}^{N_R})$ that

$$r^{(N_R-2)/2}|v(r)| \leqslant \frac{\text{const.}}{\sqrt{N_R - 2}} \quad \text{for all } r > 0,$$

or

$$r^{(N-2)/2}|u_k(r)| \leqslant \frac{\text{const.}}{\sqrt{N - 2 + 2\lambda_R}} = \epsilon_R, \tag{2.13}$$

where $\epsilon_R \to 0$ as $R \to \infty$ (recall that $\lambda_R \to \infty$ as $R \to \infty$ because M_R does so by (2.11)).

Case 2. The general case, where we do not assume that all u_k have support outside a fixed annulus, can be handled by cut-off functions. Let $\theta_R \in C^\infty(\mathbb{R})$ be a non-negative function such that

$$\theta_R(r) = \begin{cases} 1 & \text{for } r \in [0, R^{-1}] \cup [R, \infty), \\ 0 & \text{for } r \in [R^{-1} + 1, R - 1], \end{cases} \tag{2.14}$$

with $|\theta_R'(r)| \leqslant 2$ for all r. Then, defining $v_k = \theta_R u_k$ and $w_k = (1 - \theta_R)u_k$ (where, for simplicity of notation, we now drop the subscript R), we can write

$$u_k = v_k + w_k,$$

where v_k has support in the union of $B_{R^{-1}+1}$ and $\mathbb{R}^N \setminus B_{R-1}$, and w_k has support in $A_{R^{-1},R} = \bar{B}_R \setminus B_{R^{-1}}$. Now, a straightforward calculation shows that

$$\int_{\mathbb{R}^N} (|\nabla v_k|^2 + V v_k^2)\,\mathrm{d}x \leqslant 2\int_{\mathbb{R}^N} (|\nabla u_k|^2 + V u_k^2)\,\mathrm{d}x + 2\int_{A_{R-1,R} \cup A_{R^{-1},R^{-1}+1}} u_k^2\,\mathrm{d}x,$$

where the second term above goes to zero since $u_k \rightharpoonup 0$ as $k \to \infty$. Therefore, the sequence v_k is bounded in E_V. This allows us to use Case 1, in order to conclude from (2.13) that

$$\sup_{r \geqslant 0} r^{(N-2)/2}|v_k(r)| \to 0. \tag{2.15}$$

An analogous calculation shows that w_k has a bounded E_0-norm, and thus $w_k \to 0$ uniformly on its support in $A_{R^{-1},R-1}$ by compactness in the Morrey embedding. Recalling that $u_k = v_k + w_k$, we extend the estimate (2.15) to the sequence u_k, which proves the theorem for $N > 2$.

We now consider $u_k \rightharpoonup 0$ in the case $N = 2$.

Case 3. Assume first that all the u_k have support outside the annulus $A_\eta = \{x \in B \mid \mathrm{e}^{-1/\eta} \leqslant |x| \leqslant \mathrm{e}^{-\eta}\}$, with a fixed small $\eta > 0$. Since $(\log(1/r))^2 r^2 V(r) \to \infty$ as $r \to 0$ or $r \to 1$, there exists $M_\eta > 0$, $\lim_{\eta\to 0} M_\eta = +\infty$, such that

$$V(r) \geqslant \frac{M_\eta}{r^2(\log(1/r))^2}$$

whenever $r \in [0, \mathrm{e}^{-1/\eta}]$ or $r \in [\mathrm{e}^{-\eta}, 1]$.

Also, letting $\lambda = \lambda_\eta > 0$ be any of the two roots of the equation

$$\lambda_\eta(\lambda_\eta - 1) = M_\eta, \tag{2.16}$$

it follows that $\varphi_\eta(r) = (\log(1/r))^{\lambda_\eta}$ solves

$$-\Delta\varphi + \frac{M_\eta}{r^2(\log(1/r))^2}\varphi = 0 \quad \text{in } B \setminus \{0\},$$

and we obtain

$$\int_B \varphi_\eta^2 |\nabla(\varphi_\eta^{-1} u_k)|^2 r \,\mathrm{d}r = \int_B \left(|\nabla u_k|^2 + \frac{M_\eta}{r^2(\log(1/r))^2} u_k^2\right) r \,\mathrm{d}r \leqslant C,$$

where $C > 0$ is such that

$$\|u_k\|^2 \leqslant C$$

for all $k \in \mathbb{N}$. Therefore, we have

$$\int_B |\nabla v|^2 \left(\log\frac{1}{r}\right)^{2\lambda_\eta} r \,\mathrm{d}r \leqslant C, \tag{2.17}$$

where $v = (\log(1/r))^{-\lambda_\eta} u_k$. From Lemma 2.5, we have

$$\sup_{r\in(0,1)} |v(r)| \left(\log\frac{1}{r}\right)^{\lambda_\eta - 1/2} \leqslant \frac{\text{const.}}{\sqrt{|\lambda_\eta - 1|}}, \tag{2.18}$$

or

$$|u_k(r)| \left(\log\frac{1}{r}\right)^{-1/2} \leqslant \frac{\text{const.}}{\sqrt{|\lambda_\eta - 1|}} = \epsilon_\eta, \tag{2.19}$$

where $\epsilon_\eta \to 0$ as $\eta \to 0$ (recall that $\lambda_\eta \to \infty$ as $\eta \to 0$ because M_η does so by (2.16)).

Case 4. The general case for $N = 2$, where we do not assume that all u_k have support outside a fixed annulus, can be handled by means of cut-off functions.

Let

$$\theta_\eta(r) = \begin{cases} 1 & \text{for } r \in [0, \mathrm{e}^{-1/\eta}] \cup [\mathrm{e}^{-\eta}, 1], \\ 0 & \text{for } r \in [\mathrm{e}^{-1/(2\eta)}, \mathrm{e}^{-\eta/2}], \\ \dfrac{2}{\eta}\log\dfrac{\mathrm{e}^{-\eta/2}}{r} & \text{for } r \in (\mathrm{e}^{-\eta}, \mathrm{e}^{-\eta/2}), \\ 2\eta\log\dfrac{\mathrm{e}^{-1/(2\eta)}}{r} & \text{for } r \in (\mathrm{e}^{-1/\eta}, \mathrm{e}^{-1/(2\eta)}), \end{cases} \tag{2.20}$$

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and observe that, apart from a set of measure zero,

$$|\theta_\eta'(r)| \leqslant \begin{cases} \dfrac{2\eta}{r} & \text{for } r \in (\mathrm{e}^{-1/\eta}, \mathrm{e}^{-1/(2\eta)}), \\ \dfrac{2}{\eta r} & \text{for } r \in (\mathrm{e}^{-\eta}, \mathrm{e}^{-\eta/2}), \\ 0 & \text{otherwise.} \end{cases} \tag{2.21}$$

The choice of the cut-off function involving logarithmic terms is common in optimal cut-off functions for $N = 2$ and has been used in the literature (see, for example, [**6**]). Then, setting $v_k^{(\eta)} = \theta_\eta u_k$ and $w_k^{(\eta)} = (1 - \theta_\eta) u_k$, we can write

$$u_k = v_k^{(\eta)} + w_k^{(\eta)},$$

and proceed analogously to Case 2, above. Our choice of θ_η yields a bound on the gradient norm for both $v_k^{(\eta)}$ and $w_k^{(\eta)}$, which is uniform in η. The contribution of $v_k^{(\eta)}$ is small by the estimate (2.19) (similar to (2.13)), while $w_k^{(\eta)}$ converges uniformly to zero on the annulus $\mathrm{e}^{-1/\eta} \leqslant r \leqslant \mathrm{e}^{-\eta}$ by compactness in the Morrey embedding. □

Now we can give the proof of Theorem 1.1.

Proof of Theorem 1.1. For $N = 2$, the theorem follows from the corresponding statement in Theorem 2.6, and the equivalence of the Zygmund norm (1.6) (the same as $L^\infty(B, (\log(1/r))^{-1/2}\,\mathrm{d}x)$ and the Orlicz norm of $\exp L^2$. For $N > 2$, this is a particular case of Theorem 2.4 with $p = 2^*$ corresponding to $\alpha_p = 0$. □

3. Applications

Let $f \in C(\mathbb{R})$ and assume that there exists $A > 0$, such that, for all $s \in \mathbb{R}$,

$$|f(s)| \leqslant \begin{cases} A|s|^{2^*-1} & \text{for } N > 2, \\ \lambda |s| \mathrm{e}^{As^2} & \text{for } N = 2, \end{cases} \tag{3.1}$$

where $\lambda > 0$ will be specified later. Let

$$F(s) = \int_0^s f(t)\,\mathrm{d}t$$

and

$$\psi(u) := \int_\Omega F(u(x))\,\mathrm{d}x. \tag{3.2}$$

We recall that $\Omega = \mathbb{R}^N$ if $N > 2$ and $\Omega = B$ if $N = 2$. Then, clearly, $\psi \in C^1(E_0)$ and therefore $\psi \in C^1(E_V)$.

Proposition 3.1. *Assume that f satisfies (3.1) and let ψ be the functional (3.2). Then ψ is weakly continuous on E_V and ψ' is weakly continuous as a map from E_V into $E_V' \simeq E_V$.*

Proof. Let $u_k \rightharpoonup u$. Consider the case $N > 2$ first. By the compactness given by Theorem 2.4, $u_k \to u$ in L^{2^*}. The assertion then follows from the well-known continuity of $\psi\colon L^{2^*} \to \mathbb{R}$ and of $\psi'\colon L^{2^*} \to \mathcal{D}^{1,2}$.

Now, let $N = 2$. It suffices to show that $f(u_k) \to f(u)$ in $L^p(B)$ for any p. This follows from Lebesgue dominated convergence once we show that $|f(u_k)|^p$ is bounded by an L^1- function for all k sufficiently large. Indeed, combining an elementary inequality with Theorem 2.4, we have with some $\epsilon_k \to 0$,

$$u_k^2 \leqslant 2u_k^2 + 2(u_k - u)^2 \leqslant 2u_k^2 + C\epsilon_k \log(1/r).$$

Applying the bound (3.1), we obtain

$$\begin{aligned} |f(u_k)|^p &\leqslant |u_k|^p \exp(pAu_k^2) \\ &\leqslant \exp(p(A+1)u_k^2) \\ &\leqslant \exp(p(A+1)2u_k^2) \exp(p(A+1)C\epsilon_k \log(1/r)) \\ &= \frac{\exp(2p(A+1)u_k^2)}{r^{p(A+1)C\epsilon_k}} \\ &\leqslant \frac{\exp(2p(A+1)u_k^2)}{r}, \end{aligned}$$

where the last inequality holds for all k sufficiently large. □

Let

$$J(u) = \tfrac{1}{2}\|u\|^2 - \psi(u)$$

and define

$$\Phi := \{u_t \in C([0,1], E_V)\colon u_0 = 0 \text{ and } J(u_1) \leqslant 0\}.$$

We recall that the notation of the norm without further specification refers to the space E_V.

Theorem 3.2. *Assume that (3.1) holds and that the functional J satisfies the following functional-analytic conditions:*

$$\Phi \neq \emptyset, \tag{3.3}$$

$$\text{for every } \rho > 0 \text{ sufficiently small,} \quad c_\rho := \inf_{\|u\|=\rho} J(u) > 0, \tag{3.4}$$

and

$$J(u_k) \to c, \quad J'(u_k) \to 0 \quad \Longrightarrow \quad \|u_k\| \text{ is bounded}, \tag{3.5}$$

where

$$c := \inf_{u_t \in \Phi} \max_{0 \leqslant t \leqslant 1} J(u_t) > 0. \tag{3.6}$$

Then there exists $u \in E_V$ such that $J(u) = c$, $J'(u) = 0$. In particular, u is a weak solution of

$$-\Delta u + V(r)u = f(u). \tag{3.7}$$

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Moreover, the conditions (3.3)–(3.5) follow, respectively, from the following explicit conditions:

$$\sup_s F(s) > 0 \quad \textit{for } N > 2, \qquad \lim_{s\to\infty} \frac{F(s)}{s^2} = +\infty \quad \textit{for } N = 2, \tag{3.8}$$

$$\lambda < 4\alpha\lambda_1, \tag{3.9}$$

where λ is as in (3.1), α is as in (V2), λ_1 is the first eigenvalue of the Dirichlet Laplacian in B and

$$\textit{there exists } \theta > 2 \textit{ such that } f(s)s \geqslant \theta F(s) \textit{ for all } s \in \mathbb{R}. \tag{3.10}$$

Proof. From the standard Mountain Pass Theorem and conditions (3.3), (3.4), it follows that there exists a sequence $(u_k) \in E_V$ such that $J(u_k) \to c$ and $u_k - \psi'(u_k) \to 0$ in E_V. By (3.5), u_k is bounded in E_V and, therefore, for a renumbered subsequence, $u_k \rightharpoonup u$ in E_V. Then, by (3.1) and Theorem 1.1, $\psi'(u_k) \to \psi'(u)$ in E_V and thus $u_k \to u$ strongly in E_V. Consequently, u is a critical point with $J(u) = c$.

Next, note that (3.5) follows from (3.10) by repeating the classical argument in [**2**]. So, it remains to derive (3.3) from (3.8), and (3.4) from (3.9).

In the case $N = 2$ condition (3.3) follows from (3.8) immediately. Now, let $N > 2$. let us show that the path $u_t := u(t^{-1}\,\cdot)$, for a suitable $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$, is in the class Φ. Continuity of the path in $\mathcal{D}^{1,2}$ is shown, for example, in [**4**]. In order to verify continuity of the path in $L^2(\mathbb{R}^N, V\,\mathrm{d}x)$, note first that if $t_k \to t > 0$, then

$$\int_{\mathbb{R}^N} V(x)u_{t_k}^2\,\mathrm{d}x = \int_{\mathbb{R}^N} t_k^N V(t_k x)u^2\,\mathrm{d}x \to \int_{\mathbb{R}^N} t^N V(tx)u^2\,\mathrm{d}x.$$

Furthermore, if $t_k \to 0$, then, since u has compact support in $\mathbb{R}^N \setminus \{0\}$, we have $t_k^N V(t_k x) \to 0$ uniformly on $\operatorname{supp} u$. Using the change of variables $x = ty$ in $\int F(u_t)\,\mathrm{d}x$, we have

$$\lim_{t\to\infty} t^N \int_{\mathbb{R}^N} F(u(y))\,\mathrm{d}y = \infty,$$

once we observe that, by (3.8), the integral above becomes positive for a suitable u.

Finally, since the proof of (3.4) for $N > 2$ is clear from (3.1), let us then prove (3.4) for $N = 2$. From (V2) it follows that

$$\|u\|^2 \geqslant 4\alpha\|\nabla u\|_2^2 \geqslant 4\alpha\lambda_1 \int_B u^2\,\mathrm{d}x.$$

Therefore, the quadratic part of $J(u)$ is non-negative whenever the constant λ in (3.1) does not exceed $4\alpha\lambda_1$. We conclude that, once $\lambda < 4\alpha\lambda_1$, one has

$$J(u) \geqslant (4\alpha\lambda_1 - \lambda)\|u\|^2 + o(\|u\|^2),$$

from which (3.4) follows immediately. □

Remark 3.3. It is possible to provide sufficient conditions, alternative to (3.10), by following an argument based on Pohozhaev identity and similar to that of Jeanjean and Tanaka [**4**].

We note that we have considered only autonomous nonlinearities. Analogous results based on compact embeddings of E_V into appropriate weighted L^p-spaces are also possible. Due to the pointwise estimates (2.2) and (2.3), a nonlinearity satisfying (3.1) may be multiplied by $h(ur^{(N-2)/2})$ when $N > 2$ or by $h(u/\sqrt{\log(1/r)})$ when $N = 2$, with a general h, without a major change in the character of the problem. In particular, this includes nonlinearities of Hénon type (see [**8**]). We elaborate this case below as an illustration.

So now let $N > 2$ and $f \in C([0,\infty) \times \mathbb{R})$. Assume that there exists $C > 0$, $p > 2$ and $\lambda > 0$ such that, for all $r \geqslant 0$ and $s \in \mathbb{R}$,

$$|f(r,s)| \leqslant Cr^{\alpha_p}|s|^{p-1} + \lambda|s|r^{-2}. \tag{3.11}$$

As we did for the definitions at the start of this section we set

$$F(r,s) = \int_0^s f(r,t)\,\mathrm{d}t \quad \text{and} \quad \psi(u) := \int_{\mathbb{R}^N} F(r,u)\,\mathrm{d}x.$$

Then $\psi \in C^1(\mathcal{D}^{1,2}(\mathbb{R}^N))$ by (2.4), and therefore $\psi \in C^1(E_V)$. Moreover, due to Theorem 2.4, $\psi' \colon E_V \to E_V$ is continuous in E_V with respect to weak convergence. We will use the same notation as above for the analogous functional $J(u) = \frac{1}{2}\|u\|^2 - \psi(u)$ and for the class of paths $\varPhi$.

Theorem 3.4. *Assume that (3.11) holds and that the functional J satisfies the functional-analytic conditions (3.3)–(3.5) of Theorem 3.2, where c is given by (3.6). Then there exists $u \in E_V$ such that $J(u) = c$, $J'(u) = 0$. In particular, u is a weak solution of*

$$-\Delta u + V(r)u = f(r,u). \tag{3.12}$$

Moreover, the conditions (3.3)–(3.5) follow, respectively, from the following explicit conditions:

$$\lim_{r\to\infty} r^2 F(r,s) = \infty \quad \text{for } 0 < s < s_0 \text{ with some } s_0 > 0, \tag{3.13}$$

$$\text{the constant } \lambda \text{ in (3.11) is less than the constant } \alpha \text{ in (VN)} \tag{3.14}$$

and

$$\text{there exists } \theta > 2 \text{ such that } f(r,s)s \geqslant \theta F(r,s) \text{ for all } r > 0,\ s \in \mathbb{R}. \tag{3.15}$$

Proof. The assertion of the theorem follows from the same argument as in Theorem 3.2, with reference to (3.11) and Theorem 2.4. Moreover, (3.5) follows from (3.10) by repeating the classical argument in [**2**]. It only remains to verify the functional-analytic conditions (3.3), (3.4) from (3.13) and (3.14).

To prove (3.3), we use the same path $u_t := u(t^{-1}\,\cdot)$ as in Theorem 3.2. Then, with $t_k \to 0$, taking u that vanishes near the origin, we have $t_k^N V(t_k x) \to 0$ uniformly on $\operatorname{supp} u$. Therefore, (3.3) will follow once we prove (after using the change of variables $y = tx$) that

$$\lim_{t\to\infty} t^N \int_{\mathbb{R}^N} F(ty, u(y))\,\mathrm{d}y = \infty,$$

which in turn is immediate from (3.13) once we require that $0 \leqslant u \leqslant s_0$.

We turn now to (3.4). From (3.11) it follows that

$$J(u) \geqslant \frac{\|u\|^2}{2} - \frac{\lambda}{2}\int_{\mathbb{R}^N} \frac{|u|^2}{r^2}\,\mathrm{d}x - \frac{C}{p}\int_{\mathbb{R}^N} r^{\alpha_p}|u|^p\,\mathrm{d}x.$$

Note that $\gamma := \lambda/\alpha < 1$ in view of (3.9), so that, by Hardy's inequality and by (VN), we obtain

$$J(u) \geqslant \frac{1-\gamma}{2}\|u\|^2 - C_p\int_{\mathbb{R}^N} r^{\alpha_p}|u|^p\,\mathrm{d}x.$$

Therefore, (3.4) is immediate in view of Theorem 2.4. □

Appendix A. Proof of Proposition 2.1 for $N > 2$

By one-dimensional Morrey embedding, noting that the weights under the integrals are bounded and bounded away from zero, we have

$$\int_{r\in[1,2]} |u|^{2^*}\,\mathrm{d}x \leqslant C\int_{r\in[1,2]} (|\nabla u|^2 + r^{-2}u^2)\,\mathrm{d}x \biggl(\int_{r\in[1,2]} |u|^{2^*}\,\mathrm{d}x\biggr)^{1-(2/2^*)}.$$

Applying this inequality to $2^{(N-2)j/2}u_k(2^j x)$, $j \in \mathbb{Z}$, and rescaling the variables of integration, we obtain

$$\int_{r\in[2^j,2^{j+1}]} |u_k|^{2^*}\,\mathrm{d}x \leqslant C\int_{r\in[2^j,2^{j+1}]} (|\nabla u_k|^2 + r^{-2}u_k^2)\,\mathrm{d}x \biggl(\int_{r\in[2^j,2^{j+1}]} |u|^{2^*}\,\mathrm{d}x\biggr)^{1-(2/2^*)}.$$

Summation over $j \in \mathbb{Z}$ gives

$$\int_{\mathbb{R}^N} |u_k|^{2^*}\,\mathrm{d}x \leqslant C\int_{\mathbb{R}^N} (|\nabla u_k|^2 + r^{-2}u_k^2)\,\mathrm{d}x \sup_{j\in\mathbb{Z}} \biggl(\int_{r\in[2^j,2^{j+1}]} |u_k|^{2^*}\,\mathrm{d}x\biggr)^{1-(2/2^*)}.$$

With suitable $j_k \in \mathbb{Z}$, we arrive at

$$\int_{\mathbb{R}^N} |u_k|^{2^*}\,\mathrm{d}x \leqslant C\biggl(\int_{r\in[1,2]} |2^{-(N-2)j_k/2}u_k(2^{-j_k}x)|^{2^*}\,\mathrm{d}x\biggr)^{1-(2/2^*)},$$

where the last term converges to zero by the one-dimensional Morrey embedding.

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