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RANDOM GROUPS AND NONARCHIMEDEAN LATTICES

SYLVAIN BARRÉ[1] and MIKAËL PICHOT[2]

[1] Université de Bretagne Sud, Université Européenne de Bretagne, France;
email: sylvain.barre@univ.ubs.fr

[2] Department of Mathematics & Statistics, McGill University, Montréal, Québec, Canada H3A 2K6;
email: pichot@math.mcgill.ca

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Abstract

We consider models of random groups in which the typical group is of intermediate rank (in particular, it is not hyperbolic). These models are parallel to Gromov's well-known constructions, and include for example a 'density model' for groups of intermediate rank. The main novelty is the higher rank nature of the random groups. They are randomizations of certain families of lattices in algebraic groups (of rank 2) over local fields.

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1. Introduction

This paper introduces models of random groups 'of higher rank'. The construction, basic properties, and applications are detailed in § 2 to § 8 below, which we now summarize.

The construction (see § 2) is rather general. If Γ' is a group which acts properly on a simply connected complex X' of dimension 2 with X'/Γ' compact, and $\Gamma'' \subset \Gamma'$ is a subgroup of 'very large' finite index, then one can choose at random a family of Γ''-orbits of 2-cells $Y \subset X'$ inside X'. Then let X denote the universal cover of $X'' := X' \setminus Y$. The random group Γ is the group of transformations of the Galois covering

$$X \twoheadrightarrow X''/\Gamma''.$$

This construction leads to several distinct models of random groups including a 'density model', following Gromov. The initial structural data $(\Gamma', X', \ldots)$ for the model is called the *deterministic data*. The basic properties of Γ depend on the deterministic data.

An idea of groups 'of intermediate rank' was introduced in [2] in particular to address the following question, where X is CAT(0) and X/Γ is compact:

$$\mathbb{R}^2 \hookrightarrow X \Rightarrow \mathbb{Z}^2 \hookrightarrow \Gamma?$$

(This is the 'periodic flat plane problem' which has been formulated in many places; see [12] for an early reference.) Since the assumption $\mathbb{R}^2 \hookrightarrow X$ is equivalent to X being nonhyperbolic, the new models are relevant to the study of this question. We will see that in some cases (depending on the deterministic data, the density parameter, etc.) the answer is positive 'generically', but that the precise relation between the two conditions '$\mathbb{R}^2 \hookrightarrow X$' and '$\mathbb{Z}^2 \hookrightarrow \Gamma$' remains mysterious even for random groups associated with lattices in PSL_3.

Before turning to these models, let us discuss briefly Gromov's original construction of random groups and the density model introduced in [14] (see also [12, Section 6], [13], or [15]).

Let Γ_1 be a hyperbolic group in the sense of Gromov, and take successive quotients $\Gamma_1 \twoheadrightarrow \Gamma_2 \twoheadrightarrow \cdots$, say

$$\Gamma_{n+1} := \Gamma_n / \langle\langle R_n \rangle\rangle,$$

where $R_n \subset \Gamma_n$ is a finite set of additional relations. As explained in [14], the set R_n can 'in general' be chosen so that the following hold:

(i) Γ_{n+1} stays hyperbolic;

(ii) $\Gamma_n \twoheadrightarrow \Gamma_{n+1}$ is injective on larger and larger balls.

Property (ii) ensures the existence of an infinite limit group Γ_∞, while the attribute 'in general' accounts for the oversupply of choices in the construction; its precise meaning depends on the size and the nature of R_n. For example, if Γ_1 is torsion free and the sets R_n consist of a single relation which is a 'higher and higher' power of the nth element in a list exhausting Γ_1, then the limit Γ_∞ is a finitely generated infinite torsion group [14, Section 4.5.C]. Here, (i) and (ii) become geometric assertions relying on $K < 0$, and the construction offers almost total freedom. (Gromov's construction is related to the Burnside problem—the existence of infinite torsion groups was established by Golod, and the existence of infinite groups of finite exponent by Adian and Novikov, and by Olshanskii using the small cancellation theory.)

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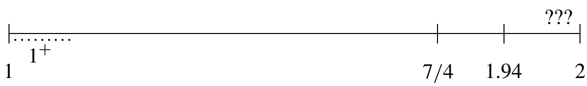
A prominent feature is the *genericity of hyperbolic groups*, as put forward in [14] and illustrated by (i) above. Gromov has since invented several models of random groups and constructed many exotic infinite groups using them [11, 15–17, 21]. We are interested here in his so-called *density model*, which studies 'one-step' random quotients $\Gamma_n \twoheadrightarrow \Gamma_{n+1}$ for 'very large' random sets R_n of 'very long' relations. If one starts with a free group F_r on r generators, and we let δ denote the density parameter, then the random group in the density model is a quotient of the form $F_r/\langle\langle W_p\rangle\rangle$, where W_p is a set of $\approx|S_p|^\delta$ words chosen uniformly independently at random in the sphere S_p of radius p in F_r. Gromov shows that, if $\delta < 1/2$, then the resulting random group is hyperbolic with overwhelming probability as $p \to \infty$, while if $\delta > 1/2$ it is trivial (meaning 1 or $\pm$) with overwhelming probability. In this model, small cancellations occur for $\delta < 1/12$. (One can also start here with a nonelementary hyperbolic group Γ and take random quotients by elements in the spheres $S_p \subset \Gamma$, $p \to \infty$; the same phase transition '$\delta < 1/2 \Rightarrow$ hyperbolic' and '$\delta > 1/2 \Rightarrow 1/\pm$' is then valid provided that Γ is torsion free [22].) An earlier model of Gromov, called the 'few-relator model', studies the situation where $|W_p|$ is bounded. We refer to [21] for a survey of these groups.

The groups of intermediate rank constructed in [2] can be put on a 'rank interpolation line':



(we emphasize again that this only has a schematic value: as discussed in [2], the phenomenon of rank interpolation is not unidimensional.) The two extreme cases in this picture are the hyperbolic groups (rank 1, or more generally the groups with isolated flats, of rank 1^+) and the lattices in nonarchimedean groups (rank 2). The value 1.94 refers to the bowtie group $\Gamma_{\bowtie}$ introduced in [2] and further studied in [4]. The present paper constructs many groups whose rank is arbitrarily close to 2: if the deterministic data arises from a nonarchimedean Lie group of rank 2, then the 'rank' of the random group (for example, the local rank in the sense of [3, Definition 4.5]) is as close to 2 as desired.

We now formulate our main result in the special case of the density model with deterministic data the Cartwright–Steger lattices in $\mathrm{PGL}_3(\mathbf{F}_q((y)))$ and their congruence subgroups. The techniques and constructions involved in the proof of this result apply in more general situations, and we will state and establish more general statements along the text. In fact, most of the assertions in Theorem 1, with the notable exception of the fact that '$\delta < \frac{5}{8} \Rightarrow \mathbb{Z}^2 \hookrightarrow \Gamma$', will be proved under less restrictive assumptions.

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The Cartwright–Steger lattices [7] are uniform lattices $\Gamma_n < \mathrm{PGL}_n(\mathbf{F}_q((y)))$ associated with the ring $R = \mathbf{F}_q[y, 1/y, 1/(1+y)] \hookrightarrow \mathbf{F}_q(y)$. Their congruence subgroups $\Gamma_n(I)$ correspond to ideals $I \lhd R$. The groups Γ_n act transitively on the vertexes of the Bruhat–Tits building X_n of $\mathrm{PGL}_n(\mathbf{F}_q((y)))$. Below $n = 3$; the random groups discussed in Theorem 1 have as deterministic data the lattices $(X_3, \Gamma_3, \{\Gamma_3(I_p)\})$ associated with the Cartwright–Steger lattices of rank 2.

THEOREM 1. *Let q be a prime power. Fix two sequences $(f_p)_{p\geqslant 1}$ and $(s_p)_{p\geqslant 1}$, where $f_p \in \mathbf{F}_q[y]$ is a monic irreducible polynomial prime to y and $y+1$ and $s_p \geqslant 1$ is an integer. If the density parameter δ satisfies*

$$\delta < \tfrac{5}{8},$$

then the random group Γ in the density model of parameter δ with (Cartwright–Steger) deterministic data $(X_3, \Gamma_3, \{\Gamma_3(I_p)\})$ satisfies

$$\mathbb{Z}^2 \hookrightarrow \Gamma$$

with overwhelming probability, where the congruence subgroups $\Gamma_3(I_p)_{p\geqslant 1}$ are associated with the ideal $I_p = \langle f_p^{s_p} \rangle$ generated by $f_p^{s_p}$ in $\mathbf{F}_q[y, 1/y, 1/(1+y)]$. If in addition $s_p \geqslant k$ for p large enough, then $\mathbb{Z}^2 \hookrightarrow \Gamma$ with overwhelming probability whenever

$$\delta < \frac{7k-3}{7k+1}$$

(which can be made as close to 1 as desired, independently of q). Furthermore, if

$$\delta < \frac{q-1}{q-2}$$

(which can be made as close to 1 as desired, independently of k), then Γ acts freely uniformly on a space X of dimension 2 with the geodesic extension property, and if

$$\delta < \tfrac{1}{2} \quad \text{and} \quad q \geqslant 5,$$

then Γ has Kazhdan's property T with overwhelming probability. In addition, if δ_0 is an arbitrary real number <1 given in advance, then there exists q_0 such that, if $q \geqslant q_0$, then Γ has Kazhdan's property T with overwhelming probability for every $\delta < \delta_0$. If, on the other hand,

$$\delta > \frac{q}{q-1},$$

then Γ does not have property FA. Finally, if

$$\delta < \tfrac{1}{2}$$

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(and q is arbitrary), then (X, Γ) has the extension rigidity property of [3] *(namely, it 'remembers' the building X_3 it comes from). Finally, if $\delta_0 < 1$ and $\varepsilon > 0$ are real numbers given in advance, then there exists q_0 such that, if $q \geqslant q_0$, then (X, Γ) has local rank (in the sense of* [3]*) uniformly $\geqslant 2 - \varepsilon$ with overwhelming probability for every $\delta < \delta_0$.*

The paper is structured as follows. An analog of Gromov's few-relator model is studied in Section 4 in relation with the periodic flat plane problem. The density model is studied from Section 5 onwards, where we introduce 'critical densities' for various properties of groups in this model. In our framework, the density parameter δ regulates the size of the random subset $Y \subset X'$. For example, we have a critical density δ_T for Kazhdan's property T, δ_{FA} for property FA, etc., and most importantly the critical density $\delta_{\mathbb{Z}^2}$ for the property that $\mathbb{Z}^2 \hookrightarrow \Gamma$. The critical densities depend on the deterministic data. In Section 6, we discuss various analogs of Gromov's $\delta = 1/2$ phase transition theorem in the density model, while Section 7 is devoted to estimating $\delta_{\mathbb{Z}^2}$ for nonarchimedean lattices (in positive characteristic). Finally, Section 8 derives other properties of the random group (for example property T) and its 'intermediate rank' behavior. The reason why property T arises only 'for sufficiently large residue fields' is the same as in Garland's paper [10], namely that the spectral gap is large enough only for q large enough.

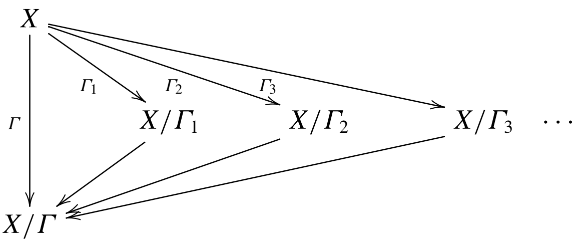
We conclude with a question, the issue of which seems hard to predict at this stage.

QUESTION 2. Is there (X, Γ) of local rank $> r$ such that $\mathbb{Z}^2 \not\hookrightarrow \Gamma$ for every $r < 2$?

Here, X is a CAT(0) of dimension 2 and $\Gamma \curvearrowright X$ freely with X/Γ compact, and the local rank is defined in [3]. The question can be considered both in the general case or when the order q is bounded. See [3, Question 0.2] for a related question (some aspects of the 'local to global problem' implicit in Question 2 are also discussed in [3]) and also [4, Problem 3].

2. Description of the random group

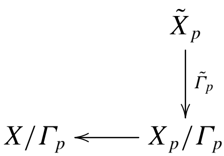
Let Γ be a discrete group acting freely simplicially on a 2-complex X with X/Γ compact, and let $\Gamma_1, \Gamma_2, \ldots$ be a family of finite index subgroups of Γ with $[\Gamma : \Gamma_p] \to \infty$, $p \to \infty$. The random group defined below is a 'randomization' of the following (deterministic) sequence of covering maps:



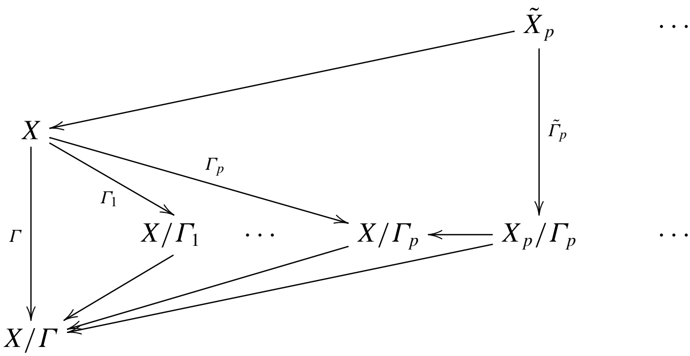
associated with X and $(\Gamma_p)_{p\geqslant 1}$. (The family may be nested $\Gamma \geqslant \Gamma_1 \geqslant \Gamma_2 \cdots$, and may correspond to a tower $X/\Gamma \leftarrow X/\Gamma_1 \leftarrow X/\Gamma_2 \leftarrow X/\Gamma_3 \leftarrow \cdots$ of compact spaces.) The randomization is achieved by inserting a 'random topological noise' to the spaces X/Γ_p (where p is very large) which is detected by the fundamental group.

DEFINITION 3. We call $(X, \Gamma, \{\Gamma_p\})$ the *deterministic data*.

The topological perturbation is implemented as follows. For each $p \geqslant 1$, remove a family of 2-cells in X/Γ_p (equivalently, a family of Γ_p-orbits of 2-cells in the *fixed* space X) at random (with respect to a probability scheme for removing 2-cells, for example, Bernoulli). The universal cover $\tilde{X}_p$ of the resulting (random) space $X_p \subset X$ has a (random) group $\tilde{\Gamma}_p$ of deck transformations:



as described in the following overall diagram:



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We study the properties of $(\tilde{X}_p, \tilde{\Gamma}_p)$ when the order of approximation p is very large. The construction provides a random group $\tilde{\Gamma}_p$, a random space $\tilde{X}_p$, a random action $\tilde{\Gamma}_p \curvearrowright \tilde{X}_p$, and a random compact space $\tilde{X}_p/\tilde{\Gamma}_p$.

At this stage, the model will be precisely defined as long as the random scheme removing 2-cells is specified; several options are possible. We study here analogs of Gromov's well-known models for adding relators at random to a finitely generated group (more precisely, analogs of the 'few-relator model' and the 'density model' of [14, 15]).

Fix the deterministic data $(X, \Gamma, \{\Gamma_p\})$ (different choices for the triples $(X, \Gamma, \{\Gamma_p\})$ give rise to different models of random groups and lead to a priori distinct random objects). The set

$$\mathscr{C}_p := \{\Gamma_p\text{-orbits of faces in } X\}$$

(whose element are called *equivariant faces* of X with respect to Γ_p, or sometimes *equivariant chambers* when we have in mind a Bruhat–Tits building) is finite, and in many interesting cases it is rapidly growing. It plays the role of the set

$$W_p := \text{ all words (or reduced words) of length } p \text{ (or at most } p) \text{ in } \Gamma$$

in Gromov's models, from which the relations are picked up at random and added to the given group Γ (e.g. the free group $\mathbf{F}_2$).

For a finite subset $A = \{C_1, \ldots, C_k\}$ of $\mathscr{C}_p$, we set

- $X_A := X - \bigcup_{l=1}^{k} \mathring{C}_l$.
- $K_A := \pi_1(X_A)$ and $\tilde{X}_A \twoheadrightarrow X_A$ is the corresponding universal cover.
- Γ_A is the Galois group of the covering map:

$$\tilde{X}_A \twoheadrightarrow X_A/\Gamma_p.$$

The random group is defined by the following.

DEFINITION 4. Fix for every $p \geqslant 1$ a process $\mathbb{P}_p$ for selecting random subsets of elements in $\mathscr{C}_p$. The *random group of order* p in the $(X, \Gamma, \{\Gamma_p\}, \{\mathbb{P}_p\})$-model is the group Γ_A associated by the construction above to the $\mathbb{P}_p$-generic subset $A \subset \mathscr{C}_p$. We say that a property P occurs *with overwhelming probability* in this model if the probability that the random group Γ_A of order p satisfies P converges to 1 as $p \to \infty$.

REMARK 5. (1) In all cases considered below, the random process $\{\mathbb{P}_p\}$ is *universal* in that it does not depend on the deterministic data $(X, \Gamma, \{\Gamma_p\})$. More

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precisely, a predetermined process $\mathbb{P}$ is chosen for selecting a random finite subset in an arbitrary finite set, and this process $\mathbb{P}$ is applied recursively to the terms of the sequence $(\mathscr{C}_p)_p$.

(2) Most of the results of the present paper extend easily to the case of *proper* actions on *cell* complexes with compact quotient. We also note that these models of random group only take into account the 'profinite information' contained in the deterministic data.

We study two special cases of Definition 4: the 'bounded model' and the 'density model'. In the first model, a uniformly bounded number of elements of $\mathscr{C}_p$ is chosen at random.

DEFINITION 6 (The bounded model). Fix an integer parameter $c \geqslant 1$. The *bounded model* over $(X, \Gamma, \{\Gamma_p\})$ is the $(X, \Gamma, \{\Gamma_p\}, \{\mathbb{P}_p\})$-model associated with the process

$$\mathbb{P}_p := \text{ 'choose } c \text{ chambers in } \mathscr{C}_p\text{, uniformly and independently at random'.}$$

This corresponds to Gromov's 'few-relator model'.

In the second model, the number of chosen chambers in $\mathscr{C}_p$ is unbounded and rapidly growing.

DEFINITION 7 (The density model). Fix a real parameter $\delta > 0$ (the density). The *density model* over $(X, \Gamma, \{\Gamma_p\})$ is the $(X, \Gamma, \{\Gamma_p\}, \{\mathbb{P}_p\})$-model associated with the process

$$\mathbb{P}_p := \text{ 'choose } |\mathscr{C}_p|^\delta \text{ chambers in } \mathscr{C}_p\text{, uniformly and independently at random'.}$$

This corresponds to the Gromov 'density model'. As in Gromov models, the bounded model can be seen as a manifestation of the 'density model with $\delta = 0$' (see also Section 5).

REMARK 8. In a sense, the new models can be thought of as 'mirror images' of the Gromov models: rather than starting with a group with a large supply of quotients, for example nonabelian-free groups, and gradually adding relations at random, we typically start (see below) with lattices in some algebraic group of rank 2, which have 'as many relations as is conceivable' for an infinite group (in particular, they are just infinite up to center), and remove them at random. It is unclear how to randomize (say, residually finite) discrete groups of higher cohomological dimension; for example, the above construction provides a precise meaning for the expression 'Γ is a random extension of a lattice in $\mathrm{PSL}_n(K)$' for $n = 3$, where K is a local field—what about $n > 3$?

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3. Preliminary results

Let $(X, \Gamma, \{\Gamma_p\})$ be the deterministic data, and let $p \geqslant 1$, $A \subset \mathscr{C}_p$, X_A, Γ_A, K_A (following the notation of Section 2) be fixed throughout the section.

Observe that the sequence of covering spaces $\tilde{X}_A \twoheadrightarrow X_A \twoheadrightarrow X_A/\Gamma_p$ provides an exact sequence (nonsplit in general)

$$1 \to K_A \to \Gamma_A \to \Gamma_p \to 1$$

of discrete groups.

LEMMA 9. *If X is contractible, then the following hold.*

(1) *The homology groups $H_i(X_A, \mathbb{Z})$ vanish for $i \geqslant 2$, and the covering space $\tilde{X}_A$ is contractible.*

(2) *The group K_A is a free group on countably many generators.*

(3) *The group Γ_A is a finitely presented group of geometric dimension 2. If $|A| \neq 0$, then Γ_A is a strict extension of Γ.*

Proof. The first part of the first assertion is clear, and the second part classically follows from the first ($\tilde{X}_A$ is weakly homotopy equivalent to a point, and therefore contractible). The universal coefficient theorem

$$0 \to \mathrm{Ext}(H_1(X_A, \mathbb{Z}), \mathbb{Z}) \to H^2(X_A, \mathbb{Z}) \to \mathrm{Hom}(H_2(X_A, \mathbb{Z})) \to 0,$$

where the module $H_1(X_A, \mathbb{Z})$ is free and thus projective, shows that the second integral cohomology $H^2(X_A, \mathbb{Z})$ vanishes. Thus, K_A is a free group. This follows from the Stallings–Swan theorem that a group of cohomological dimension 1 is free. (Note that the group K_A does not act 'naturally' on a tree in general.) Let us prove (3). Since Γ acts freely on X contractible, X/Γ is an Eilenberg–MacLane space, $K(\Gamma, 1)$, and it follows from (1) that $\tilde{X}_A/\Gamma_A$ is a $K(\Gamma_A, 1)$ and in particular Γ_A is finitely presented of geometric dimension 2. (If the action $\Gamma \curvearrowright X$ is only assumed to be proper, then the groups Γ_A are of proper (Bredon) geometric dimension 2.) Assume that $|A| \neq 0$. Let γ be the boundary of an element of A. If $\Gamma_A \to \Gamma$ is an isomorphism, then γ is a boundary in X_A, and therefore is an equator in a 2-sphere of X. This contradicts the fact that X_A is aspherical, so $K_A \neq 1$, and, by equivariance, K_A is infinitely generated. □

REMARK 10. If Γ is of higher cohomological dimension, then K_A is not necessarily free. The density model is interesting to study in this situation, and it seems to be working properly only in 'high density regimes' (this to compensate,

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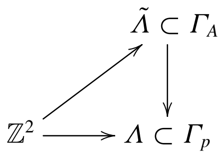
especially in the case of SL_n, $n \geqslant 4$, for the very strong ambient rigidity properties of the deterministic data).

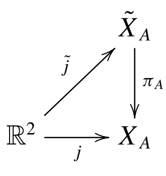
The following 'flat plane correspondence' assertion is useful in connection with Gromov's periodic flat plane problem.

LEMMA 11. *Assume that X is a CAT(0) space and that $\Gamma \curvearrowright X$ is isometric. For every flat $\Pi \subset X_A$, there exists a flat $\tilde{\Pi} \subset \tilde{X}_A$ such that the restriction of $\tilde{\pi}_A : \tilde{X}_A \to X_A$ to $\tilde{\Pi}$ is an isometry onto Π. Furthermore, if there exists a subgroup $\Lambda \subset \Gamma$, $\Lambda \simeq \mathbb{Z}^2$, such that Π/Λ is a compact torus, then there is a corresponding subgroup $\tilde{\Lambda} \subset \Gamma_A$, $\tilde{\Lambda} \simeq \mathbb{Z}^2$, such that $\tilde{\Pi}/\tilde{\Lambda}$ is a compact torus, and the following diagram commutes:*



Conversely, if $\tilde{\Pi} \subset \tilde{X}_A$ is a flat in $\tilde{X}_A$, then $\tilde{\pi}_A(\tilde{\Pi})$ is a flat in X_A. If in addition there exists a subgroup $\tilde{\Lambda} \subset \Gamma_A$, $\tilde{\Lambda} \simeq \mathbb{Z}^2$, such that $\tilde{\Pi}/\tilde{\Lambda}$ is a compact torus, then its projection Λ in Γ is a subgroup isomorphic to $\mathbb{Z}^2$, such that Π/Λ is a compact torus.

Proof. Since $\pi_A : \tilde{X}_A \to X_A$ is a locally isometric covering map and $\mathbb{R}^2$ is contractible, the map $j : \mathbb{R}^2 \to \Pi \subset X_A$ admits a unique isometric lifting $\tilde{j} : \mathbb{R}^2 \to \tilde{\Pi} \subset \tilde{X}_A$ through any point $\tilde{j}(0) = \tilde{x} \in \tilde{X}_A$ such that $\pi_A(\tilde{x}) = j(0) =: x$, such that the following diagram commutes:



The map $\tilde{\pi}_A$, being a local isometry, restricts to an isometry from $\tilde{\Pi}$ onto Π. Let $\Lambda \subset \Gamma$ be a subgroup isomorphic to $\mathbb{Z}^2$ such that Π/Λ is a compact torus. Let $s \in \Lambda$, and let $g_s \in \Gamma_A$ be any lift of $s \in \Lambda$. By equivariance of π_A, we have $\pi_A(g_s(\tilde{x})) = sx$. Choose $k_s \in K_A$ such that $k_s g_s \tilde{x} \in \tilde{\Pi}$. Then $(k_s g_s)^{-1}(\tilde{\Pi})$ is an isometric lifting of Π which contains $\tilde{x}$. Uniqueness of liftings implies that $k_s g_s(\tilde{\Pi}) = \tilde{\Pi}$. Let $\{a, b\}$ be a generating set of Λ, and let $\tilde{a}, \tilde{b} \in \Gamma_A$ be such that $\tilde{a}(\tilde{\Pi}) = \tilde{\Pi}$ and $\tilde{b}(\tilde{\Pi}) = \tilde{\Pi}$. Since $\pi_A(\tilde{a}\tilde{b}\tilde{x}) = ab\pi_A(\tilde{x}) = ba\pi_A(\tilde{x}) = \pi_A(\tilde{b}\tilde{a}\tilde{x})$

and $\pi_A : \tilde{\Pi} \to \Pi$ is isometric, the elements $\tilde{a}$ and $\tilde{b}$ of Γ_A commute and therefore generate an (infinite torsion-free) abelian group $\tilde{\Lambda} \subset \Gamma_A$. If $\Lambda \simeq \mathbb{Z}$, let $\tilde{c}$ be a generator, and let $c \in \Gamma_p$ be its image. Then $c(\Pi) = \Pi$ and $\Lambda \subset \langle c \rangle$, contradicting $\Lambda \simeq \mathbb{Z}^2$. Therefore $\tilde{\Lambda} \curvearrowright \tilde{\Pi}$ is conjugate to a discrete action of $\mathbb{Z}^2$ on $\mathbb{R}^2$, and hence $\tilde{\Pi}/\tilde{\Lambda}$ is compact.

Conversely, let Π be the image of $\tilde{\Pi}$ under π_A. Since $\pi_{A|\tilde{\Pi}}$ is a locally isometric covering map, Π is either a torus, a cylinder, or a flat plane. The first two possibilities are incompatible with the fact that $\Pi \subset X_A \subset X$, where X is a CAT(0) space of dimension 2. In particular, we see that, if $\tilde{\Lambda} = \langle \tilde{a}, \tilde{b} \rangle$ is a subgroup isomorphic to $\mathbb{Z}^2$ acting on $\tilde{\Pi}$ with $\tilde{\Pi}/\tilde{\Lambda}$ compact, then $K_A \cap \tilde{\Lambda} = \{e\}$. But this implies that its image Λ in Γ is isomorphic to $\mathbb{Z}^2$, acting freely on Π with Π/Λ compact. □

LEMMA 12. *Assume that X is a CAT(0) space and that Γ acts isometrically. Assume that $A \subset B \subset \mathscr{C}_p$.*

(1) *If Γ_A is Gromov hyperbolic, then Γ_B is Gromov hyperbolic.*

(2) *If Γ_B contains a subgroup isomorphic to $\mathbb{Z}^2$, then Γ_A contains a subgroup isomorphic to $\mathbb{Z}^2$.*

Proof. (1) By invariance under quasiisometry, it is enough to prove that $\tilde{X}_B$ is hyperbolic. If it is not hyperbolic, then, being a CAT(0) space, it contains a flat plane $\Pi \simeq \mathbb{R}^2 \hookrightarrow \tilde{X}_B$ by the flat plane theorem. Since the image of Π in X_B under the covering map $\pi_B : \tilde{X}_B \twoheadrightarrow X_B$ is a flat plane, the space X_A contains a flat plane, and therefore $\tilde{X}_A$ contains a flat plane by Lemma 11. Therefore, $\tilde{X}_A$ is not hyperbolic and Γ_A is not a hyperbolic group, contrary to assumption. Thus Γ_B is hyperbolic.

(2) Since $A \subset B$, the map $\tilde{X}_B \twoheadrightarrow X_B \hookrightarrow X_A$ lifts to a map

$$\tilde{X}_B \to \tilde{X}_A \twoheadrightarrow X_A.$$

Let $Y = \tilde{X}_A$, $\Delta = \Gamma_A$, and let C be the pull-back of $B \setminus A \subset X$ in Y. Let also $\tilde{\Lambda} \subset \Gamma_B$ be a subgroup isomorphic to $\mathbb{Z}^2$, and let $\tilde{\Pi} \subset \tilde{X}_B$ be the corresponding periodic flat. We apply Lemma 11 to the isometric action $\Delta \curvearrowright Y$, and the subset C of Y. Then Y_C coincides with the image of $\tilde{X}_B$ in Y and $\tilde{Y}_C = \tilde{X}_B$, $\Delta_C = \Gamma_B$. It follows that $\Pi = \tilde{\pi}_C(\tilde{\Pi})$ is a flat in $Y_C \subset Y = \tilde{X}_A$ and that the projection Λ of $\tilde{\Lambda}$ in Δ is a subgroup isomorphic to $\mathbb{Z}^2$, such that Π/Λ is compact. This shows that Γ_A contains a copy of $\mathbb{Z}^2$. □

The same is true for the isolated flats property.

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LEMMA 13. *Assume that X is a CAT(0) space and that Γ acts isometrically. If X has the isolated flats property, then $\tilde{X}_A$ has the isolated flats property.*

4. Random periodic flat plane problems I

We consider the bounded model first.

Recall that a countable group Γ is called *virtually indicable* if some finite index subgroup of Γ admits an infinite abelian quotient.

The main theorem in this section is a result that is significantly more general than Theorem 1 but is restricted to the bounded model. It produces groups which are *infinitesimal perturbations* of the deterministic data.

LEMMA 14. *Assume that the deterministic data $(X, \Gamma, \{\Gamma_p\})$ satisfies:*

- *X is a CAT(0) space and Γ acts isometrically;*
- *$\Gamma_p \triangleleft \Gamma$, and $[\Gamma : \Gamma_p] \to_p \infty$;*
- *Γ is not virtually indicable;*
- *$\mathbb{Z}^2 \hookrightarrow \Gamma$;*

then the random group in the bounded $(X, \Gamma, \{\Gamma_p\})$-model contains a copy of $\mathbb{Z}^2$ with overwhelming probability.

Proof. Let Λ be a subgroup of Γ isomorphic to $\mathbb{Z}^2$. Let $\Pi \hookrightarrow X$ be a flat plane on which Λ acts freely with X/Λ a compact torus (which exists by the flat torus theorem). Consider the subgroup

$$\Lambda_p = \Gamma_p \cap \Lambda,$$

and let F_p be the set of Λ_p-orbits of chambers in Π. Since

$$[\Lambda : \Lambda_p] \leqslant [\Gamma : \Gamma_p],$$

the set F_p is finite. We denote by $F'_p \subset \mathscr{C}_p$ the image of F_p into $\mathscr{C}_p$ under the map which associates to a Λ_p-orbit in Π its corresponding Γ_p-orbit in X.

Let $(\Gamma_{A_p}, \tilde{X}_{A_p})$ be the random group associated with a random subset $A_p \subset \mathscr{C}_p$ in the bounded model (so $|A_p| = c \geqslant 1$ is a fixed integer). We will show that

$$\mathbb{P}(A_p \cap \Pi \neq \emptyset) \xrightarrow[p\to\infty]{} 0.$$

Assume toward a contradiction that the limit is not zero. Then there exist $\delta > 0$ and a sequence $p_1 < p_2 < p_3 < \cdots$ of integers such that, for all $i \geqslant 1$, the random set A_{p_i} in $\mathscr{C}_{p_i}$ contains, with probability at least $\delta > 0$, an equivariant chamber C such that $C \cap \Pi \neq \emptyset$:

$$\mathbb{P}(\exists C \in A_{p_i} \mid C \cap \Pi \neq \emptyset) \geqslant \delta.$$

A fortiori,

$$\sum_{C \in A_{p_i}} \mathbb{P}(C \cap \Pi \neq \emptyset) \geqslant \delta.$$

Thus, for all $i \geqslant 1$,

$$|\{C \in \mathscr{C}_{p_i} \mid C \cap \Pi \neq \emptyset\}| \geqslant \frac{\delta}{c} |\mathscr{C}_{p_i}|.$$

Since the condition $C \cap \Pi \neq \emptyset$ is equivalent to the fact that $C \in F'_{p_i}$, we obtain

$$|F_{p_i}| \geqslant |F'_{p_i}| \geqslant |\{C \in \mathscr{C}_{p_i} \mid C \cap \Pi \neq \emptyset\}| \geqslant \frac{\delta}{c} |\mathscr{C}_{p_i}|.$$

Consider the groups $G_p = \Gamma / \Gamma_p$ and $H_p = \Lambda / \Lambda_p$, which act freely on $\mathscr{C}_p$ and F_p, respectively.

It follows that there exists a constant κ, which depends only on δ, c, the number of Γ-orbit of chambers in X, and the number of Λ-orbits of chambers in Π, such that

$$[G_{p_i} : H_{p_i}] \leqslant \kappa.$$

Hence, for every i, the subgroup $\Lambda \Gamma_{p_i}$ of Γ is (by the isomorphism theorems) of index

$$[\Gamma : \Lambda \Gamma_{p_i}] \leqslant \kappa.$$

As Γ is finitely generated (for it acts freely on X with X/Γ compact), the family $\mathfrak{F}$ of subgroups of index at most κ is finite.

A finitely generated group Γ is not virtually indicable if and only if every finite index subgroup Γ_0 of Γ has finite abelianization $\Gamma_0/[\Gamma_0, \Gamma_0]$. Since $\Lambda \Gamma_{p_i} \in \mathfrak{F}$, we have

$$|\Lambda \Gamma_{p_i} / [\Lambda \Gamma_{p_i}, \Lambda \Gamma_{p_i}]| < \gamma$$

for some fixed $\gamma > 0$, not depending on i. On the other hand, since $\Lambda \simeq \mathbb{Z}^2$,

$$H_{p_i} \subset \Lambda \Gamma_{p_i} / [\Lambda \Gamma_{p_i}, \Lambda \Gamma_{p_i}],$$

and as

$$|H_{p_i}| \xrightarrow[|G_{p_i}| \to \infty]{} \infty,$$

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we have

$$|\Lambda\Gamma_{p_i}/[\Lambda\Gamma_{p_i},\Lambda\Gamma_{p_i}]| \xrightarrow[p_i\to\infty]{} \infty,$$

which gives the desired contradiction. Thus,

$$\mathbb{P}(A_p \cap \Pi \neq \emptyset) \xrightarrow[p\to\infty]{} 0.$$

We now apply Lemma 11, whose assumptions are satisfied with overwhelming probability. This shows that $\mathbb{Z}^2 \hookrightarrow \Gamma_{A_p}$ with overwhelming probability. □

Theorem 16 below can be expressed by saying that 'the periodic flat plane problem is stable under randomization in the bounded model', under suitable assumptions. More precisely, we have the following.

DEFINITION 15. We say that a property P is *stable under randomization* in the $(X, \Gamma, \{\Gamma_p\}, \{\mathbb{P}_p\})$-model if the random group has property P with overwhelming probability provided that the deterministic data $(X, \Gamma, \{\Gamma_p\})$ has property P.

THEOREM 16. *Let $(X, \Gamma, \{\Gamma_p\})$ be the deterministic data, where Γ acts isometrically on X CAT(0), and $\Gamma_p \triangleleft \Gamma$ is a sequence of normal subgroups such that $[\Gamma : \Gamma_p] \to \infty$. If Γ is not virtually indicable, then the periodic flat plane alternative is stable under randomization in the bounded $(X, \Gamma, \{\Gamma_p\})$-model.*

Proof. Assume that Γ satisfies the periodic flat plane alternative and let us prove that the random group does. If Γ contains $\mathbb{Z}^2$, then Lemma 14 shows more: every copy of $\mathbb{Z}^2$ eventually survives in the random group. If Γ does not contain $\mathbb{Z}^2$, then it is hyperbolic, and Lemma 12 shows that the random group is hyperbolic as well. □

According to the 'flat closing conjecture' (which is equivalent to the hypothesis that the periodic flat plane problem always has a positive answer), one expects that virtual indicability does not play a role regarding the existence of a periodic flat plane in the bounded model. The conjecture implies that, for arbitrary deterministic data $(X, \Gamma, \{\Gamma_p\})$, where X is CAT(0) and Γ acts isometrically, the periodic flat plane alternative is stable under randomization in the bounded $(X, \Gamma, \{\Gamma_p\})$-model. This can be checked easily in a few situations, including spaces with isolated flats, using Lemma 13.

Let us conclude this section with a statement which (in the case of the bounded model) weakens the assumption on the deterministic data for the $\mathbb{Z}^2 \hookrightarrow \Gamma$ assertion in Theorem 1.

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Let k be a nonarchimedean local field with discrete valuation, and let $\mathbf{G}$ be an algebraic group of rank 2 over k. Let X be the associated Bruhat–Tits building, let Γ be a uniform lattice in $\mathbf{G}$ acting freely on X, let $\{\Gamma_p\}_p$ be a sequence of finite index normal subgroups, and let us take $(X, \Gamma, \{\Gamma_p\})$ as the deterministic data.

COROLLARY 17. *If Γ is the random group in the bounded $(X, \Gamma, \{\Gamma_p\})$-model, then $\mathbb{Z}^2 \hookrightarrow \Gamma$ with overwhelming probability.*

Note that such deterministic data can be constructed. Indeed, let Γ' be any uniform lattice in $\mathbf{G}$ (see [19, Ch. IX.3]; if the characteristic of k is zero, then all lattices are all uniform). Then, by a well-known result of Selberg, Γ has a torsion-free subgroup of finite index Γ, and since $\mathbf{G}$ over k is a linear group, Malcev's theorem shows that Γ is residually finite. Now Γ_p may be taken to be any sequence of finite index subgroups of Γ, for example a sequence of normal subgroups with trivial intersection.

Proof. Being an algebraic group of rank 2, $\mathbf{G}$ satisfies Kazhdan's property T (see [5, Ch. 1, Theorem 1.6.1]), and so does the lattice Γ. *A fortiori*, Γ is not virtually indicable, and therefore Lemma 14 applies. □

5. Critical densities, factors, and lifts

This section defines two relevant critical densities for studying the periodic flat plane problem in the density model: $\delta_{\mathbb{R}^2}$ (the critical density for being hyperbolic) and $\delta_{\mathbb{Z}^2}$ (the critical density for containing $\mathbb{Z}^2$). Existence follows from the 'lift/quotient stability in the density model' of the corresponding group property (compare Proposition 23).

We call a group property P *lift stable* (respectively, *factor stable* or *quotient stable*), if, for every surjective morphism $\Gamma \twoheadrightarrow \Gamma'$ where Γ' (respectively, Γ) has property P, then Γ (respectively, Γ') has property P. It is clear that the following hold.

- P lift stable $\Leftrightarrow \neg P$ factor stable.
- P lift stable and P factor stable implies that P trivial. (In this respect, factor-stable and lift-stable properties are 'half-trivial' properties.)

The following result explains the origin, for lift-/factor-stable group properties, of transition thresholds observed in the density models.

PROPOSITION 18. *Let P be a lift-stable (respectively, P' be a factor-stable) group property and let $(\Gamma, X, \{\Gamma_p\})$ be the deterministic data. There exists a*

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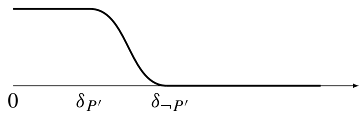
critical parameter $\delta_P = \delta_P(\Gamma, X, \{\Gamma_p\})$ *(respectively,* $\delta_{P'} = \delta_{P'}(\Gamma, X, \{\Gamma_p\})$*) such that the random group* Γ *in the density model of parameter* δ *over* $(\Gamma, X, \{\Gamma_p\})$ *satisfies the following:*

- Γ *has property* P *if* $\delta > \delta_P$ *(respectively,* Γ *has property* P' *if* $\delta < \delta_{P'}$*);*
- Γ *does not have property* P *if* $\delta < \delta_P$ *(respectively,* Γ *does not have property* P' *if* $\delta > \delta'_P$*);*

(with overwhelming probability). In particular, $\delta_{\neg P} \leqslant \delta_P$ *and* $\delta_{P'} \leqslant \delta_{\neg P'}$ *respectively—leading to a 'threshold':*



'Half-trivial' properties have one threshold

It seems that in known cases $\delta_P = \delta_{\neg P}$. For P such that $\delta_P > \delta_{\neg P}$, it could be informative to determine the true behavior of the probabilities on the interval between $\delta_{\neg P}$ and δ_P.

EXAMPLE 19. The proposition shows the existence of critical densities for the following.

- Kazhdan's property T (factor stable), say δ_T.
- Serre's property FA (factor stable), say δ_{FA}.
- Largeness, which is the existence of a finite index subgroup of Γ surjecting to a nonabelian-free group (lift stable), say $\delta^{\text{virt}}_{\twoheadrightarrow F_2}$.
- Virtual indicability (lift stable), say $\delta^{\text{virt}}_{\twoheadrightarrow \mathbb{Z}}$.

Here,

$$\delta_T \leqslant \delta_{FA} \leqslant \delta^{\text{virt}}_{\twoheadrightarrow F_2} \leqslant \delta^{\text{virt}}_{\twoheadrightarrow \mathbb{Z}}.$$

Observe that, if $P \Rightarrow P'$, where both P, P' are factor stable/both are lift stable, or if $P \Rightarrow \neg P'$, where P is factor stable and P' is lift stable, then $\delta_P \leqslant \delta_{P'}$.

The proof of Proposition 18 will follow from an assertion that 'the random group at density δ' is a quotient of the random group at a higher density $\delta > \delta'$', which we make precise in Lemma 21. The existence of transition thresholds does not generalize to properties which are not lift stable or factor stable. For example, a property like $T \vee \overset{\text{virt}}{\twoheadrightarrow} \mathbb{Z}$ admits *a priori* two thresholds. In general, the above

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result can be extended to any finite conjunction of properties which are lift stable or factor stable, resulting in a finite number of phase transitions.

We prove Lemma 21 in a more general model, called the 'α-model', which extends both the bounded model (where α is bounded) and the density model of parameter δ for any $\delta > 0$ (where $\alpha_p = |\mathscr{C}_p|^\delta$).

DEFINITION 20. Let $(X, \Gamma, \{\Gamma_p\})$ be the deterministic data, and let $\alpha : \mathbb{N} \to \mathbb{N}$ be an increasing sequence. The *α-model* over $(X, \Gamma, \{\Gamma_p\})$ is the $(X, \Gamma, \{\Gamma_p\}, \{\mathbb{P}_p\})$-model of random groups in which $\mathbb{P}_p$ selects α_p chambers in $\mathscr{C}_p$, uniformly and independently at random. Given $\alpha, \beta : \mathbb{N} \to \mathbb{N}$, we say that the α-model is dominated by the β-model if $\alpha_p \leqslant \beta_p$ for every n sufficiently large.

Zero density models, in which say $\alpha_p := o(|\mathscr{C}_p|^\delta)$ for every $\delta > 0$ (this includes the bounded model), can be used to provide variations on the model density $\delta > 0$, for example: $\alpha_p := |\mathscr{C}_p|^\delta + o(|\mathscr{C}_p|^\delta)$.

The bounded model is dominated by the density model of parameter δ, which is dominated by the density model of parameter $\delta' > \delta$. Lemma 21 makes explicit the fact that, if α dominates β, then 'the random group in the α-density model is a quotient of the random group in the β-density model'. We prove the case of lift-stable properties only, since the corresponding statement for factor-stable properties is analogous.

LEMMA 21. *Let P be a lift-stable group property, and let $(\Gamma, X, \{\Gamma_p\})$ be the deterministic data. If the α-model on $(\Gamma, X, \{\Gamma_p\})$ is dominated by the β-model on $(\Gamma, X, \{\Gamma_p\})$, then the probability that the random group has property P in the α-model is dominated by the probability that the random group has property P in the β-model.*

In particular, if P occurs with overwhelming probability in the α-model, then it does so in the β-model, establishing Proposition 18.

Proof. If P is lift stable, then

$$\begin{aligned} &|\{c_1, \ldots, c_{\alpha_p} \in \mathscr{C}_p \mid \Gamma_{\{c_1,\ldots,c_{\alpha_p}\}} \text{ has property P}\}| \\ &\quad = \frac{|\{c_1, \ldots, c_{\beta_p} \in \mathscr{C}_p \mid \Gamma_{\{c_1,\ldots,c_{\alpha_p}\}} \text{ has property P}\}|}{|\mathscr{C}_p|^{\beta_p - \alpha_p}} \\ &\quad \leqslant \frac{|\{c_1, \ldots, c_{\beta_p} \in \mathscr{C}_p \mid \Gamma_{\{c_1,\ldots,c_{\beta_p}\}} \text{ has property P}\}|}{|\mathscr{C}_p|^{\beta_p - \alpha_p}}. \end{aligned}$$

Therefore, if A_α denotes the random subset of $\mathscr{C}_p$ in the α-model and A_β denotes the random subset of $\mathscr{C}_p$ in the β-model, then

$$\mathbb{P}(\Gamma_{A_\alpha} \text{ has property P}) \leqslant \mathbb{P}(\Gamma_{A_\beta} \text{ has property P}).$$

This proves the assertion. □

However, a property like 'hyperbolicity' is neither factor stable nor lift stable, even though Gromov establishes a sharp phase transition in his density model.

The following weakening of lift/factor stability is consistent with the density model and allows for more general properties.

DEFINITION 22. A group property P is said to be *lift stable (respectively, factor stable or quotient stable) in the density model* if the random group at density δ satisfies property P with overwhelming probability whenever the random group at density $\delta' < \delta$ (respectively, $\delta' > \delta$) satisfies property P with overwhelming probability.

Thus, we have (tautologically) a critical density $\delta_P(\Gamma, X, \{\Gamma_p\})$ for every lift-stable property P in the density model, and similarly for factor-stable properties. Furthermore,

$$\delta_{\neg P}(\Gamma, X, \{\Gamma_p\}) \leqslant \delta_P(\Gamma, X, \{\Gamma_p\})$$

if P is lift stable and $\neg P$ factor stable.

PROPOSITION 23. *Let $(\Gamma, X, \{\Gamma_p\})$ be the deterministic data, where X is a CAT(0) space and Γ acts isometrically. Then the following hold.*

(1) *Gromov hyperbolicity is lift stable in the density model over $(\Gamma, X, \{\Gamma_p\})$. Therefore, there exists a critical density, denoted $\delta_{\mathbb{R}^2}(\Gamma, X, \{\Gamma_p\})$, for the random group in the density model to be hyperbolic.*

(2) *The existence of subgroups isomorphic to $\mathbb{Z}^2$ is factor stable in the density model. Therefore, there exists a critical density, denoted $\delta_{\mathbb{Z}^2}(\Gamma, X, \{\Gamma_p\})$, for the random group in the density model to contain a subgroup isomorphic to $\mathbb{Z}^2$.*

By the periodic flat plane theorem,

$$\delta_{\mathbb{R}^2}(\Gamma, X, \{\Gamma_p\}) \geqslant \delta_{\mathbb{Z}^2}(\Gamma, X, \{\Gamma_p\}).$$

PROBLEM 24. Given the deterministic data $(\Gamma, X, \{\Gamma_p\})$, where X is a CAT(0) space and Γ acts properly uniformly isometrically, is it true that

$$\delta_{\mathbb{R}^2}(\Gamma, X, \{\Gamma_p\}) = \delta_{\mathbb{Z}^2}(\Gamma, X, \{\Gamma_p\})?$$

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The flat closing conjecture implies that the answer is positive.

Proof of Proposition 23. (1) By Lemma 12(a), we have

$$\begin{aligned}|\{c_1, \ldots, c_{\alpha_p} \in \mathscr{C}_p \mid \Gamma_{\{c_1,\ldots,c_{\alpha_p}\}} \text{ is hyperbolic}\}| \\ = \frac{|\{c_1, \ldots, c_{\beta_p} \in \mathscr{C}_p \mid \Gamma_{\{c_1,\ldots,c_{\alpha_p}\}} \text{ is hyperbolic}\}|}{|\mathscr{C}_p|^{\beta_p-\alpha_p}} \\ \leqslant \frac{|\{c_1, \ldots, c_{\beta_p} \in \mathscr{C}_p \mid \Gamma_{\{c_1,\ldots,c_{\beta_p}\}} \text{ is hyperbolic}\}|}{|\mathscr{C}_p|^{\beta_p-\alpha_p}}.\end{aligned}$$

This shows that the random group in the β-model is hyperbolic provided that the random in the α-model is hyperbolic, for any β dominating α, which applies in particular to the density model.

(2) The same proof applies (using now Lemma 12(b)), with a reverse inequality. Namely, we have

$$\begin{aligned}|\{c_1, \ldots, c_{\alpha_p} \in \mathscr{C}_p \mid \Gamma_{\{c_1,\ldots,c_{\alpha_p}\}} \text{ contains } \mathbb{Z}^2\}| \\ = \frac{|\{c_1, \ldots, c_{\beta_p} \in \mathscr{C}_p \mid \Gamma_{\{c_1,\ldots,c_{\alpha_p}\}} \text{ contains } \mathbb{Z}^2\}|}{|\mathscr{C}_p|^{\beta_p-\alpha_p}} \\ \geqslant \frac{|\{c_1, \ldots, c_{\beta_p} \in \mathscr{C}_p \mid \Gamma_{\{c_1,\ldots,c_{\beta_p}\}} \text{ contains } \mathbb{Z}^2\}|}{|\mathscr{C}_p|^{\beta_p-\alpha_p}}.\end{aligned}$$

This shows that the random group in the α-model contains $\mathbb{Z}^2$ provided that the random in the β-model contains $\mathbb{Z}^2$, for any β dominating α. □

REMARK 25. The above discussion of 'lift-stable' and 'factor-stable' properties also applies to the classical Gromov models and shows the existence of critical densities, for example in the density model. Recall that there exist morphisms between the random group of different models at the *same* density, for example when comparing the usual model with the triangular model; see [**21**, I.3.g]. The above results are of a different nature in that they concern morphisms between random groups at distinct density regimes in a given model.

6. General facts on the random group at positive density

The random group Γ in Gromov's density model exhibits the following well-known phase transition at density $\delta = \frac{1}{2}$ (see [**15**] and Theorem 11 in [**21**]).

(1) If $\delta < 1/2$, then with overwhelming probability Γ is infinite hyperbolic, torsion free and of geometric dimension 2.

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(2) If $\delta > 1/2$, then with overwhelming probability Γ is trivial (either $\{e\}$ or $\mathbb{Z}/2\mathbb{Z}$).

In other words, $\delta_{\mathrm{Hyp}} = \delta_{\mathrm{triv}} = \delta_{\neg\mathrm{Hyp}} = \delta_{\neg\mathrm{triv}} = 1/2$.

The analogous result in our setting takes the following form. The proof, as in Gromov's case, is an illustration of the box principle.

PROPOSITION 26. *Let $(\Gamma, X, \{\Gamma_p\})$ be the deterministic data, and let $r \in \mathbb{N}_{\geqslant 2}$ be fixed. Then the random space X_A in the density model over $(X, \Gamma, \{\Gamma_p\})$ exhibits a phase transition with overwhelming probability:*

(1) *If $\delta < 1/2$, then the random set $A \subset X$ is r-separated (that is, the simplicial distance between any two elements of A is at least r).*

(2) *If $\delta > 1/2$, then there exist at least r pairs of adjacent elements of A.*

Proof. (1) Let $Y_1, Y_2, Y_3, \ldots$ be a sequence of independent uniformly distributed random variables with values in $\mathscr{C}_p$, and for $m \leqslant i \leqslant |\mathscr{C}_p|$ denote by $E_p^{(i)}(m)$ the event

$$E_p^{(i)}(m) = \{\mathrm{diam}(Y_{k_1}, \ldots, Y_{k_m}) \geqslant r \mid \forall 1 \leqslant k_1 < \cdots < k_m \leqslant i\}.$$

We have

$$\begin{aligned}\mathbb{P}(E_p^{(i)}(m)) &\\ = \sum_{(c_1,\ldots,c_{i-1})\in\mathscr{C}_p^{i-1}} & \mathbb{P}(E_p^{(i)}(m) \mid \{Y_1 = c_1, \ldots, Y_{i-1} = c_{i-1}\} \cap E_p^{(i-1)}(m)) \\ & \cdot \mathbb{P}(\{Y_1 = c_1, \ldots, Y_{i-1} = c_{i-1}\} \cap E_p^{(i-1)}(m)).\end{aligned}$$

Given the event $\{Y_1 = c_1, \ldots, Y_{i-1} = c_{i-1}\} \cap E_p^{(i-1)}(m)$, namely, that the faces c_1, $\ldots, c_{i-1}$ have been chosen and any m of them have diameter at least r, we obtain, letting $N_p(c_1, \ldots, c_{i-1})$ be the number of faces in X whose union with $m-1$ of the faces $c_1, \ldots, c_{i-1}$ forms a set of diameter at most r,

$$\mathbb{P}(E_p^{(i)}(m) \mid \{Y_1 = c_1, \ldots, Y_{i-1} = c_{i-1}\} \cap E_p^{(i-1)}(m)) = 1 - \frac{N_p(c_1, \ldots, c_{i-1})}{|\mathscr{C}_p|}.$$

Let N_p be the number of faces in the r neighborhood of a single chamber of X (this does not depend on the given chamber). Then

$$\begin{aligned}N_p(c_1, \ldots, c_{i-1}) &\leqslant N_p \cdot |\{(k_1, \ldots, k_{m-1}) \mid 1 \leqslant k_1 < \cdots < k_{m-1} \leqslant i-1\}| \\ &\leqslant N_p \cdot (i-1)^{m-1},\end{aligned}$$

so

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$$\mathbb{P}(E_p^{(i)}(m)) \geqslant \sum_{(c_1,\dots,c_{i-1})\in\mathscr{C}_p^{i-1}} \left(1 - \frac{N_p\cdot(i-1)^{m-1}}{|\mathscr{C}_p|}\right) \cdot \mathbb{P}(\{Y_1 = c_1, \dots, Y_{i-1} = c_{i-1}\} \cap E_p^{(i-1)}(m))$$
$$= \left(1 - \frac{N_p\cdot(i-1)^{m-1}}{|\mathscr{C}_p|}\right)\mathbb{P}(E_p^{(i-1)}(m)).$$

Using $e^{-2x} \leqslant 1 - x$ for all $x < 0.79$, we get

$$\mathbb{P}(E_n^{(i)}) \geqslant e^{-2\sum_{j=1}^{i}(N_p\cdot(j-1)^{m-1}/|\mathscr{C}_p|)} \geqslant e^{-2N_p(i(i-1)/|\mathscr{C}_p|)}.$$

Therefore, if $i = |\mathscr{C}_p|^\delta$, where $\delta < 1/2$, then $\mathbb{P}(E_n)$ is arbitrarily close to 1. This concludes the proof of (1). The proof of (2) is of a similar nature. □

Then the density model exhibits several extra phase transitions as the parameter δ increases. The transitions depend on the local structure of the initial space X:

Proposition 27. *Let $(\Gamma, X, \{\Gamma_p\})$ be the deterministic data, and let $r \in \mathbb{N}_{\geqslant 2}$, $k, \ell \in \mathbb{N}_{\geqslant 1}$ be fixed. Then the random space X_A in the density model over $(X, \Gamma, \{\Gamma_p\})$ admits a phase transition at density $\delta = k/(k+1)$. Namely, the following hold with overwhelming probability.*

(1) *If $\delta < k/(k+1)$, every ball B of radius r in X^2 contains at most k elements of A.*

(2) *If $\delta > (k-1)/k$, there exists an r-separated set $E \subset X^{(1)}$ of cardinality at least ℓ, such that every edge $e \in E$ is included in exactly* $\min(k, q_e + 1)$ *elements of A.*

Proof. (1) Let $Y_1, Y_2, Y_3, \dots$ be a sequence of independent uniformly distributed random variables with values in $\mathscr{C}_p$. For a ball $B \subset X^{(2)}$ of radius r, we write $Y_i \sim_B Y_j$ to mean that both Y_i and Y_j are included in B.

Let $B \subset X^{(2)}$ be a ball of radius r, and fix $1 \leqslant m_1 < \cdots < m_k \leqslant i$. Since

$$\mathbb{P}(Y_{m_1} \sim_B Y_{m_2} \sim_B \cdots \sim_B Y_{m_k}) \leqslant \left(\frac{b^*}{|\mathscr{C}_p|}\right)^k,$$

where $b^* := \max_{B\subset X^{(2)}} |B_r|$ (which is finite as X/Γ is compact), we obtain

$$\mathbb{P}(\exists 1 \leqslant m_1 < \cdots < m_k \leqslant i, \exists B, Y_{m_1} \sim_B Y_{m_2} \sim_B \cdots \sim_B Y_{m_k})$$
$$\leqslant \sum_{1\leqslant m_1<\cdots<m_k\leqslant i}\ \sum_{f\in\mathscr{C}_p} \mathbb{P}(Y_{m_1} \sim_{B_f} Y_{m_2} \sim_{B_f} \cdots \sim_{B_f} Y_{m_k})$$

$$\leqslant |\{(m_1, \ldots, m_k) \mid 1 \leqslant m_1 < \cdots < m_k \leqslant i\}| \cdot |\mathscr{C}_p| \cdot \left(\frac{b^*}{|\mathscr{C}_p|}\right)^k$$
$$\leqslant O\left(\frac{i^k}{|\mathscr{C}_p|^{k-1}}\right).$$

Therefore, if $i = |\mathscr{C}_p|^\delta$, where $\delta < k/(k-1)$, then with overwhelming probability every ball in $X^{(2)}$ of radius at most r contains at most k elements of the random subset A of $\mathscr{C}_p$, at density δ.

(2) Let $Y_1, Y_2, Y_3, \ldots$ be a sequence of independent uniformly distributed random variables with values in $\mathscr{C}_p$. Let $\mathscr{F}_{p,k}$ be the set of all k-tuples of elements of $\mathscr{C}_p$ whose faces are mutually adjacent to a same edge. For every $\psi \in \mathscr{F}_{p,k}$, we write $E_{i,k}(\psi)$ for the event

$$E_{i,k}(\psi) = \{\exists 1 \leqslant m_1, \ldots, m_k \leqslant i \ (\forall r \neq s,\ m_r \neq m_s) \mid Y_{m_1} = \psi_1, \ldots, Y_{m_k} = \psi_k\},$$

and let $\chi_{i,k}(\psi)$ be the characteristic function of $E_{i,k}(\psi)$. We will estimate the expectation of the random variable

$$Z_{p,k} := \sum_{\psi \in \mathscr{F}_{p,k}} \chi_{|\mathscr{C}_p|^\delta,k}(\psi)$$

whenever $\delta \in [(k-1)/k, k/(k+1)]$. Fix k and $\psi \in \mathscr{F}_{p,k}$. Write

$$\mathbb{P}(E_{i,k}(\psi)) = \mathbb{P}\left(\bigcup_{m_1,\ldots,m_k=1 \mid \forall r\neq s, m_r\neq m_s}^{i} Y_{m_1} = \psi_1, \ldots, Y_{m_k} = \psi_k\right).$$

We have

$$\mathbb{P}(E_{i,k}(\psi)) \geqslant \sum_{m_1,\ldots,m_k=1 \mid \forall r\neq s, m_r\neq m_s}^{i} \mathbb{P}(Y_{m_1} = \psi_1, \ldots, Y_{m_k} = \psi_k)$$
$$- \sum_{m_1,\ldots,m_k=1 \mid \forall r\neq s, m_r\neq m_s}^{i} \sum_{m'_1,\ldots,m'_k=1 \mid \forall r\neq s, m'_r\neq m'_s}^{i} \mathbb{P}(Y_{m_1} = \psi_1, \ldots, Y_{m_k} = \psi_k, Y_{m'_1} = \psi_1, \ldots, Y_{m'_k} = \psi_k)$$

by inclusion–exclusion, where

$$\mathbb{P}(Y_{m_1} = \psi_1, \ldots, Y_{m_k} = \psi_k) = \frac{1}{|\mathscr{C}_p|^k}$$

and

$$\mathbb{P}(Y_{m_1} = \psi_1, \ldots, Y_{m_k} = \psi_k, Y_{m'_1} = \psi_1, \ldots, Y_{m'_k} = \psi_k) \leqslant \frac{1}{|\mathscr{C}_p|^{2k}}.$$

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Therefore,

$$\mathbb{P}(E_{i,k}(\psi)) \geqslant \frac{i(i-1)\cdots(i-k-1)}{|\mathscr{C}_p|^k} - \frac{(i(i-1)\cdots(i-k-1))^2}{|\mathscr{C}_p|^{2k}}.$$

Substituting $|\mathscr{C}_p|^\delta$ for i, we deduce

$$\begin{aligned}\mathbb{E}(Z_{p,k}) &= \sum_{\psi\in\mathscr{F}_{p,k}} \mathbb{E}(\chi_{|\mathscr{C}_p|^\delta,k}(\psi))\\ &\geqslant |\mathscr{F}_{p,k}|\bigg(\frac{|\mathscr{C}_p|^\delta(|\mathscr{C}_p|^\delta-1)\cdots(|\mathscr{C}_p|^\delta-k-1)}{|\mathscr{C}_p|^k}\\ &\quad - \frac{|\mathscr{C}_p|^{2d}(|\mathscr{C}_p|^\delta-1)^2\cdots(|\mathscr{C}_p|^\delta-k-1)^2}{|\mathscr{C}_p|^{2k}}\bigg).\end{aligned}$$

Since, for p large enough,

$$\frac{|\mathscr{C}_p|^\delta - j}{|\mathscr{C}_p|} < \frac{1}{2},$$

whenever $\delta < 1$ and $j \leqslant k+1$, we have

$$\frac{|\mathscr{C}_p|^{2d}(|\mathscr{C}_p|^\delta-1)^2\cdots(|\mathscr{C}_p|^\delta-k-1)^2}{|\mathscr{C}_p|^{2k}} < \frac{|\mathscr{C}_p|^\delta(|\mathscr{C}_p|^\delta-1)\cdots(|\mathscr{C}_p|^\delta-k-1)}{2^k|\mathscr{C}_p|^k},$$

and therefore

$$\begin{aligned}\mathbb{E}(Z_{p,k}) &\geqslant |\mathscr{F}_{p,k}|\bigg(1-\frac{1}{2^k}\bigg)\bigg(\frac{|\mathscr{C}_p|^\delta(|\mathscr{C}_p|^\delta-1)\cdots(|\mathscr{C}_p|^\delta-k-1)}{|\mathscr{C}_p|^k}\bigg)\\ &\geqslant |\mathscr{F}_{p,k}|\bigg(1-\frac{1}{2^k}\bigg)\bigg(\frac{(|\mathscr{C}_p|^\delta-k-1)^k}{|\mathscr{C}_p|^k}\bigg).\end{aligned}$$

Now,

$$|\mathscr{F}_{p,k}| \geqslant \frac{3|\mathscr{C}_p|}{(q^*+1)},$$

so

$$\mathbb{E}(Z_{p,k}) \geqslant \frac{3}{(q^*+1)}\bigg(1-\frac{1}{2^k}\bigg)\bigg(\frac{(|\mathscr{C}_p|^\delta-k-1)^k}{|\mathscr{C}_p|^{k-1}}\bigg).$$

This shows that

$$\mathbb{E}(Z_{p,k}) \to \infty$$

whenever $\delta > (k-1)/k$. Since is X uniformly locally bounded, this shows that we can find at least ℓ edges $e_1, \ldots, e_\ell$ in X (for any ℓ fixed in advance), having

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at least k chambers adjacent to e_j removed. In addition, since the number of such edges grows to infinity, we may suppose that the pairwise distance between these edges is at least r from each other in X (for any r fixed in advance), and since the chambers are chosen independently at random, we may also suppose that exactly $\min(k, q_{e_j}+1)$ distinct chambers will be removed on $e_1, \ldots, e_\ell$. □

PROPOSITION 28. *Let* $(\Gamma, X, \{\Gamma_p\})$ *be the deterministic data, and let* $\ell \in \mathbb{N}_{\geqslant 2}$. *Then the random group* Γ_A *in the density model over* $(X, \Gamma, \{\Gamma_p\})$ *satisfies the following with overwhelming probability.*

(1) *If* $\delta < q_*(X)/(q_*(X)+1)$, *then* Γ_A *acts freely on a simply connected space* $\tilde{X}_A$ *without boundary with* X/Γ_A *compact.*

(2) *If* $\delta > q_*(X)/(q_*(X)+1)$, *then* Γ_A *splits off a free factor isomorphic to a free group* F_ℓ *on* ℓ *generators:*

$$\Gamma_A \simeq \Delta * F_\ell,$$

where Δ *is a finitely presented group, and* $q_*(X) := \min_{e\in X^{(1)}} q_e$.

In particular,

$$\delta_{FA}((\Gamma, X, \{\Gamma_p\})) \leqslant \frac{q_*(X)}{q_*(X)+1} < 1,$$

where δ_{FA} *is the critical density for Serre's property FA (existence of fixed point for actions on trees).*

Proof. The first assertion follows directly from Lemma 27, so we prove (2). Since X/Γ is compact, there exists a constant $\gamma > 0$ such that, if $q_*(X)$ denotes the lowest order of an edge in X, then

$$E_p := \{e \in \mathscr{C}_p^{(1)} \mid q_e = q_*(X)\}$$

satisfies

$$|E_p| \geqslant \gamma |\mathscr{C}_p^{(1)}|$$

for every $p > 0$. Let $k = q_*(X)$, and let $\tilde{\mathscr{F}}_{p,k}$ be the set of all k-tuples of elements of $\mathscr{C}_p$ whose faces are mutually adjacent to a same edge $e \in E_p$. The proof of Lemma 27, with the same notation except for $\tilde{\mathscr{F}}_{p,k}$ replacing $\mathscr{F}_{p,k}$, shows that

$$\mathbb{E}(Z_{p,k}) \geqslant |\tilde{\mathscr{F}}_{p,k}|\left(1-\frac{1}{2^k}\right)\left(\frac{(|\mathscr{C}_p|^\delta - k - 1)^k}{|\mathscr{C}_p|^k}\right),$$

and since now

$$|\tilde{\mathscr{F}}_{p,k}| \geqslant \frac{3\gamma|\mathscr{C}_p|}{(q^*+1)},$$

we obtain $\mathbb{E}(Z_{p,k}) \to \infty$. Therefore there exists, with overwhelming probability, an r-separated set $E \subset E_p$ of cardinal ℓ, such that every edge $e \in E$ is included in exactly $q_*(X)+1$ elements of $A \subset \mathscr{C}_p$. Let V_p be the quotient space X/Γ_p, let T_p be a maximal tree of the 1-skeleton of V_p, and choose a root $s_p \in T_p$. Let V_p' be the topological space obtained from V_p by collapsing T_p to s_p, which is homotopy equivalent to V_p. The elements of E/Γ, being face-free edges in

$$V_A := X_A/\Gamma_p \subset V_p,$$

form a wedge sum B_ℓ of ℓ edges in the collapse V_A' of V_A in V_p', and V_A' is the wedge sum of B_ℓ and a compact complex V_A''. The van Kampen theorem shows that $\Gamma_A \simeq \pi_1(V_A')$ splits as

$$\pi_1(V_A') = \pi_1(V_A'') * \pi_1(B_\ell),$$

where $\pi_1(B_\ell) \simeq F_\ell$ and $\pi_1(V_A'')$ is finitely presented.

In particular, we have

$$\delta_{\neg FA}((\Gamma, X, \{\Gamma_p\})) \leqslant \frac{q_*(X)}{q_*(X)+1},$$

proving the proposition. □

7. Random periodic flat plane problems II

We now prove the implication

$$\delta < \tfrac{5}{8} \Rightarrow \mathbb{Z}^2 \hookrightarrow \Gamma$$

in Theorem 1, and similar assertions over local rings.

In Theorem 1, the fixed global field (denoted k) is $\mathbf{F}_q(y)$, and the uniform lattice Γ in the deterministic data $(X, \Gamma, \{\Gamma_p\})$ is the Cartwright–Steger lattice [7] which is of type $\tilde{A}_2$ over $\mathbf{F}_q((y))$. The presentation of Γ by Lubotzky, Samuels, and Vishne in [18] is especially useful, where the congruence subgroups $\{\Gamma_p\}$ are described explicitly to illustrate the Ramanujan property of their quotients. We first review briefly the construction for notational purposes, referring to [18] for complete details.

Let D be the central simple algebra of degree 3 over k defined by

$$D = \bigoplus_{i,j=0}^{2} k\xi_i z^j$$

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with relations $z\xi_i = \phi(\xi_i)z$ and $z^3 = 1 + y$, where ϕ is a generator of $\mathrm{Gal}(\mathbf{F}_{q^3}/\mathbf{F}_q)$ and $\xi_i = \phi^i(\xi_0)$ is a basis for $\mathbf{F}_{q^3}$ over $\mathbf{F}_q$. Associated with D are algebraic groups over k defined by $\tilde{\mathbf{G}} = D^\times$ and $\mathbf{G} = D^\times/k^\times$. For a valuation ν on k, we let $D_\nu = D \otimes_k k_\nu$, and we say that D_ν splits whenever $D_\nu \simeq \mathrm{M}_3(k_\nu)$.

Let $T = \{\nu_{1/y}, \nu_{1+y}\}$ consisting of the degree valuation $\nu_{1/y}$ on k, and the valuation ν_{1+y} associated with the prime $1 + y$, namely, $\nu_{1+y}((1 + y)^i f/g) = i$ where the polynomials f, g are prime to $(1 + y)$. Then, by [18, Proposition 3.1], the algebra D_ν splits for all valuations $\nu \notin T$ on k, while it is a division algebra for $\nu \in T$.

Let $\nu_0 = \nu_y \notin T$ and $R_0 = \{x \in k \mid \nu(x) \geqslant 0,\ \forall \nu \neq \nu_0\} = \mathbf{F}_q[1/y]$. Since D_{ν_0} splits, we have $\mathbf{G}(k_{\nu_0}) \simeq \mathrm{PGL}_3(\mathbf{F}_q((y)))$, so $\mathbf{G}(R_0)$ embeds as a discrete subgroup of $\mathrm{PGL}_3(\mathbf{F}_q((y)))$. Since $T \neq \emptyset$ and $\mathbf{G}(k_\nu)$ is compact for $\nu \in T$, then Strong Approximation (see [18, Section 4]) shows that $\mathbf{G}(R_0)$ is a cocompact lattice in $\mathrm{PGL}_3(\mathbf{F}_q((y)))$.

The Cartwright–Steger lattice [7, Section 2] is the subgroup Γ of $\mathbf{G}(R_0)$ defined as follows. We note that, as stated above, the lattice $\mathbf{G}(R_0)$ is well defined only up to commensurability. A strict definition of $\mathbf{G}(R_0)$ depends upon fixing a embedding $\mathbf{G}$ into a linear group over k, which is chosen here to be $\mathrm{GL}_9(k)$ (see [18, Proposition 3.3]), so that $\mathbf{G}(R_0) := \mathbf{G}(k) \cap \mathrm{M}_9(R_0)$. Then Γ consists of matrices of $\mathbf{G}(R_0)$ whose reductions modulo $1/y$ are upper triangular with 3×3 identity blocks on the diagonal.

Another description of Γ from [18, Section 4] is as follows. Let R be the subring of k given by $R = \mathbf{F}_q[y, 1/y, 1/(1 + y)]$, and let $A(R)$ be the R-algebra $A(R) = \bigoplus_{i,j=0}^{2} R\xi_i z^j$ having the same defining relations as D, so that D appears as the algebra of central fractions of $A(R)$, namely, $D = (R\backslash\{0\})^{-1}A(R)$. We have $A^\times(R)/R^\times = \mathbf{G}(R_0)$ [18, Proposition 4.9]. Let

$$b = 1 - z^{-1} \in A^\times(R),$$

and, for $u \in \mathbf{F}_{q^3}^\times \subset A^\times(R)$, set $b_u = ubu^{-1} \in A^\times(R)$. Let $\tilde{\Gamma}$ be the subgroup of $A^\times(R)$ generated by the elements b_u. Then Γ is the quotient of $\tilde{\Gamma}$ modulo $R^\times$.

Write X for the Bruhat–Tits building associated to $\mathrm{PGL}_3(\mathbf{F}_q((y)))$. The vertexes of X are $\mathbf{F}_q[[y]]$-lattices in $\mathbf{F}_q((y))^3$, and incidence is given by flags. The action of $\mathbf{G}(R_0)$ on X (via its embedding in $\mathrm{PGL}_3(\mathbf{F}_q((y)))$) is transitive on vertexes. The group Γ is a normal subgroup of $\mathbf{G}(R_0)$ of finite index (see [7, Theorem 2.6]) which also acts transitively on the vertexes of X (in fact, the reduced norm of b is $y/(1 + y)$, which coincide with y up to the invertible element $1 + y$ of $\mathbf{F}_q[[y]]$). The congruence subgroups of Γ are defined as follows. Let I be an ideal in R. The quotient map $R \to R/I$ induces an epimorphism $A(R) \to A(R/I)$, which itself induces a group morphism $A^\times(R) \to A^\times(R/I)$. Denote the kernel by $A^\times(R, I)$ and its quotient modulo $R^\times$ by $(A^\times/R^\times)(R, I)$.

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Then we set $\Gamma(I) = \Gamma \cap (A^\times/R^\times)(R, I)$, and let $\Gamma_I = \Gamma/\Gamma(I)$ be the quotient group.

Let $f_p \in \mathbf{F}_q[y]$ be a monic irreducible polynomial prime to y and $y+1$, let $s_p \geqslant 1$ be an integer, and let $I_p = \langle f_p^{s_p} \rangle$. The deterministic data for Theorem 1 is $(X, \Gamma, \{\Gamma(I_p)\})$, where

- X is the Bruhat–Tits building of $\mathrm{PGL}_3(\mathbf{F}_q((y)))$;
- Γ is the Cartwright–Steger lattice in $\mathrm{PGL}_3(\mathbf{F}_q((y)))$;
- $\{\Gamma(I_p)\}$ are the arithmetic lattices associated with I_p.

We suppose that $\deg(f_p^{s_p}) \to \infty$.

The assertion that $\mathbb{Z}^2 \hookrightarrow \Gamma$ for these random groups is related to the elementary algebraic structure of PGL_3 and PSL_3 over fields and local rings. As usual for a local ring L, $\mathrm{GL}_3(L)$ is the group of units of $\mathrm{M}_3(L)$, $\mathrm{SL}_3(L)$ is the subgroup of $\mathrm{GL}_3(L)$ of matrices with determinant 1, and $\mathrm{PGL}_3(L)$, $\mathrm{PSL}_3(L)$ are the projective versions.

We start with a lemma. Let (L, m) be a finite local principal ring of prime characteristic with residue field $\mathbf{F}_q = L/m$, uniformizer π, and length s (the smallest integer $s > 0$ such that $m^s = 0$).

Lemma 29. *The order of the group $\langle \gamma, \gamma' \rangle$ generated by two commuting elements $\gamma, \gamma' \in \mathrm{PGL}_3(L)$ is at most*

$$3q^{2\lceil \log_q s \rceil}(q^3 - 1),$$

where $\log_q$ is the logarithm relative to q, and $\lceil \cdot \rceil$ is the upper integer value.

Proof. In the reduction modulo m

$$\mathrm{PGL}_3(L) \ni \gamma \mapsto \bar{\gamma} \in \mathrm{PGL}_3(\mathbf{F}_q),$$

an element $\gamma \in \mathrm{PGL}_3(L)$ such that $\bar{\gamma} = 1$ has a representative of the form

$$\lambda(1 + \pi^k \gamma_0) \in \mathrm{GL}_3(L)$$

for some $\lambda \in L^\times$, $\gamma_0 \in \mathrm{M}_3(L)$, and $k \geqslant 1$. The Frobenius map gives

$$(1 + \pi^k \gamma_0)^{q^{\lceil \log_q s \rceil}} = 1,$$

and therefore $\gamma^{q^{\lceil \log_q s \rceil}} = 1$.

Thus, if γ and γ' are two commuting elements in $\mathrm{PGL}_3(L)$, then the kernel K in the exact sequence

$$1 \to K \to \langle \gamma, \gamma' \rangle \to \langle \bar{\gamma}, \bar{\gamma}' \rangle \to 1$$

is an abelian group of order at most $q^{2\lceil \log_q s \rceil}$.

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We now estimate $|\langle\bar\gamma, \bar\gamma'\rangle|$. Assume without loss of generality that $L = \mathbf{F}_q$. Let $\underline{\gamma}, \underline{\gamma}'$ be representatives of γ, γ' in $\mathrm{GL}_3(L)$, and let $f \in L[X]$ be the characteristic polynomial of $\underline{\gamma}$.

Assume that f does not have a root in L. Then f is irreducible over L (since a factorization would split off a degree-1 factor) and is therefore equal to its minimal polynomial. It follows that the rational canonical form of $\underline{\gamma}$ has only one block associated with f, and $\underline{\gamma}$ has a cyclic vector ξ. Given $\gamma' \in \mathrm{PGL}_3(L)$, there exists $P \in L[X]$ such that

$$\underline{\gamma}'\xi = P(\underline{\gamma})\xi.$$

If

$$\underline{\gamma}\,\underline{\gamma}' = \lambda\underline{\gamma}'\underline{\gamma}$$

for some $\lambda \in L^\times$, then

$$\underline{\gamma}^k\underline{\gamma}' = \lambda^k\underline{\gamma}'\underline{\gamma}^k,$$

so

$$\underline{\gamma}'\underline{\gamma}^k\xi = \lambda^{-k}\underline{\gamma}^k P(\underline{\gamma})\xi = \lambda^{-k}P(\underline{\gamma})\underline{\gamma}^k\xi.$$

Thus,

$$\underline{\gamma}' = P(\underline{\gamma})\begin{pmatrix}1 & 0 & 0\\ 0 & \lambda^{-1} & 0\\ 0 & 0 & \lambda^{-2}\end{pmatrix}$$

in the basis $(\xi, \underline{\gamma}\xi, \underline{\gamma}^2\xi)$. Taking determinants, the equality $\underline{\gamma}\,\underline{\gamma}' = \lambda\underline{\gamma}'\underline{\gamma}$ shows that λ is a cube root of 1. Thus every element of $\langle\underline{\gamma}, \underline{\gamma}'\rangle$ belong to the subspace

$$\{P(\underline{\gamma}) \mid P \in L[X]\}$$

of $M_3(L)$, possibly after multiplying by an element of the form

$$\begin{pmatrix}1 & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda^2\end{pmatrix}$$

$\lambda \in L^\times$ a cube root of 1. Since the dimension of $\{P(\underline{\gamma}) \mid P \in L[X]\}$ over L is 3, the order of $\langle\underline{\gamma}, \underline{\gamma}'\rangle$ is at most $3(q^3 - 1)$.

A similar argument works more generally if f equals the minimal polynomial. Otherwise, f has a root a in L, and we may assume up to conjugacy that $\underline{\gamma}$ has the form $\left(\begin{smallmatrix}a & 0\\ 0 & \alpha\end{smallmatrix}\right)$, where $a \in L^\times$ and $\alpha \in \mathrm{GL}_2(L)$. Write

$$\underline{\gamma}' = \begin{pmatrix}b & m\\ n & \beta\end{pmatrix},$$

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$b \in L^\times$, $m^t, n \in L^2$, and $\beta \in \mathrm{GL}_2(L)$. Since

$$\underline{\gamma\gamma'} = \lambda\underline{\gamma'}\underline{\gamma}$$

for some $\lambda \in L^\times$, we have $ab = \lambda ba$, so

$$\lambda = 1,$$

and thus $m\alpha = am$, $\alpha n = an$, and

$$\alpha\beta = \beta\alpha.$$

If $m = n = 0$, then the argument of the previous section shows that the order of $\langle\alpha, \beta\rangle$ is at most $q^2 - 1$, so the order of $\langle\gamma, \gamma'\rangle$ is at most $q^3 - 1$. Otherwise, a is an eigenvalue of α, and it is easy to see that the order of $\langle\gamma, \gamma'\rangle$ is at most $q^3 - 1$ in this case too, which proves the lemma. □

Proof of the assertion that $\mathbb{Z}^2 \hookrightarrow \Gamma$ in Theorem 1. The local ring

$$R/I_p \simeq \mathbf{F}_q[y]/\langle f_p(y)^{s_p}\rangle$$

has residue field

$$\mathbf{F}_{q_p} = R/\langle f_p\rangle \simeq \mathbf{F}_q[y]/\langle f_p(y)\rangle.$$

By [18, Theorem 6.6],

$$\mathrm{PSL}_3(R/I_p) \subset \Gamma_{I_p} \subset \mathrm{PGL}_3(R/I_p).$$

Let Λ be any subgroup of Γ isomorphic to $\mathbb{Z}^2$. Lemma 29 shows that the order of the image of Λ in Γ_{I_p} is at most

$$3q_p^{2\lceil \log_{q_p} s\rceil}(q_p^3 - 1),$$

where q_p is the order of the residue field. Denote by $\Pi \simeq \mathbb{R}^2$ the flat associated to Λ in X, and let $\mathscr{C}_p$ be the set of $\Gamma(I_p)$-orbits of chambers in X. The set F_p of $\Lambda \cap \Gamma(I_p)$ orbits of chambers in Π is finite, and there is a constant C (namely, $C = 3\times$ the number of Λ orbits of chambers in Π) such that

$$|F_p| \leqslant C q_p^{2\lceil \log_{q_p} s_p\rceil + 3}.$$

Let F'_p be the image of F_p in $\mathscr{C}_p$.

Consider a sequence $Y_1, Y_2, Y_3, \ldots$ of independent identically distributed random variable with values in $\mathscr{C}_p$. The probability $\mathbb{P}(E_n)$ of the event,

$$E_n = \{Y_i \notin F'_p, \forall i \leqslant |\mathscr{C}_p|^\delta\},$$

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can be estimated by

$$\mathbb{P}(E_n) \geqslant \left(1 - \frac{|F'_p|}{|\mathscr{C}_p|}\right)^{|\mathscr{C}_p|^\delta},$$

and therefore

$$\mathbb{P}(E_n) \geqslant e^{-2|\mathscr{C}_p|^{\delta-1}|F'_p|}.$$

Since Γ_{I_p} acts freely on $\mathscr{C}_p$ with $\mathscr{C}_p/\Gamma_{I_p}$ fixed,

$$|\mathscr{C}_p| \geqslant D|\Gamma_{I_p}|$$

for some constant $D > 0$. Let us compute $|\Gamma_{I_p}|$. The reduction map

$$R/I_p \to R/\langle f_p \rangle = \mathbf{F}_{q_p}$$

induces a surjective map

$$\pi : \mathrm{GL}_3(R/I_p) \to \mathrm{GL}_3(\mathbf{F}_{q_p}),$$

since det: $\mathrm{GL}_3(R/I_p) \to (R/I_p)^\times$ commutes with reduction modulo $\langle f_p \rangle$, and since an element is invertible in R/I_p if and only if its image in $\mathbf{F}_{q_p}$ is nonzero (as R/I_p is local). Therefore any pull-back of a matrix of $\mathrm{GL}_3(\mathbf{F}_{q_p})$ in $\mathrm{M}_3(R/I_p)$ is a matrix in $\mathrm{GL}_3(R/I_p)$. The kernel of π consists of matrices of the form $\mathrm{Id} + \gamma_0$ where all coefficient of γ_0 belong to the ideal $\langle f_p \rangle$. Therefore

$$|\mathrm{GL}_3(R/I_p)| = q_p^{9(s_p-1)}|\mathrm{GL}_3(\mathbf{F}_{q_p})| = q_p^{9s_p-6}(q_p^3-1)(q_p^2-1)(q_p-1).$$

As det is surjective and $a \in (R/I_p)^\times$ if and only if a is not a multiple of f_p, we have that

$$\begin{aligned}|\mathrm{SL}_3(R/I_p)| &= \frac{1}{|(R/I_p)^\times|}|\mathrm{GL}_3(R/I_p)| \\ &= \frac{1}{q_p^{s_p-1}(q_p-1)} q_p^{9s_p-6}(q_p^3-1)(q_p^2-1)(q_p-1) \\ &= q_p^{8s_p-5}(q_p^3-1)(q_p^2-1).\end{aligned}$$

This gives

$$|\Gamma_{I_p}| \geqslant |\mathrm{PSL}_3(R/I_p)| = \mu_p^{-1}|\mathrm{SL}_3(R/I_p)| = \mu_p^{-1}q_p^{8s_p-5}(q_p^3-1)(q_p^2-1),$$

where $\mu_p = |\{a \in (R/I_p)^\times \mid a^3 = 1\}|$. Therefore,

$$\begin{aligned}|\mathscr{C}_n|^{\delta-1}|F'_p| &\leqslant CD^{\delta-1}\mu_p^{1-\delta} q_p^{2\lceil \log_{q_p} s_p\rceil+3} q_p^{8s_p(\delta-1)} \\ &= CD^{\delta-1}\mu_p^{1-\delta} q_p^{2\lceil \log_{q_p} s_p\rceil+8s_p(\delta-1)+3}.\end{aligned}$$

Thus, since $\lceil \log_{q_p} s_p\rceil / s_p \longrightarrow_{n\to\infty} 0$ when $\deg f_p^{s_p} \to \infty$, we obtain the following.

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(i) If $s_p = 1$ for large n, then $(R/I_p)^\times$ is cyclic, $\mu_p \leqslant 3$ (for large n), and thus $\mathbb{P}(E_n) \to 1$ if

$$8(\delta - 1) + 3 < 0;$$

that is,

$$\delta < \tfrac{5}{8}.$$

(ii) If $s_p \geqslant k \geqslant 2$ for large n, then $(R/I_p)^\times$ is not cyclic in general, but using the rough estimate $\mu_p \leqslant q_p^{s_p - 1}$ (or $\mu_p \leqslant q_p - 1$ if the order satisfy $q \neq 3^\ell$, $\ell = 1, 2, 3, \ldots$), we have that $\mathbb{P}(E_n) \to 1$ if

$$(1 - \delta)(s_p - 1) + 8s_p(\delta - 1) + 3 < 0;$$

that is,

$$\delta < \frac{7k - 2}{7k + 1}.$$

(iii) If $(f_p^{s_p})$ does not satisfy (i) or (ii), we separate the cases $s_p \geqslant 2$ and $s_p = 1$, and obtain the desired conclusion when $\delta < \frac{5}{8}$ as $(7k - 2)/(7k + 1) > \frac{5}{8}$ for $k \geqslant 2$.

This shows that $\mathbb{P}(E_n) \to 1$. Therefore, Lemma 11 applies with overwhelming probability, and we have that $\mathbb{Z}^2 \hookrightarrow \Gamma$ with overwhelming probability. $\square$

REMARK 30. We have proved the stronger result that every embedding $\mathbb{Z}^2 \hookrightarrow \Gamma_3$ eventually lifts to an embedding $\mathbb{Z}^2 \hookrightarrow \Gamma$ with overwhelming probability. Recall that, according to Mostow and Prasad, the set of embeddings $\mathbb{Z}^2 \hookrightarrow \Gamma_3$ is dense [20].

QUESTION 31. Are these random groups residually finite? (For comparison, note that there exist nonresidually finite [8] central extensions of the form $1 \to \mathbb{Z} \to \widetilde{\mathrm{Sp}(2n, \mathbb{Z})} \to \mathrm{Sp}(2n, \mathbb{Z}) \to 1$.)

QUESTION 32. What is the value of $\delta_{\mathbb{Z}^2}(X_3, \Gamma_3, \{\Gamma_3(I_p)\})$? What is the value of $\delta_{\mathbb{R}^2}(X_3, \Gamma_3, \{\Gamma_3(I_p)\})$?

QUESTION 33 ('Perforated buildings'). If X is a Bruhat–Tits building of rank 2 (possibly exotic) and $Y \subset X$ is an R-separated subset of chambers, does $\mathbb{R}^2 \hookrightarrow X \setminus Y$ for R large enough?

REMARK 34. It is proved in [18] that the congruence quotients of X_3 by $\Gamma_3(I_p)$ are Ramanujan complexes. This fact, whose proof relies on the 'Jacquet–Langlands correspondence' (in positive characteristic), is not used in this paper.

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(Nevertheless, the random quotients X/Γ as in Theorem 1 are Ramanujan complexes 'up to ε', at least if the density is not too large.) The main input from algebraic groups used in the proof is Strong Approximation, and so it is most likely that results in the spirit of Theorem 1 can also be established for other algebraic groups (e.g. Sp) and over other fields (e.g. characteristic 0), the first step being to construct explicit families of lattices. We also note that the case of positive characteristic is easier to handle from an algorithmic perspective (the construction of f_p can be done in polynomial time using the Shoup algorithm).

8. Further properties of the random group

The critical density $\delta_T(X, \Gamma, \{G_p\})$ for Kazhdan's property T can be estimated using the so-called 'spectral criterion for property T', which is local. This is done by making (spectral) bounds on the local geometric perturbations explicit, following [10, Section 6] and [9].

We recall (see [5] for more details) that the 'spectral criterion' refers to the following well-known result of Garland [10, Theorem 5.9].

THEOREM 35 (Garland). *Let V be finite simplicial complex of dimension ℓ in which every simplex belongs to an ℓ-cell, and let $r \leqslant \ell$. Assume that, for every vertex $v \in V^{(0)}$, the rth reduced cohomology group $\tilde{H}^r(\mathrm{Lk}\, v, \mathbb{R})$ [i.e. $\tilde{H}^r = H^r$ if $r \geqslant 1$ and $\tilde{H}^0 = \mathrm{coker}(H^0(\star) \to H^0)$] of the link $\mathrm{Lk}\, v$ of V vanishes, and denote by $\lambda_r(\mathrm{Lk}\, v)$ the smallest nonzero eigenvalue of $\Delta^+ = d\partial$ acting on $C^r(\mathrm{Lk}\, v)$. Set $\lambda_r := \min_v \lambda_r(\mathrm{Lk}\, v)$. If*

$$\lambda_{r-1} \geqslant \frac{\ell - r}{r+1},$$

then $H^r(V, \rho) = 0$ for every finite-dimensional unitary representation ρ of $\pi_1(V)$.

(If $\ell = 2$, $r = 1$, this says that

$$\lambda_0 > 1/2 \Rightarrow H^1(V, \rho) = 0$$

for any ρ.)

The statement in [10] deals only with those V whose universal coverings are Bruhat–Tits buildings, but Borel points out [6, Theorem 1.5] that Garland's argument is general. (While we are mostly interested in Bruhat–Tits buildings here, it is the general case that is useful for us.) Borel also notes in Section 2.2 that the assumption that V is finite can be replaced by 'uniformly locally finite', provided that the conclusion refers to the vanishing of $L^2H^r(V, \rho)$ (L^2-summable cochains). It was later shown that the family of admissible ρ with vanishing H^r could be substantially enlarged; in particular, the linearity assumption on ρ can be disposed of (in the smooth category [24]), as can the finite-dimensionality assumption on ρ (see [1, 23, 25]). The latter leads to the conclusion that $H^r(V, \rho) = 0$ for every unitary representation ρ of $\pi_1(V)$ which, when $r = 1$, is equivalent to Γ having Kazhdan's property T by a well-known result of Delorme and Guichardet (see [5]). Garland studies in particular the two-dimensional case, exploiting the computation of the exact value of λ_0 for Tits buildings (associated with BN-pairs, the computation appears in [9] and [10, Proposition 7.10]) to derive the vanishing of H^1 'when the order is large enough'. (As Garland observes, his cohomology vanishing results for lattices—with finite-dimensional unitary targets—were in fact already known in degree 1, as they were covered by Kazhdan's work.)

We now explain why property T holds for random groups 'when the order is large enough' (in particular, when $q \geqslant 5$ if $\delta < 1/2$, and the deterministic data is of type $\tilde{A}_2$, and for $q \gg 1$ if δ is only assumed to be bounded away from 1).

The idea is that λ_0 is controlled if the number of chambers removed is significantly smaller than the order (defined as the minimal number of chambers on an edge minus 1) of V.

LEMMA 36. *For every $\delta > 0$ and every integer $n \geqslant 0$, there exists a constant $\delta' > 0$ such that, if G is a graph of order $q \geqslant 5n/\delta$ with $\lambda_0(G) > 1 - \delta'$, and $G' \subset G$ is a subgraph with $|G'^{(1)}| \geqslant |G^{(1)}| - n$, then $\lambda_0(G') \geqslant 1 - \delta$.*

Indeed, by the Cheeger inequality (for graphs), it is enough to prove the corresponding statement for the Cheeger constant; then the numerators of the two isoperimetric ratios differ by a constant that is washed out as $q \to \infty$.

More explicitly, we have the following.

Proof. For a subset $A \subset V$, let $h_G(A) := |\partial_G A|/(\min(|A|_G, |X - A|_G))$, where $|\partial_G A|$ is the number of edges with extremities belonging to both A and $X - A$, and $|A|_G := \sum_{x\in A} \mathrm{val}_G(x)$. The Cheeger constant of G is $h(G) := \min_{A\subset G} h_G(A)$. Let A be a subset of V' such that $h(G') = h(A)$. By the Cheeger inequality,

$$\lambda_0(G) \leqslant 2h(G) \leqslant \frac{2|\partial_G A|}{\min(|A|_G, |X - A|_G)}$$
$$\leqslant \frac{2|\partial_{G'} A| + 2n}{\min(|A|_{G'}, |X - A|_{G'})} = 2h(G') + \frac{2n}{\min(|A|_{G'}, |X - A|_{G'})}.$$

Since $q \geqslant 5n/\delta$, we obtain $\lambda_0(G) \leqslant 2h(G') + \delta/2$. Thus

$$\lambda_0(G) \leqslant 2\sqrt{2\lambda_0(G')} + \delta/2,$$

using again the Cheeger inequality. □

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Let X be a thick irreducible Euclidean building of rank 2. The links of X are spherical buildings, and there exist integers $q^* \geqslant q_* \geqslant 2$ such that the degree of an edge in X is either $q^* + 1$ or $q_* + 1$. We call q_* the order of X. In the $\tilde{A}_2$ case, $q^* = q_*$.

THEOREM 37 (Generic property T). *For every $\delta_0 < 1$, there exists q_0 such that, if $\delta < \delta_0$ and if in the deterministic data $(X, \Gamma, \{\Gamma_p\})$, the space X is a thick classical (i.e. associated with an algebraic group over a local field) irreducible Bruhat–Tits building of rank 2 and order $q_* \geqslant q_0$, then the random group at density δ with deterministic data $(X, \Gamma, \{\Gamma_p\})$ has Kazhdan's property T with overwhelming probability. Thus*

$$\delta_T(X, \Gamma, \{\Gamma_p\}) \to 1$$

as $q_*(X) \to \infty$.

REMARK 38. The groups appearing in the above result (and in Corollary 17) are buildings with chambers missing in the sense of [3]. Some (weak) buildings with chambers missing have the 'opposite' Haagerup property; see [3]. We also note that random groups in the density model admit an infinite quotient with property T, *independently of the order*, as soon as we are given deterministic data with property T.

Proof. Let k be sufficiently large that $\delta_0 < k/(k+1)$. Let $0 < \varepsilon < 1/2$ be given. By Lemma 36, we can find $\varepsilon' > 0$ such that, if $\lambda_0(G) > 1 - \varepsilon'$, then $\lambda_0(G - k) > 1 - \varepsilon > 1/2$. Feit and Higman [9] and Garland [10] show that, if G_q is a spherical building of order q, then $\lambda_0(G_q)$ converges to 1 as $q \to \infty$. Let q_0 be larger than $10k$ and be such that $\lambda_0(G_q) > 1 - \varepsilon'$ for any spherical building G_1 of order $q \geqslant q_0$. If, in the deterministic data $(X, \Gamma, \{\Gamma_p\})$, the space X is a thick classical irreducible building of rank 2 such that $q_*(X) \geqslant q_0$, and if $\delta < \delta_0$, then, by Proposition 27, the links of X contain at most k chambers missing at every vertex, with overwhelming probability. Therefore, the random perforated building $X \setminus \Gamma A$ at density δ satisfies $\lambda_0(X \setminus \Gamma A) \geqslant 1 - \varepsilon > 1/2$ with overwhelming probability. Since the random universal cover $\widetilde{X \setminus \Gamma A}$ is locally isomorphic to $X \setminus \Gamma A$, Theorem 35 applies. Thus the random group has property T (and the spectral gap is arbitrarily close to 1) with overwhelming probability. □

It is natural to wonder if the bound on the order can be estimated. It seems difficult to give a general answer, but we can do it at least when the density is $<1/2$ (or in the model with few chambers missing). In fact, one can compute the *exact* value of $\lambda_0(X)$, in the spirit of Feit and Higman's paper [9].

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More precisely, we have the following.

THEOREM 39. *If $(X, \Gamma, \{\Gamma_p\})$ is of type $\tilde{A}_2$, then, both in the bounded model and in the density model of parameter $\delta < 1/2$, the random group has property T (with overwhelming probability) when $q \geqslant 5$.*

This follows from the following.

PROPOSITION 40. *Let G be a spherical building of type A_2 and order q with a single chamber missing. Then*

$$\lambda_0(G) = 1 - \frac{\sqrt{q + 1/4} + 1/2}{q + 1}.$$

In particular, $\lambda_0(G) > 1/2$ whenever $q \geqslant 5$.

We recall (see, e.g., [5]) that the value computed by Feit and Higman in the A_2 case is

$$\lambda_0(G) = 1 - \frac{\sqrt{q}}{q + 1};$$

so, in particular, $\lambda_0(G) > 1/2$ whenever $q \geqslant 2$.

Proof. Let $G \rightsquigarrow G'$ be an extension into an A_2 building of order q (see [3]), and let P be the projective plane corresponding to G'. We choose the following basis for the space of functions on the vertex set of G:

$$(\delta_{p_1}, \delta_{p_2}, \ldots \delta_{p_{q^2+q+1}}, \delta_{l_1}, \delta_{l_2}, \ldots, \delta_{l_{q^2+q+1}}),$$

where $p_1 \in G$ (respectively $l_1 \in G$) corresponds to the point p (respectively, the line l) of P associated to the missing chamber, while $p_2, \ldots, p_{q+1}$ (respectively, $l_2, \ldots, l_{q+1}$) correspond to an enumeration of the points of l distinct from p (respectively, the lines adjacent to p distinct from l) in P.

The Laplace operator Δ is of the form

$$\Delta = \mathrm{Id} - \begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix},$$

where A is the normalized adjacency matrix, namely $A = D_q^{-1/2} A_0 D_q^{-1/2}$, where A_0 is the usual (bipartite) adjacency matrix (we recall that $(A_0)_{i,j} = 1$ when l_i is adjacent to p_j), and D_q is the diagonal matrix having $q + 1$ down the diagonal, except on the first entry which is q. Denote $\tilde{q} = \sqrt{q(q + 1)}$. A computation shows

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that

$$AA^t = A^tA = \begin{pmatrix} q\tilde{q}^{-2} & 0_q & (q+1)^{-1}\tilde{q}^{-1}\mathbf{1}_{1\times q^2} \\ 0_q & B_q & (q+1)^{-2}\mathbf{1}_{q\times q^2} \\ (q+1)^{-1}\tilde{q}^{-1}\mathbf{1}_{q^2\times 1} & (q+1)^{-2}\mathbf{1}_{q^2\times q} & C_{q^2} \end{pmatrix},$$

where B_q is the $q \times q$ matrix with

$$\frac{q}{(q+1)^2} + \frac{1}{q(q+1)}$$

on the diagonal, and $(q-1)/q(q+1)$ elsewhere, while C_{q^2} is the $q^2 \times q^2$ matrix with $(q+1)^{-1}$ on the diagonal and $(q+1)^{-2}$ elsewhere. Set

$$\gamma_q = (q+1)\tilde{q}^{-1} = \sqrt{1+\frac{1}{q}}.$$

We have

$$(q+1)^2A^tA - q\mathrm{Id} = \begin{pmatrix} 1 & 0_q & \gamma_q\mathbf{1}_{1\times q^2} \\ 0_q & \gamma_q^2\mathbf{1}_{q\times q} & \mathbf{1}_{q\times q^2} \\ \gamma_q\mathbf{1}_{q^2\times 1} & \mathbf{1}_{q^2\times q} & \mathbf{1}_{q^2\times q^2} \end{pmatrix},$$

whose kernel is of codimension 3. Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0_q \\ \gamma_q\mathbf{1}_{q^2\times 1} \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ \gamma_q^2\mathbf{1}_{q\times 1} \\ \mathbf{1}_{q^2\times 1} \end{pmatrix} \quad v_3 = \begin{pmatrix} \gamma_q \\ \mathbf{1}_{q^2+q\times 1} \end{pmatrix}.$$

A direct computation shows that

$$\begin{aligned}
(q+1)^2A^tAv_1 &= (q+1)v_1 + q^2\gamma_q v_3 \\
(q+1)^2A^tAv_2 &= (q+q\gamma_q^2)v_2 + q^2v_3 \\
(q+1)^2A^tAv_3 &= \gamma_q v_1 + qv_2 + (q^2+q)v_3.
\end{aligned}$$

Thus the problem reduces to computing the spectrum of the 3×3 matrix

$$\begin{pmatrix} q+1 & 0 & \gamma_q \\ 0 & 2q+1 & q \\ q^2\gamma_q & q^2 & q^2+q \end{pmatrix}.$$

The characteristic polynomial is

$$-x^3 + (q^2+4q+2)x^2 - (2q^3+6q^2+4q+1)x + q^2(q+1)^2,$$

where $(q+1)^2$ is an obvious root. The two other roots are

$$x = (2q + 1 - \sqrt{4q + 1})/2$$

and

$$x = (2q + 1 + \sqrt{4q + 1})/2.$$

Therefore, the eigenvalue of $\begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix}$ we are interested in is

$$\frac{1}{q+1}\sqrt{q + \frac{1}{2} + \sqrt{q + \frac{1}{4}}} = \frac{\frac{1}{2} + \sqrt{q + \frac{1}{4}}}{q+1}. \qquad \square$$

A similar computation can be worked out when two chambers are missing (arguing according to the respective positions of the chambers in an apartment), but the value of

$$\lambda_{0,k}(G) := \inf_{E \subset G^{(1)}, |E| = k} \lambda_0(G \setminus E)$$

when G is a spherical building (of dimension 1) might be more challenging to find. It would also be interesting to make similar estimates with more general (e.g. smooth) targets $\lambda_{0,k}(G, Y)$ and deduce the corresponding fixed points theorem for the random groups following [17, 24].

The results in [3] on buildings with chambers missing and [2] on property RD can be applied to our random groups in the $\tilde{A}_2$ case. This shows in particular that the random group is *rigid* in the sense that it remembers the building that it comes from (see [3, Section 5]). Furthermore, by [2, Corollary 6], the random group satisfies the Baum–Connes conjecture without coefficients.

THEOREM 41. *If the deterministic data is of type $\tilde{A}_2$, then the following hold.*

(1) *The random group (Γ, X) at density $\delta < \frac{1}{2}$ admits the unique extension $(\Gamma, X) \rightsquigarrow (\Gamma', X')$ (in the sense of [3, Section 1]) into a Euclidean building.*

(2) *The random group Γ at arbitrary density satisfies the Baum–Connes conjecture without coefficients.*

The first assertion is a consequence of Proposition 26 and [3, Theorem 5.11]. It is unknown if Γ satisfies the Baum–Connes conjecture with coefficients. The conjecture without coefficients is also unknown for the other Coxeter types (in the irreducible case), but would follow from the 'interpolation of property RD to intermediate rank situations' (see [2]), that is, between hyperbolic groups and (uniform) higher rank lattices of the corresponding type, provided that the latter groups satisfy property RD (which is conjectured by Valette).

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